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Charecterization of Gaussian semigroups on a Lie group

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. It is shown that a convolution semigroup of probability measures on a Lie group is Gaussian if and only if the infinitesimal generator of the corresponding semigroup of Fourier transforms satisfies some equation. The result is similar to the characterization of Gaussian measures on a compact Lie group due to Carnal [1].

1. Introduction

Carnal [1] has proved the following characterization of Gaussian measures on a compact Lie group G: let μ be a probability measure on G embeddable into a convolution semigroup; then μ is a Gaussian measure if and only if its Fourier transform $\hat{\mu}$ satisfies the equation

$$|\det(\hat{\mu}(D \otimes D))| \cdot |\det(\hat{\mu}(D \otimes \overline{D}))| = |\det(\hat{\mu}(D))|^{4n(D)}$$

for any irreducible unitary representation D of G where $n(D)$ is the dimension of the representation space of D. The aim of the present note is to give a similar characterization of Gaussian semigroups on arbitrary Lie groups.

2. Preliminaries

Let G be a Lie group of dimension $m \geq 1$ with neutral element e. Let $G^{\times} := G \setminus \{e\}$. Let $\mathcal{U}(e)$ denote the system of all neighborhoods

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of e. By $\mathcal{C}^b(G)$ we denote the space of bounded continuous complex-valued functions on G equipped with the supremum norm $\|\cdot\|_\infty$. Let $\mathcal{D}(G)$ be the space of infinitely differentiable complex-valued functions with compact support on G. The space $\mathcal{E}(G)$ of bounded *regular functions* on G is defined by

$$\mathcal{E}(G) := \left\{ f \in \mathcal{C}^b(G) : f \cdot g \in \mathcal{D}(G) \quad \text{for all} \quad g \in \mathcal{D}(G) \right\}.$$

Let $\mathcal{G}$ be the Lie algebra of G and $\exp : \mathcal{G} \mapsto G$ the exponential mapping. An element $X \in \mathcal{G}$ can be regarded as a (left-invariant) differential operator on G: for $f \in \mathcal{D}(G)$ we put

$$Xf(x) = \lim_{t \to 0} \frac{f(x \exp tX) - f(x)}{t}.$$

$\mathcal{M}_+(G)$ is the space of positive Radon measures on G, $\mathcal{M}_+^b(G)$ the subspace of bounded measures and $\mathcal{M}^1(G)$ the set of probability measures on G which, furnished with the operation of convolution $*$ and the weak topology, is a topological semigroup. The Dirac measure in $x \in G$ is denoted by ε_x.

3. Convolution semigroups of probability measures

A family $(\mu_t)_{t \geq 0}$ in $\mathcal{M}^1(G)$ is said to be a *(continuous) convolution semigroup* if we have $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$, and $\lim_{t \downarrow 0} \mu_t = \mu_0 = \varepsilon_e$. Its *generating functional* $(A, \mathcal{A})$ is defined by

$$\mathcal{A} := \left\{ f \in \mathcal{C}^b(G) : A(f) := \lim_{t \downarrow 0} t^{-1} \left(\int f(x) \mu_t(dx) - f(e) \right) \quad \text{exists} \quad \right\}.$$

We have $\mathcal{E}(G) \subset \mathcal{A}$ and if $\{\zeta_1, \ldots, \zeta_m\}$ is a system of canonical coordinates of the first kind in $\mathcal{D}(G)$ adapted to the basis $\{X_1, \ldots, X_m\}$ of $\mathcal{G}$ then on $\mathcal{E}(G)$ the functional A admits the *canonical decomposition* (Lévy-Khinchin formula)

$$\begin{aligned} A(f) = {} & \sum\nolimits_{i=1}^m a_i (X_i f)(e) + \sum\nolimits_{i,j=1}^m a_{ij} (X_i X_j f)(e) \\ & + \int_{G^*} \left[f(x) - f(e) - \sum\nolimits_{i=1}^m \zeta_i(x) (X_i f)(e) \right] \eta(dx), \end{aligned}$$

where $a_1, \ldots, a_m$ are real numbers, $(a_{ij})_{1 \leq i,j \leq m}$ is a real symmetric positive semidefinite matrix and η is a Lévy measure on G, i. e. $\eta \in \mathcal{M}_+(G^*)$ with $\int_{G^*} \varphi(x) \eta(dx) < \infty$, where φ is a Hunt function for G (see Heyer [3], p. 268, Siebert [5] and Hunt [4]). We shall also say that the generating functional A admits the canonical decomposition $(a_i, a_{ij}, \eta)_{1\le i,j\le m}$.

A convolution semigroup $(\mu_t)_{t\ge 0}$ of non-degenerate measures in $\mathcal{M}^1(G)$ is called a *Gaussian semigroup* if $\lim_{t\downarrow 0} t^{-1}\mu(G\setminus U) = 0$ for all $U\in\mathcal{U}(e)$. A non-degenerate convolution semigroup $(\mu_t)_{t\ge 0}$ in $\mathcal{M}^1(G)$ with canonical decomposition $(a_i, a_{ij}, \eta)_{1\le i,j\le m}$ is a Gaussian semigroup if and only if $\eta = 0$. A non-degenerate measure $\mu\in\mathcal{M}^1(G)$ is called a *Gaussian measure* if there exists a Gaussian semigroup $(\mu_t)_{t\ge 0}$ such that $\mu_1 = \mu$. (For information on Gauss semigroups cf. Heyer [3].)

4. Unitary representations and Fourier transforms

A (continuous) *unitary representation* of G is a homomorphism D of G into the group of unitary operators on a complex Hilbert space $\mathcal{H}$ such that the mapping $x\to D(x)u$ of G into $\mathcal{H}$ is continuous for all $u\in\mathcal{H}$. The space $\mathcal{H}$ is called the representation space of D and is denoted by $\mathcal{H}(D)$. The inner product and the norm in $\mathcal{H}(D)$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively.

The class of all (continuous) unitary representations of G is denoted by $\mathrm{Rep}(G)$. A representation $D\in\mathrm{Rep}(G)$ is said to be *irreducible* if the only closed subspaces of $\mathcal{H}(D)$ invariant under D are $\{0\}$ and $\mathcal{H}(D)$. By $\mathrm{Irr}(G)$ we denote the class of all irreducible representations in $\mathrm{Rep}(G)$.

If $(D,\mathcal{H}(D))$ is a representation of G, the *conjugate representation* $\overline{D}$ is modeled in $\overline{\mathcal{H}(D)}$, the $\mathbb{C}$-linear dual of $\mathcal{H}(D)$. For $u\in\mathcal{H}(D)$ define $\bar u\in\overline{\mathcal{H}(D)}$ via $\bar u(v) = \langle v,u\rangle$. This map $\mathcal{H}(D)\to\overline{\mathcal{H}(D)}$ is bijective, but conjugate linear. The inner product in $\overline{\mathcal{H}(D)}$ is $\langle\bar u,\bar v\rangle := \overline{\langle u,v\rangle}$, and the conjugate representation $\overline{D}$ of G is given by $\overline{D}(x)\bar u := \overline{D(x)u}$. Thus the matrix elements of $\overline{D}(x)$ are the complex conjugates of those for $D(x)$.

If $(D_1,\mathcal{H}(D_1))$ and $(D_2,\mathcal{H}(D_2))$ are representations of G, we define the *tensor product* $\mathcal{H}(D_1)\bar\otimes\mathcal{H}(D_2)$ of Hilbert spaces to be the spaces of all Hilbert-Schmidt operators $S:\overline{\mathcal{H}(D_2)}\to\mathcal{H}(D_1)$. If $\mathcal{H}(D_1)\otimes\mathcal{H}(D_2)$ is the algebraic tensor product of $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ as vector spaces, it corresponds to a dense subspace of $\mathcal{H}(D_1)\bar\otimes\mathcal{H}(D_2)$ if we identify $u\otimes v$ with the rank-1 operator $(u\otimes v)(\bar w) := \langle v,w\rangle u$, and we have $\langle u_1\otimes v_1, u_2\otimes v_2\rangle = \langle u_1,u_2\rangle\langle v_1,v_2\rangle$. The *tensor product representation* $D_1\otimes D_2$ is given on $\mathcal{H}(D_1)\otimes\mathcal{H}(D_2)$ by $(D_1\otimes D_2)(x)(u\otimes v) := D_1(x)u\otimes D_2(x)v$ for all $x\in G$, $u\in\mathcal{H}(D_1)$, $v\in\mathcal{H}(D_2)$. It extends to a unitary representation on $\mathcal{H}(D_1)\bar\otimes\mathcal{H}(D_2)$ given by $(D_1\otimes D_2)(x)S := D_1(x)\circ S\circ(\overline{D}_2(x))^{-1}$ for all $x\in G$.

Let $D \in \mathrm{Rep}(G)$. The vector $u \in \mathcal{H}(D)$ is said to be *differentiable* for D if the coefficient function $x \to \langle D(x)u, v\rangle$ of G into $\mathbb{C}$ is in $\mathcal{E}(G)$ for any $v \in \mathcal{H}(D)$. By $\mathcal{H}_0(D)$ we denote the space of all vectors in $\mathcal{H}(D)$ differentiable for D.

For a probability measure μ on G we define its *Fourier transform* $\hat{\mu}$ by

$$\langle \hat{\mu}(D)u, v\rangle := \int \langle D(x)u, v\rangle \mu(dx)$$

for all $D \in \mathrm{Rep}(G)$ $(u, v \in \mathcal{H}(D))$. Then $\hat{\mu}(D)$ is a bounded linear operator on $\mathcal{H}(D)$ such that $\|\hat{\mu}(D)\| \le 1$.

Let $D \in \mathrm{Rep}(G)$. By the usual properties of the Fourier transformation $(\hat{\mu}_t(D))_{t\ge 0}$ is a strongly continuous semigroup of contractions on $\mathcal{H}(D)$. We denote its infinitesimal generator by $(A(D), \mathcal{A}(D))$. We recall some results due to Siebert (see [6]):

$$\mathcal{A}(D) = \{u \in \mathcal{H}(D) : \langle Du, v\rangle \in \mathcal{A} \quad \text{for all} \quad v \in \mathcal{H}(D)\}$$

and

$$\langle A(D)u, v\rangle = A(\langle Du, v\rangle)$$

for all $u \in \mathcal{A}(D)$ and $v \in \mathcal{H}(D)$. Moreover, $\mathcal{H}_0(D) \subseteq \mathcal{A}(D)$.

5. Characterisation of Gaussian semigroups

For any $D \in \mathrm{Rep}(G)$ and $u \in \mathcal{H}_0(D)$ we introduce a function $f_{D,u}$ on G defined by

$$f_{D,u}(x) := \mathrm{Re}[\langle u, u\rangle - \langle D(x)u, u\rangle]$$

for all $x \in G$.

The following characterisation of Gauss semigroups is similar to the result valid for Gauss measures on almost periodic Lie projective groups (cf. Heyer, [3]).

Theorem 1. *Let G be a Lie group. Let $(\mu_t)_{t\ge 0}$ be a non-degenerate convolution semigroup in $\mathcal{M}^1(G)$ with generating functional A and Lévy-measure η. The following statements are equivalent:*

(i) $(\mu_t)_{t\ge 0}$ *is a Gaussian semigroup;*

(ii) $\eta = 0$;

(iii) $\lim_{t\downarrow 0} \frac{1}{t} \int_G f(x)\mu_t(dx) = 0$ *for all* $f \in \mathcal{C}^b(G)$ *with* $e \notin \mathrm{supp}(f)$;

(iv) $A(f^2_{D,u}) = 0$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathcal{H}_0(D)$;

(v) we have the (Gauss) condition

$$\mathrm{Re}\langle A(D\otimes D)(u\otimes u),u\otimes u\rangle+\mathrm{Re}\langle A(D\otimes \overline{D})(u\otimes \bar{u}),u\otimes \bar{u}\rangle = \\ = 4\|u\|^2\mathrm{Re}\langle A(D)u,u\rangle$$

for all $D\in \mathrm{Irr}(G),\ u\in \mathcal{H}_0(D)$.

Pproof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) is well known (cf. Heyer [3]).

(ii) $\Longrightarrow$ (iv) follows immediately from the Lévy-Khinchin formula since for all $D\in \mathrm{Irr}(G),\ u\in\mathcal{H}_0(D)$ we have $f_{D,u}(e)=0$ and $(X_if_{D,u})(e)=0$ for $i=1,\dots,m$, thus

$$(X_if^2_{D,u})(e)=0,\qquad (X_iX_jf^2_{D,u})(e)=0$$

for $i,j=1,\dots,m$.

(iv) $\Longleftrightarrow$ (v). For every $D\in \mathrm{Irr}(G),\ u\in\mathcal{H}_0(D)$ one has the identities

$$\begin{aligned} f_{D\otimes D,u\otimes u}(x) &= \|u\|^4-\ \mathrm{Re}[\langle D(x)u,u\rangle^2]\\ f_{D\otimes \overline{D},u\otimes \bar{u}}(x) &= \|u\|^4-\ \mathrm{Re}[\langle D(x)u,u\rangle\langle \overline{D}(x)\bar{u},\bar{u}\rangle]\\ &= \|u\|^4-\ \mathrm{Re}[\langle D(x)u,u\rangle\overline{\langle D(x)u,u\rangle}]\\ 4\|u\|^2f_{D,u}(x)-2f^2_{D,u}(x) &= 2f_{D,u}(x)(2\|u\|^2-f_{D,u}(x))\\ &= 2\|u\|^4-2(\ \mathrm{Re}\langle D(x)u,u\rangle)^2\\ &= 2\|u\|^4-\ \mathrm{Re}[\langle D(x)u,u\rangle(\langle D(x)u,u\rangle+\overline{\langle D(x)u,u\rangle})]. \end{aligned}$$

Therefore

$$f_{D\otimes D,u\otimes u}+f_{D\otimes\overline{D},u\otimes\bar{u}}=4\|u\|^2f_{D,u}-2f^2_{D,u}$$

and

$$Af_D=-\ \mathrm{Re}\langle A(D)u,u\rangle$$

imply the assertion.

(iv) $\Longrightarrow$ (ii). For every $D\in \mathrm{Irr}(G)$ and $u\in\mathcal{H}_0(D)$ we have by the Lévy-Khinchin formula

$$0=A(f^2_{D,u})=\int_{G^\times} f^2_{D,u}(x)\eta(dx).$$

Since

$$\bigcap_{u\in\mathcal{H}_0(D)}\{x\in G: f^2_{D,u}(x)=0\}=\ \ker(D)$$

for every $D\in\ \mathrm{Irr}(G)$ (see Siebert [6], the proof of Lemma 5.2) and

$$\bigcap_{D\in\ \mathrm{Irr}(G)}\ker(D)=e$$

(cf. Hewitt, Ross [2] Vol. I, (22.12)) we conclude that $\eta = 0$.

Remark 1. Obviously the Gauss condition (v) can be formulated also for the generating functional of the convolution semigroup $(\mu_t)_{t\geq 0}$:

$$\operatorname{Re} A(\langle (D\otimes D)(u\otimes u), u\otimes u\rangle) + \operatorname{Re} A(\langle (D\otimes \overline{D})(u\otimes \bar{u}), u\otimes \bar{u}\rangle) = \\ = 4\|u\|^2 \operatorname{Re} A(\langle (D)u, u\rangle)$$

for all $D\in \operatorname{Irr}(G)$, $u\in \mathcal{H}_0(D)$.

Remark 2. Unfortunately, in general the Gauss condition (v) does not imply that the Fourier transform $\hat{\mu}_1$ itself satisfies some equation as in the case when G has only finite dimensional irreducible representation.Thus we cannot conclude, for example, that the definition of a Gaussian measure is independent of its embedding semigroup.

Remark 3. If $(\mu_t)_{t\geq 0}$ is a Gaussian semigroup on a Lie group then using the identity

$$f_{D\otimes D, u\otimes v} + f_{D\otimes \overline{D}, u\otimes \bar{v}} = 2(\|u\|^2 f_{D,v} + \|v\|^2 f_{D,u} - f_{D,u} f_{D,v})$$

we can conclude

$$\operatorname{Re}\langle A(D\otimes D)(u\otimes v), u\otimes v\rangle + \operatorname{Re}\langle A(D\otimes \overline{D})(u\otimes \bar{v}), u\otimes \bar{v}\rangle = \\ = 2(\|u\|^2 \operatorname{Re}\langle A(D)v, v\rangle + \|v\|^2 \operatorname{Re}\langle A(D)u, u\rangle)$$

valid for all $D\in \operatorname{Rep}(G)$, $u, v\in \mathcal{H}_0(D)$.

Introducing the notation

$$f_{D,u,v}(x) := \operatorname{Re}[\langle u, v\rangle - \langle D(x)u, v\rangle]$$

for all $D\in \operatorname{Rep}(G)$, $u, v\in \mathcal{H}_0(D)$ and $x\in G$ we have the identity

$$f_{D\otimes D, u_1\otimes v_1, u_2\otimes v_2} + f_{D\otimes \overline{D}, u_1\otimes \bar{v}_1, u_2\otimes \bar{v}_2} = \\ = 2(\operatorname{Re}\langle u_1, u_2\rangle f_{D,v_1,v_2} + \operatorname{Re}\langle v_1, v_2\rangle f_{D,u_1,u_2} - f_{D,u_1,u_2} f_{D,v_1,v_2})$$

and conclude that the infinitesimal generator A satisfies the equation

$$\operatorname{Re}\langle A(D\otimes D)(u_1\otimes v_1), u_2\otimes v_2\rangle + \operatorname{Re}\langle A(D\otimes \overline{D})(u_1\otimes \bar{v}_1), u_2\otimes \bar{v}_2\rangle = \\ = 2(\operatorname{Re}\langle u_1, u_2\rangle \operatorname{Re}\langle A(D)v_1, v_2\rangle + \operatorname{Re}\langle v_1, v_2\rangle \operatorname{Re}\langle A(D)u_1, u_2\rangle)$$

valid for all $D\in \operatorname{Rep}(G)$, $u_1, u_2, v_1, v_2\in \mathcal{H}_0(D)$.

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