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Covering axioms, directed GF-spaces and quasi-uniformities

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Abstract. Characterizations of covering properties (paracompactness, compactness, metacompactness, etc.) in terms of directed fractal structures are given. These characterizations suggest the systematic study of those quasi-uniform spaces $(X,\mathcal{U})$ in which various kinds of open covers have a refinement of the form $\{U^{-1}(x) : x \in X\}$. These properties are also studied for quasi-pseudometric spaces.

1. Introduction

The concept of a directed fractal structure was introduced in [3], though a similar idea was used in [8]. In fact, the concept of a uniform directed fractal structure is introduced in the latter paper. Directed fractal structures are related to transitive bases of quasi-uniformity, and uniform type properties for directed fractal structures correspond to Lebesgue type properties for quasi-uniformities.

In Section 3, we characterize regular (normal, paracompact, strongly paracompact, metacompact, compact) spaces by means of the existence of a directed fractal structure with some additional properties.

In Section 4, we use the relation between directed fractal structures and transitive bases of a quasi-uniformity to give a characterization of normal (paracompact, metacompact, compact) spaces in terms of Lebesgue type properties of some special quasi-uniformities. Moreover, we study

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some properties of Lebesgue type quasi-uniformities and relationships among these properties.

In Section 5 we prove that in the setting of quasi-pseudometric spaces many of these Lebesgue type properties are equivalent.

We recall some definitions and introduce notation that will be useful in this paper.

Let Γ be a covering. Recall that $\mathrm{St}(x,\Gamma) = \bigcup\{A \in \Gamma : x \in A\}$. We also define $U_\Gamma = \{(x,y) \in X \times X : y \in X \setminus \bigcup\{A \in \Gamma : x \notin A\}\}$. A quasi-uniformity $\mathcal{U}$ on a set X is a filter $\mathcal{U}$ of binary relations (called entourages) on X such that (a) each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. A subfamily $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A base $\mathcal{B}$ of a quasi-uniformity is called transitive if $B \circ B = B$ for all $B \in \mathcal{B}$. The theory of quasi-uniform spaces is covered in [4].

If $\mathcal{U}$ is a quasi-uniformity on X, then so is $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, where $U^{-1} = \{(y,x) : (x,y) \in U\}$. $\mathcal{U}^*$ denotes the coarsest uniformity that contains $\mathcal{U}$. A base for $\mathcal{U}^*$ is given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ induced by the quasi-uniformity $\mathcal{U}$ is that in which the sets $U(x) = \{y \in X : (x,y) \in U\}$, where $U \in \mathcal{U}$, form a neighborhood base for each $x \in X$. There is also the topology $\tau(\mathcal{U}^{-1})$ induced by the inverse quasi-uniformity.

Let Γ be a covering of X. Γ is said to be locally finite if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of elements of Γ. Γ is said to be a tiling, if all elements of Γ are regularly closed and they have disjoint interiors (see [1]). We say that Γ is quasi-disjoint if $A^\circ \cap B = \emptyset$ or $A \cap B^\circ = \emptyset$ holds for all $A \neq B \in \Gamma$. Note that if Γ is a tiling, then it is quasi-disjoint.

2. Directed GF-spaces

We recall from [3] the concept of a directed fractal structure.

Definition 2.1. Let Γ_1 and Γ_2 be coverings of a set X. We write $\Gamma_1 \prec\!\!\prec \Gamma_2$ if Γ_1 is a refinement of Γ_2 (that is, $\Gamma_1 \prec \Gamma_2$) and for each $B \in \Gamma_2$ it holds that $B = \bigcup\{A \in \Gamma_1 : A \subseteq B\}$.

A base of a directed fractal structure over a set X is a family of coverings $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$ such that for each $i, j \in I$ there exists $k \in I$ such that $\Gamma_k \prec\!\!\prec \Gamma_i$ and $\Gamma_k \prec\!\!\prec \Gamma_j$.

A base of a directed fractal structure over a set X is said to be a directed fractal structure if given a covering Δ with $\Gamma \prec\!\!\prec \Delta$ for some $\Gamma \in \mathbf{\Gamma}$ it holds that $\Delta \in \mathbf{\Gamma}$.

If $\mathbf{\Gamma}$ is a base of a directed fractal structure over a set X then it is clear that the family of coverings $\{\Gamma : \text{there exists } \Gamma' \in \mathbf{\Gamma} \text{ with } \Gamma' \prec\!\!\prec \Gamma\}$ is a directed fractal structure.

If $\mathbf{\Gamma}$ is a directed fractal structure over X, we will say that $(X, \mathbf{\Gamma})$ is a directed GF-space. If there is no confusion about $\mathbf{\Gamma}$, we will say that X is a directed GF-space. Whenever the index set I is $\mathbb{N}$, the set of natural numbers with its usual order, we drop the word "directed" in this definition (this notion was introduced in [2]).

It was shown in [3] that directed fractal structures are related to transitive quasi-uniformities. We summarize this relationship below.

1. We define $\mathcal{U}_{\mathbf{\Gamma}}$ as the quasi-uniformity with a base $\mathcal{B} = \{U_\Gamma : \Gamma \in \mathbf{\Gamma}\}$. Then it is easy to check that $\mathcal{B}$ is a transitive base of quasi-uniformity and hence $\mathcal{U}_{\mathbf{\Gamma}}$ is a transitive quasi-uniformity. $\mathcal{U}_{\mathbf{\Gamma}}$ is called the (transitive) quasi-uniformity induced by the directed fractal structure $\mathbf{\Gamma}$.

   We will use the notations U_Γ^{-1} instead of $(U_\Gamma)^{-1}$ and U_Γ^* instead of $(U_\Gamma)^*$ in order to avoid using unneeded parentheses (the terms $(U^{-1})_\Gamma$ and $(U^*)_\Gamma$ have no special meaning here).

2. If $\mathcal{B}$ is a transitive base for a quasi-uniformity $\mathcal{U}$, then we define $\mathbf{\Gamma}_{\mathcal{U}}$ as the directed fractal structure for which $\{\Gamma_V : V \in \mathcal{B}\}$ is a base, where $\Gamma_V = \{V^{-1}(x) : x \in X\}$ for each $V \in \mathcal{B}$. $\mathbf{\Gamma}_{\mathcal{U}}$ is called the directed fractal structure induced by the transitive quasi-uniformity $\mathcal{U}$.

   We note that $\mathbf{\Gamma}_{\mathcal{U}}$ depends only on the quasi-uniformity $\mathcal{U}$ and not on the transitive base $\mathcal{B}$.

In [3] it is also proved that $\mathcal{V} = \mathcal{U}_{\mathbf{\Gamma}_{\mathcal{V}}}$, whenever $\mathcal{V}$ is a transitive quasi-uniformity and $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}}$ whenever $\mathbf{\Gamma}$ is a directed fractal structure.

The topology induced by a directed fractal structure $\mathbf{\Gamma}$ on a set X is defined as the topology induced by the quasi-uniformity $\mathcal{U}_{\mathbf{\Gamma}}$.

Let $(X, \mathcal{T})$ be a topological space. We say that $\mathbf{\Gamma}$ is compatible with $\mathcal{T}$ if the topology induced by $\mathbf{\Gamma}$ ($\mathcal{T}(\mathcal{U}_{\mathbf{\Gamma}})$) is equal to $\mathcal{T}$.

If $\mathbf{\Gamma}$ is a directed fractal structure over X, and $\{\mathrm{St}(x, \Gamma) : \Gamma \in \mathbf{\Gamma}\}$ is a neighborhood base of x for all $x \in X$, we will call $(X, \mathbf{\Gamma})$ a starbase directed GF-space and $\mathbf{\Gamma}$ a starbase directed fractal structure. (A starbase base of a directed fractal structure is similarly defined.)

If Γ is a finite (resp. locally finite, quasi-disjoint, tiling) covering whenever $\Gamma \in \mathbf{\Gamma}$, and $\mathbf{\Gamma}$ is a base of a directed fractal structure over X, we will say that $\mathbf{\Gamma}$ is finite (resp. locally finite, quasi-disjoint, tiling). A directed fractal structure is said to be finite (resp. locally finite, quasi-disjoint, tiling) if it has a finite (resp. locally finite, quasi-disjoint, tiling) base.

Note that any member of a finite directed fractal structure is a finite covering.

A directed fractal structure can be induced in subspaces as follows. If $A \subseteq X$, and $(X, \mathbf{\Gamma})$ is a directed GF-space, then the induced directed fractal structure over A is denoted by $\mathbf{\Gamma}_A$ and is defined by $\mathbf{\Gamma}_A = \{\Gamma_A : \Gamma \in \mathbf{\Gamma}\}$, where $\Gamma_A = \{B \cap A : B \in \Gamma\}$.

We summarize some basic properties of directed fractal structures.

Proposition 2.2 ([3]).

1. *If $\mathbf{\Gamma}$ is a directed fractal structure over X, then Γ is closure preserving for each $\Gamma \in \mathbf{\Gamma}$. Moreover, A is closed for every $A \in \Gamma$ and $\Gamma \in \mathbf{\Gamma}$.*
2. *If $\mathbf{\Gamma}$ is a directed fractal structure over X, then $U_\Gamma^{-1}(x) = \bigcap\{A \in \Gamma : x \in A\}$.*

Proposition 2.3 ([3]). *Let $\mathcal{U}$ be a transitive (base of) quasi-uniformity on a topological space X and let $\mathbf{\Gamma}_{\mathcal{U}}$ be the directed fractal structure induced by $\mathcal{U}$.*

1. *$\mathbf{\Gamma}_{\mathcal{U}}$ is finite if and only if $\mathcal{U}$ is totally bounded.*
2. *$\mathbf{\Gamma}_{\mathcal{U}}$ is starbase if and only if $\mathcal{U}$ is locally symmetric.*

3. Directed fractal structures and covering properties

We introduce two tools, the regularization and the quasi-disjointification of a covering.

Definition 3.1. Let Γ be a covering of a topological space X. We define the regularization of Γ by $\text{reg}(\Gamma) = \{\overline{A^\circ} : A \in \Gamma\}$.

The proof of the following lemma is straightforward, so we omit it.

Lemma 3.2. *Let $\{F_i : i \in I\}$ be a finite family of closed sets of a topological space X. Then $(\bigcup_{i\in I} F_i)^\circ \subseteq (\bigcup_{i\in I} \overline{F_i^\circ})^\circ$.*

Proposition 3.3. *Let* Γ *be a locally finite closed covering of a topological space* X*. Then*

1. $\mathrm{reg}(\Gamma)$ *is a locally finite closed covering of* X.
2. $U_{\mathrm{reg}(\Gamma)}(x) \subseteq \mathrm{Cl}(U_\Gamma(x))$.
3. $\mathrm{St}(x, \mathrm{reg}(\Gamma)) \subseteq \mathrm{St}(x, \Gamma)$.
4. *If* $\Gamma_2 \prec\!\prec \Gamma_1$, *then* $\mathrm{reg}(\Gamma_2) \prec\!\prec \mathrm{reg}(\Gamma_1)$ *(where* Γ_1 *and* Γ_2 *are locally finite closed covering).*

Proof. 1. Since Γ is locally finite and by the previous lemma, we have that $x \in \mathrm{St}(x,\Gamma)^\circ = (\bigcup_{x\in A; A\in\Gamma} A)^\circ \subseteq \bigcup_{x\in A; A\in\Gamma} \overline{A^\circ}$, $\mathrm{reg}(\Gamma)$ is a covering. The rest is clear.

2. Let $y \in U_{\mathrm{reg}(\Gamma)}(x)$, and suppose that $y \notin \mathrm{Cl}(U_\Gamma(x))$. Then there exists an open neighborhood U of y such that $U \cap U_\Gamma(x) = \emptyset$, $U \subseteq U_{\mathrm{reg}(\Gamma)}(x)$ and U meets only a finite number of elements of Γ.

Since $U \cap U_\Gamma(x) = \emptyset$, $U \subseteq (\bigcup\{A \in \Gamma : x \notin A;\ A \cap U \neq \emptyset\})^\circ \subseteq \bigcup\{\overline{A^\circ} : A \in \Gamma;\ x \notin A; A \cap U \neq \emptyset\} \subseteq U \cap \bigcup\{\overline{A^\circ} : A \in \Gamma;\ x \notin \overline{A^\circ}\} \neq \emptyset$ (by the previous lemma), and hence $U \cap \bigcup\{\overline{A^\circ};\ A \in \Gamma; x \notin \overline{A^\circ}\} \neq \emptyset$.

On the other hand, since $U \subseteq U_{\mathrm{reg}(\Gamma)}(x)$ it follows that $U \cap \bigcup\{\overline{A^\circ} : A \in \Gamma;\ x \notin \overline{A^\circ}\} = \emptyset$. The contradiction shows that $U_{\mathrm{reg}(\Gamma)}(x) \subseteq \mathrm{Cl}(U_\Gamma(x))$.

3. The result is obvious.

4. If Γ_2 is a refinement of Γ_1, it is clear that $\mathrm{reg}(\Gamma_2)$ is a refinement of $\mathrm{reg}(\Gamma_1)$.

Now, let $x \in A_1^\circ$ with $A_1 \in \Gamma_1$. Then there exists an open neighborhood of x such that $U \subseteq A_1$ and U meets only a finite number of elements of Γ_1. By the previous lemma, we have that $x \in U \subseteq (\bigcup\{A_2 \in \Gamma_2 : A_2 \subseteq A_1;\ A_2 \cap U \neq \emptyset\})^\circ \subseteq \bigcup\{\overline{A_2^\circ} : A_2 \in \Gamma_2 : A_2 \subseteq A_1;\ A_2 \cap U \neq \emptyset\}$, and hence $A_1^\circ \subseteq \bigcup\{\overline{A_2^\circ} : A_2 \subseteq A_1;\ A_2 \in \Gamma_2\} \subseteq \overline{A_1^\circ}$, and taking closures we have that $\overline{A_1^\circ} = \bigcup\{\overline{A_2^\circ} : A_2 \subseteq A_1;\ A_2 \in \Gamma_2\}$ (because Γ_2 is locally finite and hence closure-preserving). □

If $\mathbf{\Gamma}$ is a locally finite base of a directed fractal structure over a regular space, we denote $\mathrm{reg}(\mathbf{\Gamma}) = \{\mathrm{reg}(\Gamma) : \Gamma \in \mathbf{\Gamma}\}$, and call it the regularization of $\mathbf{\Gamma}$.

The next result follows directly from the previous proposition.

Proposition 3.4. *Let* $\mathbf{\Gamma}$ *be a locally finite base of a directed fractal structure over a regular space* X*. Then* $\operatorname{reg}(\mathbf{\Gamma})$ *is a locally finite base of a directed fractal structure over* X *compatible with the topology induced by* $\mathbf{\Gamma}$*. If* $\mathbf{\Gamma}$ *is starbase, then so is* $\operatorname{reg}(\mathbf{\Gamma})$*.*

Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be locally finite bases of a directed fractal structure $\mathbf{\Gamma}$ over a regular space X. By the fourth item of Proposition 3.3, it follows easily that $\operatorname{reg}(\mathbf{\Delta}_1)$ and $\operatorname{reg}(\mathbf{\Delta}_2)$ generate the same directed fractal structure, which will be denoted by $\operatorname{reg}(\mathbf{\Gamma})$ and called the regularization of $\mathbf{\Gamma}$.

As an immediate consequence of Proposition 3.4 we have the following result.

Theorem 3.5. *Let* $\mathbf{\Gamma}$ *be a locally finite directed fractal structure over a regular space* X*. Then* $\operatorname{reg}(\mathbf{\Gamma})$ *is a locally finite directed fractal structure over* X *compatible with the topology induced by* $\mathbf{\Gamma}$*. If* $\mathbf{\Gamma}$ *is starbase, so is* $\operatorname{reg}(\mathbf{\Gamma})$*.*

Definition 3.6. Let $\Gamma = \{A^\lambda : \lambda \in \Lambda\}$ be a covering of a topological space X. For each $w \in P(\Lambda)$ (the set of nonempty subsets of Λ) we define $A^w = \operatorname{Cl}(\bigcap_{\lambda \in w} A^\lambda \setminus (\bigcup_{\lambda \notin w} A^\lambda))$. We define $\operatorname{qdi}(\Gamma) = \{A^w : w \in P(\Lambda)\} \setminus \{\emptyset\}$, and call it the quasi-disjointification of Γ.

The notation $\operatorname{qdi}(\Gamma)$ is motivated by the observation that $\operatorname{qdi}(\Gamma)$ is quasi-disjoint, see 4 bellow.

Proposition 3.7. *Let* $\Gamma = \{A^\lambda : \lambda \in \Lambda\}$ *be a closed covering of a topological space* X*. Then*

1. *For all* $w \in P(\Lambda)$*, there exists* $x \in X$ *such that* $A^w = \operatorname{Cl}(U^*_\Gamma(x))$ *or* $A^w = \emptyset$*, and for all* $x \in X$*, there exists* $w \in P(\Lambda)$ *such that* $A^w = \operatorname{Cl}(U^*_\Gamma(x))$*.*
2. $A^\lambda = \bigcup\{A^w : \lambda \in w; w \in P(\Lambda)\}$ *and hence* $\operatorname{qdi}(\Gamma) \prec\!\prec \Gamma$*.*
3. $U_{\operatorname{qdi}(\Gamma)} \subseteq U_\Gamma$*.*
4. $\operatorname{qdi}(\Gamma)$ *is quasi-disjoint.*
5. $\operatorname{qdi}(\Gamma)$ *is a closed covering and if* Γ *is locally finite, so is* $\operatorname{qdi}(\Gamma)$*.*
6. *If* $\Gamma_2 \prec\!\prec \Gamma_1$*, then* $\operatorname{qdi}(\Gamma_2) \prec\!\prec \operatorname{qdi}(\Gamma_1)$ *(where* Γ_1 *and* Γ_2 *are locally finite closed coverings).*

Proof. 1. Let $w \in P(\Lambda)$, and suppose that there exists $x \in \bigcap_{\lambda \in w} A^\lambda \setminus (\bigcup_{\lambda \notin w} A^\lambda)$. Then it is clear that $x \in A^\lambda$ if and only if $\lambda \in w$. Hence $A^w = \operatorname{Cl}(\bigcap_{\lambda \in w} A^\lambda \setminus (\bigcup_{\lambda \notin w} A^\lambda)) = \operatorname{Cl}(\bigcap_{x \in A^\lambda} A^\lambda \setminus (\bigcup_{x \notin A^\lambda} A^\lambda)) = \operatorname{Cl}(U^*_\Gamma(x))$.

Let $x \in X$, and let $w = \{\lambda \in \Lambda : x \in A^\lambda\}$. Then it is clear that $x \in A^\lambda$ if and only if $\lambda \in w$, and hence $A^w = \mathrm{Cl}(U^*_\Gamma(x))$, analogously to the preceding paragraph.

2. It is clear that $\bigcup\{A^w : \lambda \in w; w \in P(\Lambda)\} \subseteq A^\lambda$, since $A^w \subseteq A^\mu$ for all $\mu \in w$.

Let $x \in A^\lambda$. Then $x \in A^w = \mathrm{Cl}(U^*_\Gamma(x))$ (by the first item and for some $w \in P(\Lambda)$) and since $x \in A^\lambda$, $\lambda \in w$. This proves the equality.

3. This result follows easily from the previous item.

4. $A^w = \mathrm{Cl}(\bigcap_{\lambda\in w} A^\lambda \setminus (\bigcup_{\lambda\notin w} A^\lambda)) \subseteq X \setminus (\bigcup_{\lambda\notin w} A^\lambda)^\circ$, and hence $A^w \cap (\bigcup_{\lambda\notin w} A^\lambda)^\circ = \emptyset$.

On the other hand, if there exists $\lambda \in v \setminus w$, then $A^v \subseteq A^\lambda$, and $(A^v)^\circ \subseteq (A^\lambda)^\circ \subseteq (\bigcup_{\lambda\notin w} A^\lambda)^\circ$. Therefore $A^w \cap (A^v)^\circ = \emptyset$, and hence Γ is quasi-disjoint.

5. It follows from the first item that $\mathrm{qdi}(\Gamma)$ is a covering and it is obvious that each A^w is closed. Suppose that Γ is locally finite, and let us see that $\mathrm{qdi}(\Gamma)$ is also.

Let $x \in X$. Then there exists an open neighborhood U of x and a finite set $\{\lambda_1, \ldots, \lambda_n\}$ such that $U \cap A^{\lambda_i} \neq \emptyset$ for each $i \in \{1, \ldots, n\}$, but $U \cap A^\mu = \emptyset$ for all $\mu \notin \{\lambda_1, \ldots, \lambda_n\}$. Since $A^w \subseteq A^\lambda$ for all $\lambda \in w$, we have that $w \in P(\{\lambda_1, \ldots, \lambda_n\})$ for all w such that $U \cap A^w \neq \emptyset$. Therefore there are only a finite number of w for which $U \cap A^w \neq \emptyset$ and hence $\mathrm{qdi}(\Gamma)$ is locally finite.

6. It is easy to see that $U^*_{\Gamma_2}(x) \subseteq U^*_{\Gamma_1}(x)$ for all $x \in X$. Therefore $\mathrm{qdi}(\Gamma_2)$ is a refinement of $\mathrm{qdi}(\Gamma_1)$.

Using that Γ_2 is locally finite, it is easy to check that the family $\{U^*_{\Gamma_2}(y) : y \in X\}$ is closure preserving. Since $y \in U^*_{\Gamma_1}(x)$ if and only if $U^*_{\Gamma_2}(y) \subseteq U^*_{\Gamma_1}(x)$ for all $x \in X$, we have that $U^*_{\Gamma_1}(x) = \bigcup\{U^*_{\Gamma_2}(y) : y \in U^*_{\Gamma_1}(x)\} = \bigcup\{U^*_{\Gamma_2}(y) : U^*_{\Gamma_2}(y) \subseteq U^*_{\Gamma_1}(x)\}$. It follows that $\mathrm{qdi}(\Gamma_2) \prec\!\!\prec \mathrm{qdi}(\Gamma_1)$. □

Let $\mathbf{\Gamma}$ be a locally finite base of a directed fractal structure over X. We denote $\mathrm{qdi}(\mathbf{\Gamma}) = \{\mathrm{qdi}(\Gamma) : \Gamma \in \mathbf{\Gamma}\}$ and call it the quasi-disjointification of $\mathbf{\Gamma}$.

Corollary 3.8. *Let $\mathbf{\Gamma}$ be a locally finite base of a directed fractal structure over X. Then* $\mathrm{qdi}(\mathbf{\Gamma})$ *is a locally finite quasi-disjoint base of a directed fractal structure over X compatible with the topology induced by $\mathbf{\Gamma}$.*

Proof. Consider on X the topology induced by $\mathbf{\Gamma}$. By the fourth and fifth items of Proposition 3.7, $\mathrm{qdi}(\Gamma)$ is a locally finite quasi-disjoint closed covering for each $\Gamma \in \mathbf{\Gamma}$ and by the third item $U_{\mathrm{qdi}(\Gamma)}(x) \subseteq U_\Gamma(x)$. Therefore $\{U_{\mathrm{qdi}(\Gamma)}(x) : \Gamma \in \mathbf{\Gamma}\}$ is an open neighborhood base of x for all $x \in X$ (note that $U_{\mathrm{qdi}(\Gamma)}(x)$ is open since $\mathrm{qdi}(\Gamma)$ is locally finite).

By the sixth item of Proposition 3.7 and from what we have already proved, we have that $\mathrm{qdi}(\mathbf{\Gamma})$ is a locally finite quasi-disjoint base of a directed fractal structure over X compatible with the topology induced by $\mathbf{\Gamma}$. □

Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be locally finite bases of a directed fractal structure $\mathbf{\Gamma}$ over X. By the sixth item of Proposition 3.7, it easily follows that $\mathrm{qdi}(\mathbf{\Delta}_1)$ and $\mathrm{qdi}(\mathbf{\Delta}_2)$ generate the same directed fractal structure, which will be denoted by $\mathrm{qdi}(\mathbf{\Gamma})$ and called the quasi-disjointification of $\mathbf{\Gamma}$.

As an immediate consequence of Corollary 3.8 we have the following result.

Corollary 3.9. *Let $\mathbf{\Gamma}$ be a locally finite directed fractal structure over X. Then* $\mathrm{qdi}(\mathbf{\Gamma})$ *is a locally finite quasi-disjoint directed fractal structure over X compatible with the topology induced by $\mathbf{\Gamma}$.*

Definition 3.10. Let Γ be a covering of a topological space X. We define $\mathrm{til}(\Gamma) = \mathrm{reg}(\mathrm{qdi}(\Gamma))$.

Proposition 3.11. *Let Γ be a closure preserving tiling of a topological space X. Then* $\overline{U_\Gamma(x)} = \mathrm{St}(x, \Gamma)$.

Proof. Let $y \in \mathrm{St}(x, \Gamma)$ and let $A \in \Gamma$ such that $x, y \in A$. If $z \in A^\circ$ then A is the unique member of Γ which contains z and since $x \in A$ then it follows that $x \in U_\Gamma^{-1}(z) = \bigcap\{B \in \Gamma : z \in B\} = A$ and hence $z \in U_\Gamma(x)$. Therefore, $y \in A = \overline{A^\circ} \subseteq \overline{U_\Gamma(x)}$ and hence $\mathrm{St}(x, \Gamma) \subseteq \overline{U_\Gamma(x)}$. Since Γ is closure preserving, $\mathrm{St}(x, \Gamma)$ is closed and since $U_\Gamma(x) \subseteq \mathrm{St}(x, \Gamma)$, it follows that $\overline{U_\Gamma(x)} = \mathrm{St}(x, \Gamma)$. □

Theorem 3.12. *Let* $\mathbf{\Gamma}$ *be a locally finite directed fractal structure over a regular space* X*. Then* $\mathrm{til}(\mathbf{\Gamma})$ *is a starbase locally finite tiling directed fractal structure over* X*.*

Proof. Since $\mathrm{qdi}(\mathbf{\Gamma})$ is a locally finite quasi-disjoint directed fractal structure over X, by Corollary 3.9, it is clear from Theorem 3.5 that $\mathrm{til}(\mathbf{\Gamma}) = \mathrm{reg}(\mathrm{qdi}(\mathbf{\Gamma}))$ is a locally finite tiling directed fractal structure over X. Finally, $\mathrm{til}(\mathbf{\Gamma})$ is starbase by Proposition 3.11. □

The next result is easily anticipated.

Lemma 3.13. *Let* Γ *be a tiling of a topological space* X*. Then* $\mathrm{til}(\Gamma) = \Gamma$.

Proof. Let $\Gamma = \{A_\lambda : \lambda \in \Lambda\}$ be a tiling. We will use the notation of Definition 3.6. Let $w \in P(\Lambda)$. Suppose that there exist $\lambda_1 \neq \lambda_2$ with $\{\lambda_1, \lambda_2\} \subseteq w$. Then $A^w \subseteq A_{\lambda_1} \cap A_{\lambda_2}$, whence $(A^w)^\circ \subseteq (A_{\lambda_1} \cap A_{\lambda_2})^\circ = \emptyset$. On the other hand it is clear that if $w = \{\lambda\}$ then $(A^\lambda)^\circ \subseteq (A^\lambda)^\circ \setminus \bigcup_{\mu \neq \lambda} A_\mu \subseteq A^w$, since Γ is a tiling, and hence $(A^\lambda)^\circ \cap A^\mu = \emptyset$ for all $\mu \neq \lambda$. Therefore, since Γ is a tiling, we have that $A^\lambda = \mathrm{Cl}((A^\lambda)^\circ) \subseteq A^w \subseteq A^\lambda$, whence $A^w = A^\lambda$. Consequently, we have that $\mathrm{til}(\Gamma) = \Gamma$. □

Lemma 3.14 ([3])**.** *Let* $\mathbf{\Gamma}$ *be the family of all finite closed coverings of a topological space* X*. Then* $\mathbf{\Gamma}$ *is a finite directed fractal structure. Moreover* $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathcal{P}}$*, where* $\mathcal{P}$ *is the Pervin quasi-uniformity (hence* $\mathcal{U}_{\mathbf{\Gamma}}$ *is the Pervin quasi-uniformity).*

We give the description of the finest finite tiling starbase base of a directed fractal structure over a regular space.

Proposition 3.15. *Let* X *be a regular space and let* $\mathbf{\Gamma}_{\mathcal{P}}$ *be the directed fractal structure induced by the Pervin quasi-uniformity. Then* $\mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$ *is a finite tiling starbase base of a directed fractal structure compatible with the topology of* X*. Moreover* $\mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$ *is the family of all finite tilings of* X*.*

Proof. It is clear that $\mathbf{\Gamma}_{\mathcal{P}}$ is a finite directed fractal structure over X. Since X is regular, the Pervin quasi-uniformity is locally symmetric, and hence $\mathbf{\Gamma}_{\mathcal{P}}$ is starbase by Proposition 2.3. Then $\mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$ is a finite tiling starbase base of a directed fractal structure over X.

If $\mathcal{A}$ is a finite tiling of X, then, by Lemma 3.14, $\mathcal{A} \in \mathbf{\Gamma}_{\mathcal{P}}$, and by Lemma 3.13 it follows that $\mathrm{til}(\mathcal{A}) = \mathcal{A}$, and hence $\mathcal{A} \in \mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$. Conversely, if $\mathcal{A} \in \mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$, it is clear that $\mathcal{A}$ is a finite tiling of X. Therefore, $\mathrm{til}(\mathbf{\Gamma}_{\mathcal{P}})$ is the family of all finite tilings of X. □

Lemma 3.16. *Let $\mathbf{\Gamma}$ be the family of all locally finite closed coverings of a topological space X. Then $\mathbf{\Gamma}$ is a locally finite base of a directed fractal structure compatible with the topology of X.*

Proof. It is clear that if $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}$ then $\Gamma_1 \wedge \Gamma_2 \in \mathbf{\Gamma}$, and since $\Gamma_1 \wedge \Gamma_2 \prec\!\prec \Gamma_1, \Gamma_2$, $\mathbf{\Gamma}$ is a base of a directed fractal structure that is clearly locally finite.

Let $x \in X$ and U open with $x \in U$. Then $\Gamma = \{X, X \setminus U\} \in \mathbf{\Gamma}$ and $U_\Gamma(x) = U$. Therefore, the topology induced by $\mathbf{\Gamma}$ is compatible with the topology of X (note that $U_\Gamma(x)$ is open for each $\Gamma \in \mathbf{\Gamma}$). □

Proposition 3.17. *Let X be a regular space and let $\mathbf{\Gamma}$ be the family of all locally finite tilings of X. Then $\mathbf{\Gamma}$ is a locally finite tiling starbase base of a directed fractal structure over X.*

Proof. Let $\mathbf{\Delta}$ be the base of directed fractal structure consisting of all locally finite closed coverings of X (see Lemma 3.16). Then $\operatorname{til}(\mathbf{\Delta})$ is a locally finite tiling starbase base of a directed fractal structure over X by Theorem 3.12.

Let us show that $\mathbf{\Gamma} = \operatorname{til}(\mathbf{\Delta})$. It is clear that $\operatorname{til}(\mathbf{\Delta}) \subseteq \mathbf{\Gamma}$. Let $\mathcal{A} \in \mathbf{\Gamma}$, by Lemma 3.13 it follows that $\mathcal{A} = \operatorname{til}(\mathcal{A})$, and since $\mathcal{A} \in \mathbf{\Gamma} \subseteq \mathbf{\Delta}$ it follows that $\mathcal{A} \in \operatorname{til}(\mathbf{\Delta})$, and hence $\mathbf{\Gamma} = \operatorname{til}(\mathbf{\Delta})$. □

Lemma 3.18 ([3])**.** *Let $\mathbf{\Gamma}$ be the family of all closure preserving closed coverings of a topological space X. Then $\mathbf{\Gamma}$ is a directed fractal structure compatible with the topology of X. Moreover $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathcal{FT}}$, where $\mathcal{FT}$ is the finest transitive quasi-uniformity on X.*

Proposition 3.19. *Let X be a regular space and let $\mathbf{\Gamma}$ be the family of all closure preserving tilings of X. Then $\mathbf{\Gamma}$ is a tiling starbase base of a directed fractal structure compatible with the topology of X.*

Proof. Since $\mathbf{\Gamma} = \operatorname{til}(\mathbf{\Delta})$, where $\mathbf{\Delta}$ is the family of all closure preserving closed coverings of X, and $\mathbf{\Delta}$ is a directed fractal structure over X by Lemma 3.18, $\mathbf{\Gamma}$ is a base of a directed fractal structure.

Since it is clear that $\mathbf{\Gamma}$ contains $\operatorname{til}(\mathbf{\Gamma}_{\mathcal{P}})$, which is a starbase base of a directed fractal structure by Proposition 3.15, $\mathbf{\Gamma}$ is compatible with the topology of X and $\mathbf{\Gamma}$ is a starbase and tiling. □

Proposition 3.20. *Let X be a topological space. The following statements are equivalent:*

1. *X is a regular space.*
2. *$\mathbf{\Gamma}_{\mathcal{P}}$ (where $\mathbf{\Gamma}_{\mathcal{P}}$ is the directed fractal structure induced by the Pervin quasi-uniformity) is starbase.*
3. *There exists a starbase directed fractal structure over X.*

Proof. 1) implies 2). $\mathcal{P}$ is locally symmetric, since X is a regular space, and hence $\mathbf{\Gamma}_{\mathcal{P}}$ is starbase (by Proposition 2.3).

2) implies 3). Obvious.

3) implies 1). Let $\mathbf{\Gamma}$ be a starbase directed fractal structure over X. Let $x \in X$ and U be an open set containing x. Since $\mathbf{\Gamma}$ is a starbase, there exists $\Gamma \in \mathbf{\Gamma}$ such that $x \in \mathrm{St}(x, \Gamma) \subseteq U$. Since Γ is closure preserving by Proposition 2.2, we have that $x \in U_\Gamma(x) \subseteq \overline{U_\Gamma(x)} \subseteq \mathrm{St}(x, \Gamma) \subseteq U$. Therefore X is regular. □

We recall that a covering $\mathcal{A}$ is said to be directed if $A \cup B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$.

Definition 3.21 (compare [8]). Let $\mathbf{\Gamma}$ be a (base of a) directed fractal structure over a topological space X. We say that $\mathbf{\Gamma}$ is uniform (resp. F-uniform, LF-uniform, C-uniform, D-uniform) if every open covering (resp. finite, locally finite, countable, directed open covering) has a refinement in $\mathbf{\Gamma}$.

The following result easily follows from [8, Proposition 1.4] and its proof.

Lemma 3.22. *For a topological space X, the following statements are equivalent:*

1. *X is normal*
2. *Every locally finite open covering has a closure preserving closed refinement.*
3. *Every finite open covering has a closure preserving closed refinement.*

Proposition 3.23. *Let X be a topological space. The following statements are equivalent:*

1. *X is normal.*
2. *The family of all locally finite closed coverings of X is a LF-uniform locally finite base of a directed fractal structure over X.*
3. *There exists a LF-uniform directed fractal structure over X.*
4. *There exists an F-uniform directed fractal structure over X.*

PROOF. 1) implies 2). It follows from Lemma 3.16 that the family $\mathbf{\Gamma}$ of all locally finite closed coverings of X is a locally finite base of a directed fractal structure over X. Since every locally finite open covering in a normal space has a locally finite closed refinement by [8, Proposition 1.4], it follows that $\mathbf{\Gamma}$ is LF-uniform.

2) implies 3) and 3) implies 4) are evident.

4) implies 1). Let $\mathbf{\Gamma}$ be an F-uniform directed fractal structure over X. Since $\mathbf{\Gamma}$ is F-uniform, every finite open covering has a refinement in $\mathbf{\Gamma}$, and since every element of $\mathbf{\Gamma}$ is closure preserving by Proposition 2.2, it follows from Lemma 3.22 that X is normal. □

The proof of the following result is analogous to the proof of the previous one.

Proposition 3.24. *Let X be a T_0 topological space. The following statements are equivalent:*

1. *X is a normal Hausdorff space.*
2. *The family of all locally finite tilings of X is a LF-uniform locally finite tiling starbase base of a directed fractal structure over X.*
3. *There exists a LF-uniform starbase directed fractal structure over X.*
4. *There exists an F-uniform starbase directed fractal structure over X.*

Proposition 3.25. *Let $\mathbf{\Gamma}$ be an F-uniform directed fractal structure over a regular Hausdorff space X. Then $\mathbf{\Gamma}$ is starbase and X is a normal Hausdorff space.*

PROOF. Let $x \in X$ and U be an open neighborhood of x. Since X is regular, there exists an open subset V of X such that $x \in V \subseteq \overline{V} \subseteq U$. Then $\mathcal{U} = \{U, X \setminus \overline{V}\}$ is an open covering of X, and since $\mathbf{\Gamma}$ is F-uniform, there exists $\Gamma \in \mathbf{\Gamma}$ such that Γ is a refinement of $\mathcal{U}$. Then $\mathrm{St}(x, \Gamma) \subseteq U$;

indeed, if $A \in \Gamma$ with $x \in A$, then it is clear, since Γ is a refinement of $\mathcal{U}$ and $x \in A$ but $x \notin X \setminus \overline{V}$, that $A \subseteq U$. Therefore, we have $U_\Gamma(x) \subseteq \mathrm{St}(x, \Gamma) \subseteq U$, and since $U_\Gamma(x)$ is open (since Γ is closure preserving) it follows that $\mathbf{\Gamma}$ is starbase. Note that X is a normal Hausdorff space by Propositon 3.24. □

Proposition 3.26. *Let X be a T_0 topological space. The following statements are equivalent:*

1. *X is a paracompact Hausdorff space.*
2. *The family of all locally finite tilings of X is a uniform locally finite tiling starbase base of a directed fractal structure over X.*
3. *There exists a uniform starbase directed fractal structure over X.*

Proof. 1) implies 2). It follows from Proposition 3.17 that the family $\mathbf{\Gamma}$ of all locally finite tilings of X is a locally finite starbase base of a directed fractal structure over X. Since every open covering in a paracompact Hausdorff space has a locally finite tiling refinement by [8, Lemma 1.5], it follows that $\mathbf{\Gamma}$ is uniform.

2) implies 3) is evident.

3) implies 1). Let $\mathbf{\Gamma}$ be a uniform starbase directed fractal structure over X. By Proposition 3.20, X is regular and hence it is Hausdorff. On the other hand, since $\mathbf{\Gamma}$ is uniform it follows that every open covering has a refinement in $\mathbf{\Gamma}$, and since every element of $\mathbf{\Gamma}$ is closure preserving by Proposition 2.2, it follows by [6, Theorem 9.2.3] that X is paracompact. □

Proposition 3.27. *A Hausdorff space is strongly paracompact if and only if it admits a uniform locally finite starbase base of a directed fractal structure $\mathbf{\Gamma}$ such that $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}^{-1}}$ is a locally finite base of a directed fractal structure (not necessarily compatible with the topology of the space).*

Proof. First, note that $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}^{-1}}$ is locally finite in $(X, \mathcal{T}(\mathcal{U}_{\mathbf{\Gamma}}^{-1}))$ if and only if the family $\{U_\Gamma(x) : x \in X\}$ is point finite for each $\Gamma \in \mathbf{\Gamma}$.

Suppose that X is strongly paracompact and let $\mathbf{\Gamma}$ be the family of all locally finite star-finite tilings of X. By [8, Proposition 1.6], it is easy to check that $\mathbf{\Gamma}$ is a uniform locally finite starbase base of a directed fractal structure compatible with the topology of X.

Let $\Gamma \in \mathbf{\Gamma}$ and let us prove that $\{U_\Gamma(x) : x \in X\}$ is point finite. Let $y \in X$ and $x_k \in X$ such that $y \in U_\Gamma(x_k)$ for each $k \in \mathbb{N}$. Then $x_k \in$

$U_\Gamma^{-1}(y) \subseteq \mathrm{St}(y,\Gamma)$. Since $\Gamma \in \mathbf{\Gamma}$, Γ is locally finite and star-finite and hence $\mathrm{St}(y,\Gamma)$ meets only a finite number of elements of Γ. Since $x_k \in \mathrm{St}(y,\Gamma)$, $\mathrm{St}(x_k,\Gamma) \subseteq \mathrm{St}(\mathrm{St}(y,\Gamma),\Gamma)$. It follows that $U_\Gamma(x_k) = X \setminus (L \cup M_k)$, where $L = \bigcup_{A\in\Gamma; A\cap \mathrm{St}(y,\Gamma)=\emptyset} A$ and $M_k = \bigcup_{A\in\Gamma; A\cap\mathrm{St}(y,\Gamma)\neq\emptyset; x_k\notin A} A$. Since L does not depend on x_k and there are only a finite number of combinations for the elements of the union in M_k, $\{M_k : k \in \mathbb{N}\}$ is finite and hence $\{U_\Gamma(x_k) : k \in \mathbb{N}\}$ is finite. Therefore $\{U_\Gamma(x) : x \in X\}$ is a point finite family.

Conversely, let $\mathbf{\Gamma}$ be a uniform locally finite starbase base of a directed fractal structure such that $\mathbf{\Gamma}_{\mathcal{U}_\Gamma^{-1}}$ is locally finite, and let $\mathcal{V}$ be an open covering of X. Since $\mathbf{\Gamma}$ is uniform, there exists $\Gamma \in \mathbf{\Gamma}$ such that $\Gamma \prec \mathcal{V}$. Then it is clear that $\Gamma_{U_\Gamma} = \{U_\Gamma^{-1}(x) : x \in X\}$ refines $\mathcal{V}$.

Let us show that Γ_{U_Γ} is star-finite. Let $x \in X$ and $x_k \in X$ with $k \in \mathbb{N}$ such that $U_\Gamma^{-1}(x) \cap U_\Gamma^{-1}(x_k) \neq \emptyset$, and let $y_k \in U_\Gamma^{-1}(x) \cap U_\Gamma^{-1}(x_k)$. Then $x \in U_\Gamma(y_k)$ and $x_k \in U_\Gamma(y_k)$. Since $\mathbf{\Gamma}_{\mathcal{U}_\Gamma^{-1}}$ is locally finite, the family $\{U_\Gamma(y_k) : k \in \mathbb{N}\}$ is finite. Let $n \in \mathbb{N}$ be such that $U_\Gamma(y_k) = U_\Gamma(y_i)$ for each $k > n$ and some $i \leq n$. For each k it follows that the family $\{U_\Gamma^{-1}(x_m) : y_k \in U_\Gamma^{-1}(x_m)\}$ is finite, since Γ_{U_Γ} is locally finite. Then $\{U_\Gamma^{-1}(x_k) : k \in \mathbb{N}\} = \{U_\Gamma^{-1}(x_k) : x_k \in U_\Gamma(y_k)\} = \{U_\Gamma^{-1}(x_k) : x_k \in U_\Gamma(y_m)$ for some $m \leq n\}$ is a finite family. Therefore Γ_{U_Γ} is star-finite (and locally finite since Γ is) and hence X is strongly paracompact by [8, Proposition 1.6]. □

Next we characterize compact and metacompact Hausdorff spaces in terms of uniform directed fractal structures.

Lemma 3.28. *A Hausdorff space is compact if and only if every open covering has a finite closed refinement (which is a tiling).*

Proposition 3.29. *Let X be a topological space. The following statements are equivalent:*

1. *X is a compact Hausdorff space.*
2. *The family of all finite tilings of X is a uniform finite tiling starbase base of a directed fractal structure over X.*
3. *There exists a uniform finite starbase directed fractal structure over X.*

Proof. The proof is analogous to the proof of Proposition 3.23, if we have in mind Lemma 3.28 and Proposition 3.15. □

Proposition 3.30. *Let X be a topological space. The following statements are equivalent:*

1. *X is a metacompact Hausdorff space.*
2. *The family of all closure preserving closed coverings of X is a D-uniform starbase directed fractal structure over X.*
3. *There exists a D-uniform starbase directed fractal structure over X.*

Proof. The proof is analogous to the proof of Proposition 3.23, if we have in mind [6, Theorem 9.3.5] and Proposition 3.18. □

Let $f : X \to Y$, Γ be a family of subsets of X and $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$, where Γ_i is a family of subsets of X. We denote $\{f(A) : A \in \Gamma\}$ by $f(\Gamma)$ and $\{f(\Gamma_i) : i \in I\}$ by $f(\mathbf{\Gamma})$.

The proof of the following lemma is straightforward.

Lemma 3.31. *Let X and Y be topological spaces, $f : X \to Y$ a continuous function, Γ a family of subsets of X and $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$, where Γ_i is a family of subsets of X whenever $i \in I$.*

1. *If f is closed and Γ is closure preserving, so is $f(\Gamma)$.*
2. *If f is onto and $\mathbf{\Gamma}$ is uniform (resp. LF-uniform, F-uniform, C-uniform, D-uniform), so is $f(\mathbf{\Gamma})$.*

Proposition 3.32. *Let X, Y be T_1 topological spaces and $\mathbf{\Gamma}$ an F-uniform directed fractal structure over X. Let f be a continuous closed mapping from X onto Y. Then $f(\mathbf{\Gamma})$ is a (F-uniform starbase) base of a directed fractal structure compatible with the topology of Y.*

Proof. It is easy to check that $f(\Gamma_1) \prec\!\!\prec f(\Gamma_2)$ for each covering Γ_1, Γ_2 of X with $\Gamma_1 \prec\!\!\prec \Gamma_2$, and hence it follows that $\mathbf{\Gamma}$ is a base of a directed fractal structure.

Since Γ is closure preserving (Proposition 2.2) whenever $\Gamma \in \mathbf{\Gamma}$, it follows from the previous lemma that $f(\Gamma)$ is closure preserving for each $\Gamma \in \mathbf{\Gamma}$, and hence $U_{f(\Gamma)}(x)$ is open for each $x \in X$ and $\Gamma \in \mathbf{\Gamma}$. Let $y \in Y$ and U an open neighborhood of y. Then $\mathcal{A} = \{f^{-1}(Y \setminus \{y\}), f^{-1}(U)\}$ is a finite open covering of X, and since $\mathbf{\Gamma}$ is F-uniform, there exists $\Gamma \in \mathbf{\Gamma}$ such that Γ refines $\mathcal{A}$. Let $z \in \mathrm{St}(y, f(\Gamma))$. Then there exists $A \in \Gamma$ such that $z, y \in f(A)$. Since $\Gamma \prec \mathcal{A}$, it follows that $A \subseteq f^{-1}(U)$ and hence that $z \in f(A) \subseteq U$. Thus, $\mathrm{St}(y, f(\Gamma)) \subseteq U$. Therefore $f(\mathbf{\Gamma})$ is compatible with the topology of Y and is starbase. It is F-uniform by the previous lemma. □

Note that we obtain as a corollary that the closed image of a paracompact (resp. normal) Hausdorff space is paracompact (resp. normal) and Hausdorff.

4. Lebesgue type properties for quasi-uniformities

We recall that a quasi-uniformity $\mathcal{U}$ is said to be Lebesgue if every open covering has a refinement of the form $\{U(x) : x \in X\}$ for some $U \in \mathcal{U}$, and it is said to be cofinally complete if every directed open covering has a refinement of the form $\{U(x) : x \in X\}$ for some $U \in \mathcal{U}$.

Definition 4.1. A quasi-uniformity $\mathcal{U}$ is said to be co-Lebesgue (resp. F-co-Lebesgue, LF-co-Lebesgue, C-co-Lebesgue, D-co-Lebesgue) if every open covering (resp. finite, locally finite, countable, directed open covering) has a refinement of the form $\{U^{-1}(x) : x \in X\}$ for some $U \in \mathcal{U}$.

The definition of F-Lebesgue, LF-Lebesgue, C-Lebesgue and D-Lebesgue (= cofinally complete) quasi-uniformities is apparent.

It is clear that Lebesgue implies LF-Lebesgue, D-Lebesgue, C-Lebesgue and that LF-Lebesgue (resp. C-Lebesgue) implies F-Lebesgue. An LF-Lebesgue quasi-uniformiy in a paracompact space is Lebesgue, and a C-Lebesgue quasi-uniformiy in a Lindelöf space is Lebesgue.

The analogous results for co-Lebesgue properties are also clear.

Remark 4.2. Note that a transitive quasi-uniformity $\mathcal{U}$ is co-Lebesgue (resp. F-co-Lebesgue, LF-co-Lebesgue, C-co-Lebesgue, D-co-Lebesgue) if and only if $\mathbf{\Gamma}_{\mathcal{U}}$ (the directed fractal structure associated with $\mathcal{U}$) is uniform (resp. F-uniform, LF-uniform, C-uniform, D-uniform).

Also note that if $\mathcal{U}$ is a point symmetric quasi-uniformity and $\mathcal{U}^{-1}$ is a Lebesgue quasi-uniformity, then $\mathcal{U}$ is co-Lebesgue.

Proposition 4.3. *Let $\mathbf{\Gamma}$ be a directed fractal structure over a topological space X which is uniform (resp. F-uniform, LF-uniform, D-uniform). Then so is $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}}$.*

Proof. This is evident, since Γ_{U_Γ} is a refinement of Γ for every $\Gamma \in \mathbf{\Gamma}$. □

Proposition 4.4. *A topological space X admits a directed fractal structure $\mathbf{\Gamma}$ which is uniform (resp. F-uniform, LF-uniform, C-uniform, D-uniform) if and only if it admits a transitive quasi-uniformity $\mathcal{U}$ which is co-Lebesgue (resp. F-co-Lebesgue, LF-co-Lebesgue, C-co-Lebesgue, D-co-Lebesgue).*

Proof. Let $\mathbf{\Gamma}$ be a uniform directed fractal structure over X (the cases F-uniform, LF-uniform, C-uniform and D-uniform are analogous).

By Proposition 4.3, it follows that $\mathbf{\Gamma}_{\mathcal{U}_G}$ is uniform. By Remark 4.2, it follows that the quasi-uniformity $\mathcal{U}_{\mathbf{\Gamma}}$ associated with $\mathbf{\Gamma}$ is co-Lebesgue, since $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}}$ is uniform.

Conversely, let $\mathcal{U}$ be a co-Lebesgue transitive quasi-uniformity, then the directed fractal structure $\mathbf{\Gamma}_{\mathcal{U}}$ associated with $\mathcal{U}$ is uniform by Remark 4.2. □

In order to fully understand the next corollary, we recall that a topological space is orthocompact (resp. weakly orthocompact) if and only if $\mathcal{FT}$ (the finest transitive quasi-uniformity) is Lebesgue (resp. cofinally complete) and that a Tychonoff space is paracompact if and only if the fine uniformity is Lebesgue or cofinally complete. See also [4, Proposition 5.29].

Corollary 4.5. *Let X be a Hausdorff space.*

1. *X is paracompact if and only if $\mathcal{FT}$ is co-Lebesgue.*
2. *X is paracompact if and only if it admits a co-Lebesgue (transitive) quasi-uniformity.*
3. *X is compact if and only if $\mathcal{P}$ is co-Lebesgue.*
4. *X is compact if and only if it admits a totally bounded co-Lebesgue (transitive) quasi-uniformity.*
5. *X is normal if and only if $\mathcal{FT}$ is F-co-Lebesgue (or LF-co-Lebesgue).*
6. *X is normal if and only if it admits a F-co-Lebesgue or a LF-co-Lebesgue (transitive) quasi-uniformity.*
7. *X is metacompact if and only if $\mathcal{FT}$ is D-co-Lebesgue.*
8. *X is metacompact if and only if it admits a D-co-Lebesgue transitive quasi-uniformity.*

Proof. If X is a paracompact Hausdorff space then the directed fractal structure $\mathbf{\Gamma}$ of all closure preserving closed coverings is uniform, since it contains the base of directed fractal structure consisting of all locally finite tilings of X and by Proposition 3.26. Since the quasi-uniformity induced by $\mathbf{\Gamma}$ is $\mathcal{FT}$ by Proposition 3.18, and by Proposition 4.3 it follows that $\mathbf{\Gamma}_{\mathcal{U}_{\mathbf{\Gamma}}} = \mathbf{\Gamma}_{\mathcal{FT}}$ is uniform then $\mathcal{FT}$ is co-Lebesgue by Remark 4.2.

If X admits a co-Lebesgue transitive quasi-uniformity, then X is paracompact by Proposition 3.26 and Proposition 4.4.

The rest of the items have an analogous proof for the transitive cases, using Propositions 3.29, 3.23 and 3.30 (Recall from Proposition 2.3 that $\mathcal{U}$

is totally bounded if and only if the directed fractal structure $\mathbf{\Gamma}_{\mathcal{U}}$ induced by $\mathcal{U}$ is finite).

Now, let us prove the non-transitive cases.

(2) Suppose that X admits a co-Lebesgue quasi-uniformity. By [5, Lemmas 2.3 and 3.4] every open covering of X has a cushioned refinement (see [6] for a definition of a cushioned refinement) and so by [6, Theorem 9.2.3(v)] X is paracompact.

(4) Suppose that X admits a totally bounded co-Lebesgue quasi-uniformity $\mathcal{U}$. Let $\mathcal{F}$ be an ultrafilter on X that does not converge. Then for each $x \in X$ there exists $U_x \in \mathcal{U}$ such that $U_x(x) \notin \mathcal{F}$ and $U_x(x)$ is open. There is an entourage V such that $\{V^{-1}(x) : x \in X\}$ refines $\{U_x(x) : x \in X\}$. Since $\mathcal{U}^{-1}$ is totally bounded, there exists $p \in X$ with $V^{-1}(p) \in \mathcal{F}$. But $V^{-1}(p) \subseteq U_x(x)$ for some $x \in X$ and so $U_x(x) \in \mathcal{F}$ – a contradiction.

(6) Suppose that $\mathcal{U}$ is F-co-Lebesgue. Let F_1 and F_2 be disjoint closed subspaces of X. Then $\mathcal{A} = \{X \setminus F_1, X \setminus F_2\}$ is an open covering of X, and since $\mathcal{U}$ is F-co-Lebesgue, there exists $U \in \mathcal{U}$ (we can suppose that $U(x)$ is open for each $x \in X$) such that $\{U^{-1}(x) : x \in X\}$ refines $\mathcal{A}$. Then $U(F_1) \cap U(F_2) = \emptyset$. Indeed, if $x \in U(F_1) \cap U(F_2)$ we have that $U^{-1}(x) \cap F_1 \neq \emptyset$ and $U^{-1}(x) \cap F_2 \neq \emptyset$, a contradiction. Since $U(F_1)$ and $U(F_2)$ are open, X is normal. □

Question 4.6. Is it possible to drop the word transitive in item 8 of the previous proposition?

Remark 4.7. Note that a topological space can admit a Lebesgue quasi-uniformity without admitting a Lebesgue transitive quasi-uniformity (see [4]).

It is known ([4]) that a topological space is compact if and only if each admissible quasi-uniformity is Lebesgue. The next result shows that this is also the case for co-Lebesgue and Hausdorff spaces. Note that the Hausdorff condition is necessary in this proposition: by the sixth item of the preceding corollary, an F-co-Lebesgue quasi-uniform space is normal and so a non-Hausdorff compact T_1 quasi-uniform space cannot admit a co-Lebesgue quasi-uniformity.

Proposition 4.8. *A Hausdorff space X is compact if and only if every compatible quasi-uniformity is co-Lebesgue.*

PROOF. Suppose that X is compact, and let $\mathcal{U}$ be a quasi-uniformity on X, and $\mathcal{V}$ an open covering of X. Since X is compact, we can suppose for our purposes that $\mathcal{V}$ is finite. For each $x \in X$, let $U_x \in \mathcal{U}$ be such that $U_x(x) \subseteq V$ for some $V \in \mathcal{V}$, and let $V_x \in \mathcal{U}$ with $V_x \subseteq U_x$ be such that $V_x^{-1}(V_x(x)) \subseteq U_x(x)$ (note that since X is a compact Hausdorff space every compatible quasi-uniformity is locally symmetric). Then $\{V_x(x) : x \in X\}$ is an open covering of X, so there exists a finite subcovering $\{V_{x_1}(x_1), \ldots, V_{x_n}(x_n)\}$; let $W \in \mathcal{U}$ be such that $W \subseteq V_{x_k}$ for $k = 1, \ldots, n$. Let us check that $\{W^{-1}(x) : x \in X\}$ is a refinement of $\mathcal{V}$. Indeed, let $x \in X$, then there exists $k \in \{1, \ldots, n\}$ such that $x \in V_{x_k}(x_k)$. Then we have that $W^{-1}(x) \subseteq V_{x_k}^{-1}(x) \subseteq V_{x_k}^{-1}(V_{x_k}(x_k)) \subseteq U_{x_k}(x_k) \subseteq V$ for some $V \in \mathcal{V}$. Therefore $\{W^{-1}(x) : x \in X\}$ is a refinement of $\mathcal{V}$, and hence $\mathcal{U}$ is co-Lebesgue.

Conversely, if every compatible quasi-uniformity is co-Lebesgue, then $\mathcal{P}$ is co-Lebesgue, and by Corollary 4.5 it follows that X is a compact space. □

Proposition 4.9. *Let $(X,\mathcal{U})$ be a F-Lebesgue or a F-co-Lebesgue quasi-uniform space. Then it is equinormal.*

PROOF. Suppose that $\mathcal{U}$ is F-Lebesgue (if it is F-co-Lebesgue the reasoning is analogous). Let F_1 and F_2 be disjoint closed subspaces of X. Then $\mathcal{A} = \{X \setminus F_1, X \setminus F_2\}$ is an open covering of X, and since $\mathcal{U}$ is F-Lebesgue, there exists $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{A}$. So given $x \in F_1$ we have that $U(x) \subseteq X \setminus F_2$, and hence $U(F_1) \cap F_2 = \emptyset$. Therefore $\mathcal{U}$ is equinormal. □

Example 4.10. The Pervin quasi-uniformity for a regular non-normal Hausdorff space is equinormal, but it is not F-co-Lebesgue.

Corollary 4.11. *Let $\mathcal{U}$ be a co-Lebesgue (resp. C-co-Lebesgue, LF-co-Lebesgue, F-co-Lebesgue) quasi-uniformity for a topological space X. If X is regular (resp. R_0), then $\mathcal{U}$ is locally symmetric (resp. point symmetric).*

PROOF. This follows from 4.9 and [4, Proposition 2.26]. □

Definition 4.12. Let $(X,\mathcal{U})$ be a quasi-uniform space.

A filter $\mathcal{F}$ is said to be co-Cauchy if for each $U \in \mathcal{U}$ there exists $x \in X$ such that $U^{-1}(x) \in \mathcal{F}$.

A filter $\mathcal{F}$ is said to be weakly co-Cauchy if for each $U \in \mathcal{U}$ there exists $x \in X$ such that $U^{-1}(x) \cap F \neq \emptyset$ for each $F \in \mathcal{F}$.

$\mathcal{U}$ is said to be co-complete if every co-Cauchy filter clusters, and it is said to be convergence co-complete if every co-Cauchy filter converges.

$\mathcal{U}$ is said to be co-uniformly locally compact if there exists $U \in \mathcal{U}$ such that $\overline{U^{-1}(x)}$ is compact for each $x \in X$.

Note that convergence co-complete implies co-complete and co-complete implies Right K-complete (see [10] for a definition).

It is known (see [4]) that a Lebesgue (resp. D-Lebesgue) quasi-uniformity is convergence complete (resp. complete), and following the proof of the corresponding results in [4], it can be proved that a co-Lebesgue (resp. D-co-Lebesgue) quasi-uniformity is convergence co-complete (resp. co-complete); a quasi-uniformity is D-co-Lebesgue if and only if every weakly co-Cauchy filter clusters; a locally compact quasi-uniform space is co-uniformly locally compact if and only if it is D-co-Lebesgue; the conjugate of a co-Lebesgue quasi-uniformity contains each neighborhood of the diagonal and for a Tychonoff space the conjugate of a co-Lebesgue quasi-uniformity contains the fine uniformity.

Proposition 4.13. *Let X be a locally compact Hausdorff space. The following statements are equivalent:*

1. *X is metacompact.*
2. *There exists a base $\mathbf{\Gamma}$ of a directed fractal structure over X such that A is compact for each $A \in \Gamma$ and each $\Gamma \in \mathbf{\Gamma}$.*
3. *X admits a co-uniformly locally compact transitive quasi-uniformity.*

Proof. 1) implies 2). Let $\mathbf{\Gamma}$ be the family of all closure preserving coverings of X by compact sets. Let $x \in X$ and V be an open neighborhood of x. Let U be an open neighborhood of x with $\overline{U}$ compact and $\overline{U} \subseteq V$. Let U_x be an open neighborhood of x such that $\overline{U_x} \subseteq U$. For each $y \in X \setminus \overline{U_x}$, let U_y be an open neighborhood of y with $\overline{U_y}$ compact and $\overline{U_y} \cap \overline{U_x} = \emptyset$. Then $\mathcal{B} = \{\bigcup_{i \in F} U_{y_i} : y_i \in X \setminus \overline{U_x}; F \text{ finite}\}$ is a directed open cover of $X \setminus U$. Since $X \setminus U$ is metacompact, by [6, Theorem 9.3.5], there exists a closure preserving closed covering $\mathcal{A}$ of $X \setminus U$ which refines $\mathcal{B}$. It follows that $\Gamma = \mathcal{A} \cup \{\overline{U}\}$ is a closure preserving covering of X by compact sets and hence $\Gamma \in \mathbf{\Gamma}$. It easily follows that $\mathrm{St}(x, \Gamma) = \overline{U} \subseteq V$. Therefore $\mathbf{\Gamma}$ is a starbase base of a directed fractal structure (note that $\Gamma_1 \wedge \Gamma_2 \in \mathbf{\Gamma}$ whenever $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}$, and that $\Gamma_1 \wedge \Gamma_2 \prec\!\prec \Gamma_1, \Gamma_2$) compatible with the topology of X, so $\mathbf{\Gamma}$ verifies 2).

2) implies 3). $\mathcal{U}_{\mathbf{\Gamma}}$ is a co-uniformly locally compact transitive quasi-uniformity.

3) implies 1). Suppose that X admits a co-uniformly locally compact transitive quasi-uniformity. Then it is D-co-Lebesgue by the previous comments, and hence X is metacompact by Corollary 4.5. □

Proposition 4.14. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}$ is co-Lebesgue if and only if X is paracompact and $\mathcal{U}^{-1}$ contains the fine uniformity.*

Proof. If $\mathcal{U}$ is co-Lebesgue, then X is paracompact by Corollary 4.5, and $\mathcal{U}^{-1}$ contains the fine uniformity by the previous comments.

Conversely, suppose that X is paracompact and $\mathcal{U}^{-1}$ contains the fine uniformity. Let $\mathcal{A}$ be an open covering of X. Since X is paracompact, the fine uniformity is Lebesgue and there exists a symmetric element U of the fine uniformity (and hence $U \in \mathcal{U}^{-1}$) such that $\{U(x) : x \in X\}$ refines $\mathcal{A}$. Since $U \in \mathcal{U}^{-1}$ it follows that $\mathcal{U}$ is co-Lebesgue. □

5. Quasi-pseudometrics and Lebesgue type properties

In this section we study Lebesgue type properties for quasi-pseudometric spaces and show that for regular quasi-metric spaces the notions of Lebesgue, LF-Lebesgue, C-Lebesgue, co-Lebesgue, co-LF-Lebesgue and co-C-Lebesgue are equivalent, as are the notions of equinormal, F-Lebesgue and F-co-Lebesgue.

Proposition 5.1. *Let (X, d) be a equinormal quasi-pseudometric space. Then d is D-Lebesgue and F-Lebesgue. If X is regular, then d is D-co-Lebesgue and F-co-Lebesgue.*

Proof. It is proved in [7] that if d is equinormal then it is D-Lebesgue. We are going to modify the proof of that result in order to obtain the rest of the properties.

Suppose that d is equinormal, and let $\{U_n : n \in \mathbb{N}\}$ be a countable base of quasi-uniformity for d such that $U_m \subseteq U_n$ for $m \geq n$.

Let us prove that d is F-Lebesgue.

Let $\mathcal{A}$ be a finite open covering, and suppose that d is not F-Lebesgue. For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $U_n(x_n) \nsubseteq A$ for each $A \in \mathcal{A}$. Suppose that x is a cluster point of (x_n), and let $A \in \mathcal{A}$ such that $x \in A$. Then there exists $k \in \mathbb{N}$ such that $U_k^2(x) \subseteq A$. On the other hand, there exists $m \geq k$ such that $x_m \in U_k(x)$, and hence $U_m(x_m) \subseteq U_k(x_m) \subseteq U_k^2(x) \subseteq A$, which contradicts the choice of x_m. Therefore the set $\{x_n : n \in \mathbb{N}\}$ is hereditarily closed. Since $\mathcal{A}$ is finite, there exists $A \in \mathcal{A}$ such that $x_n \in A$ for each $n \in M$, where M is an infinite subset of $\mathbb{N}$.

For each $n \in M$ let $y_n \in U_n(x_n) \cap (X \setminus A)$. Since $X \setminus A$ is closed and $\{x_n : n \in M\} \subseteq A$, it follows that $\overline{\{y_n : n \in M\}} \cap \{x_n : n \in M\} = \emptyset$, and we also have that $y_n \in U_n(x_n)$, which contradicts that d is equinormal.

Let us prove that d is F-co-Lebesgue (D-co-Lebesgue), assuming that X is regular.

Let $\mathcal{A}$ be a finite (directed) open covering, and suppose that d is not F-co-Lebesgue (D-co-Lebesgue). For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $U_n^{-1}(x_n) \nsubseteq A$ for each $A \in \mathcal{A}$. We show that $\{x_n : n \in \mathbb{N}\}$ is hereditarily closed. Suppose that x is a cluster point of (x_n), and let $A \in \mathcal{A}$ be such that $x \in A$. Since X is regular, d is locally symmetric and hence there exists $k \in \mathbb{N}$ such that $U_k^{-1} \circ U_k(x) \subseteq A$. On the other hand, there exists $m \geq k$ such that $x_m \in U_k(x)$, and hence $U_m^{-1}(x_m) \subseteq U_k^{-1}(x_m) \subseteq U_k^{-1} \circ U_k(x) \subseteq A$, which contradicts the choice of x_m. Therefore the set $\{x_n : n \in \mathbb{N}\}$ is hereditarily closed. If $\mathcal{A}$ is finite, there exists $A \in \mathcal{A}$ such that $x_n \in A$ for each $n \in M$, where M is an infinite subset of $\mathbb{N}$.

For each $n \in M$ let $y_n \in U_n^{-1}(x_n) \cap (X \setminus A)$. Since each cluster point of (y_n) is a cluster point of (x_n) it follows that $\{y_n : n \in M\}$ is closed and $\{y_n : n \in M\} \cap \{x_n : n \in M\} = \emptyset$, and we also have that $y_n \in U_n^{-1}(x_n)$, which contradicts that d is equinormal.

We now consider the case that $\mathcal{A}$ is a directed open covering. Suppose that for each $n \in \mathbb{N}$ there is an $m(n) > n$ such that $x_{m(n)} \in U_n^{-1}(x_n)$. Then we can construct two subsequences (a_n) and (b_n) of (x_n) such that $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\} = \emptyset$ and $b_n \in U_n^{-1}(a_n)$, which is not possible since d is equinormal. Therefore, we can assume, without loss of generality, that for each $n \in \mathbb{N}$, $U_n^{-1}(x_n) \cap \{x_m : m > n\} = \emptyset$. Let $y_n \in U_n^{-1}(x_n) \cap (X \setminus A_n)$, where $A_n \in \mathcal{A}$ is such that $x_n \in A_n$ for each $n \in \mathbb{N}$ and $A_{n-1} \subseteq A_n$ if $n > 1$ (note that this is possible since $\mathcal{A}$ is directed). Then it is clear that $y_n \neq x_m$ for each $n, m \in \mathbb{N}$. Since each cluster point of (y_n) is a cluster point of (x_n) it follows that $\{y_n : n \in M\}$ is closed and $\{y_n : n \in M\} \cap \{x_n : n \in M\} = \emptyset$, and we also have that $y_n \in U_n^{-1}(x_n)$, which contradicts that d is equinormal. □

Corollary 5.2. *A quasi-pseudometric is equinormal if and only if it is F-Lebesgue. A regular quasi-pseudometric space is equinormal if and only if it is F-co-Lebesgue.*

Corollary 5.3. *Let (X, d) be a F-co-Lebesgue quasi-metric space. Then X admits a cofinally complete metric.*

Proof. By Corollary 4.5 it follows that X is normal, and hence d is equinormal, so it is locally symmetric and X is metrizable. By the previous proposition d is cofinally complete and hence X admits a cofinally complete metric by [7, Corollary 1]. □

Proposition 5.4. *Let (X, d) be a C-co-Lebesgue quasi-pseudometric space. Then it is Lebesgue. Moreover, if (X, d) is a C-Lebesgue regular quasi-pseudometric space, then it is co-Lebesgue.*

Proof. Suppose that (X, d) is not Lebesgue. Then there exist an open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of X and a sequence (x_n) in X such that $U_n(x_n) \setminus G_\alpha \neq \emptyset$ for all $n \in \mathbb{N}$ and $\alpha \in \Delta$. It follows that the sequence (x_n) has no cluster point (see the proof of Proposition 5.1). Hence for each $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ with $x_j \notin U_{k_n}(x_n)$ whenever $j \neq n$.

For each $n \in \mathbb{N}$, let $G_{\alpha_n} \in \mathcal{G}$ be such that $x_n \in G_{\alpha_n}$. Construct the open cover of X, $\mathcal{A} = \{U_{k_n}(x_n) \cap G_{\alpha_n} : n \in \mathbb{N}\} \cup \{X \setminus \{x_n : n \in \mathbb{N}\}\}$.

Since (X, d) is C-co-Lebesgue, there is $m \in \mathbb{N}$ such that $\{U_m^{-1}(x) : x \in X\}$ refines $\mathcal{A}$. Let $x \in U_m(x_m) \setminus G_{\alpha_m}$. Then $U_m^{-1}(x) \nsubseteq X \setminus \{x_n : n \in \mathbb{N}\}$ because $x_m \in U_m^{-1}(x)$, $U_m^{-1}(x) \nsubseteq U_{k_m}(x_m) \cap G_{\alpha_m}$ because $x \notin G_{\alpha_m}$ and $U_m^{-1}(x) \nsubseteq U_{k_j}(x_j) \cap G_{\alpha_j}$, $j \neq m$, because $x_m \notin U_{k_j}(x_j)$ for $j \neq m$. This contradiction proves that (X, d) is Lebesgue.

The proof of the second part is analogous. □

Proposition 5.5. *Let (X, d) be a LF-co-Lebesgue quasi-pseudometric space. Then (X, d) is LF-Lebesgue.*

Proof. The proof is analogous to the proof of the previous proposition, only note that $\mathcal{A}$ is locally finite if $\mathcal{G}$ is. □

Now, we summarize the previous results.

Corollary 5.6. *Let (X, d) be a regular quasi-metric space. Then it is Lebesgue if and only if it is co-Lebesgue or LF-Lebesgue or LF-co-Lebesgue or C-Lebesgue or C-co-Lebesgue. It is equinormal if and only if it is F-Lebesgue or F-co-Lebesgue.*

Proof. Note that if (X, d) is equinormal and regular then X is metrizable since d is locally symmetric and hence paracompact.

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References

[1] F. G. Arenas, Tilings in topological spaces, *Int. Jour. of Maths. and Math. Sci.* **22** (1999), 611–616.

[2] F. G. Arenas and M. A. Sánchez-Granero, A characterization of non-archimedeanly quasimetrizable spaces, *Rend. Istit. Mat. Univ. Trieste Suppl.* **XXX** (1999), 21–30.

[3] F. G. Arenas and M. A. Sánchez-Granero, Directed GF-spaces, *Appl. Gen. Topology* **2** no. 2 (2001), 191–204.

[4] P. Fletcher and W. F. Lindgren, Quasi-Uniform Spaces, Lecture Notes in Pure and Appl. Math. 77, *Marcel Dekker*, *New York*, 1982.

[5] H. J. K. Junnila, Neighbornets, *Pacific J. Math.* **76** (1978), 83–108.

[6] K. Kunen and J. E. Vaughan (editors), Handbook of Set-Theoretic Topology, *North-Holland*, *Amsterdam–New York–Oxford*, 1984.

[7] H. P. A. Kunzi and S. Romaguera, Quasi-metric spaces, quasi-metric hyperspaces and uniform local compactness, *Rend. Istit. Mat. Univ. Trieste Suppl.* **XXX** (1999), 133–144.

[8] A. R. Pears, On quasi-order spaces, normality, and paracompactness, *Proc. London Math. Soc.* (3) **23** (1971), 428–444.

[9] A. Pears and J. Mack, Quasi-order spaces, *Proc. London Math. Soc.* (3) **29** (1974), 289–316.

[10] S. Romaguera, On hereditary precompactness and completeness in quasi-uniform spaces, *Acta Math. Hung.* **73** (1996), 159–178.

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