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# On $q$-multiplicative functions

By K.-H. INDLEKOFER (Paderborn), I. KÁTAI (Budapest)
and Y.-W. LEE (Paderborn)

**Abstract.** The analogon of Delange's theorem for $q$-multiplicative functions is investigated for some subsets of integers.

## 1. Introduction

Let $q \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \dots, q-1\}$. We shall use the standard notations: $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, denote the set of positive integers, non-negative integers, integers, real-numbers, complex numbers, respectively. For $x \in \mathbb{R}$ let $\{x\}$ be the fractional part of $x$, and $\|x\|$ be the distance of $x$ to the closest integer. The $q$-ary expansion of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \dots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j , \quad \varepsilon_j(n) \in \mathbb{A} \tag{1}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \dots$ are called the digits in the $q$-ary expansion of $n$.

Let $\mathcal{A}_q$ be the set of real-valued $q$-additive functions, and $\mathcal{M}_q$ be the set of complex-valued $q$-multiplicative functions.

A function $f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$, and for every $n \in \mathbb{N}_0$,

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n) q^j). \tag{2}$$

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A function $g : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0) = 1$, and for every $n \in \mathbb{N}_0$,

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j). \tag{3}$$

Since $f(\varepsilon_j(n)q^j) = 0$, $g(\varepsilon_j(n)q^j) = 1$ for all those $j$ for which $q^j > n$, therefore the number of summands on the right hand side of (2), and the number of factors on the right hand side of (3) is finite.

Let $\overline{\mathcal{M}}_q$ be the class of $q$-multiplicative functions with modulus 1: i.e. $g \in \overline{\mathcal{M}}_q$, if $g$ is $q$-multiplicative and $|g(n)| = 1$ $(n \in \mathbb{N}_0)$. Let $e(\alpha) = e^{2\pi i \alpha}$.

A classical theorem of H. Delange [1] asserts that for $g \in \overline{\mathcal{M}}_q$, $N_x = \left[\frac{\log x}{\log q}\right]$,

$$m(x) := \frac{1}{x}\sum_{n<x} g(n) = \prod_{j=0}^{N_x-1} \frac{1}{q}\left(\sum_{b\in\mathbb{A}} g(bq^j)\right) + o_x(1),$$

whence he deduced that $\lim_{x\to\infty} |m(x)|$ always exists and equals

$$\prod_{j=0}^{\infty} \left|\frac{1}{q}\sum_{b\in\mathbb{A}} g(bq^j)\right|,$$

which is nonzero if and only if

$$\sum_{b\in\mathbb{A}} g(bq^j) \neq 0 \quad (\text{for all } j \in \mathbb{N}_0) \tag{4}$$

and

$$\sum_{j=0}^{\infty}\sum_{b\in\mathbb{A}} \mathrm{Re}(1 - g(bq^j)) < \infty. \tag{5}$$

Furthermore, he proved that $\lim\limits_{x\to\infty} m(x)$ exists and is nonzero if and only if (4) holds and the series

$$\sum_{j=0}^{\infty}\sum_{b\in\mathbb{A}} (1 - g(bq^j)) \tag{6}$$

is convergent.

An interesting problem is to give analogues of DELANGE's theorem [1], if we sum $g(n)$ on some subsets of the integers. Let $\alpha_1, \dots, \alpha_k$ be rationally independent real numbers, i.e. such that $h_1\alpha_1 + \cdots + h_k\alpha_k + h_{k+1} \cdot 1 = 0$ has the only solution $h_1 = \cdots = h_{k+1} = 0$ in integers $h_1, \dots, h_{k+1}$. Let $I_j = [u_j, v_j) \subset [0,1)$ be arbitrary proper subintervals of $[0,1)$, let $E$ be the set of those integers $n$ for which

$$\{\alpha_1 n\} \in I_1, \dots, \{\alpha_k n\} \in I_k$$

simultaneously holds.

Let

$$l(n) = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \in \mathbb{N}_0 \setminus E. \end{cases}$$

Our purpose in this paper is to investigate the sum

$$M(x) := \sum_{n<x} g(n) l(n)$$

for $g \in \overline{\mathcal{M}}_q$.

We shall prove the following

**Theorem 1.** $\lim_{x\to\infty} \frac{|M(x)|}{x}$ *always exists. It is nonzero if there exist integers* $h_1, \dots, h_k$ *for which*

$$\sum_{l=0}^{\infty} \sum_{b\in\mathbb{A}} \mathrm{Re}(1 - g(bq^l) e((h_1\alpha_1 + \cdots + h_k\alpha_k) bq^l)) < \infty \tag{7}$$

and

$$\sum_{b\in\mathbb{A}} g(bq^l) e((h_1\alpha_1 + \cdots + h_k\alpha_k) bq^l) \neq 0 \quad l = 0, 1, \dots \tag{8}$$

*The relation* (7) *can be satisfied for at most one choice of* $h_1, \dots, h_k \in \mathbb{Z}$. *Assume that* (7) *holds. Then*

$$\frac{M(x)}{x} = c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1,\dots,h_k}(x)}{x} + o_x(1)$$

*where*

$$c_{h_j}^{(j)} = \frac{e(-h_j u_j) - e(-h_j v_j)}{2\pi i h_j} \quad \text{if } h_j \neq 0,$$

*and*

$$c_0^{(j)} = (v_j - u_j),$$

*furthermore,*

$$S_{h_1,\ldots,h_k}(x) = \sum_{n<x} g(n)e((h_1\alpha_1 + \cdots + h_k\alpha_k)n).$$

$\lim_{x\to\infty} \frac{M(x)}{x}$ *exists if and only if* $\lim_{x\to\infty} \frac{S_{h_1,\ldots,h_k}(x)}{x}$ *exists.* $\lim_{x\to\infty} \frac{S_{h_1,\ldots,h_k}(x)}{x}$ *exists if and only if* $\sum_{l=0}^{\infty}\sum_{b\in\mathbb{A}}(1 - g(bq^l)e((h_1\alpha_1 + \cdots + h_k\alpha_k)bq^l))$ *is convergent.*

## 2. Proof

Let $f_j$ $(j = 1, \ldots, k)$ be the function defined in $[0, 1)$ by

$$f_j(y) = \begin{cases} 1 & \text{if } y \in I_j, \\ 0 & \text{if } y \in [0,1) \setminus I_j, \end{cases}$$

and extended periodically mod 1. Then

$$f_j(y) \sim \sum_{m=-\infty}^{\infty} c_m^{(j)} e(my),$$

where $c_m^{(j)} = \frac{e(-mu_j) - e(-mv_j)}{2\pi i m}$, if $m \neq 0$ and $c_0^{(j)} = (v_j - u_j) = |I_j|$.

Choosing a small $\Delta > 0$, for

$$f_j^*(u) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} f_j(y+u)dy \sim \sum_{m=-\infty}^{\infty} d_m^{(j)}(\Delta)e(mu)$$

we obtain that$d_0^{(j)}(\Delta) = |I_j|$ and $|d_m^{(j)}(\Delta)| \leq \frac{c}{\Delta m^2}$, with an absolute positive constant $c$. Thus the Fourier series is absolutely convergent and represents $f_j^*(u)$. Let now $K$ be a large integer, $\widetilde{f}_j(u) = \sum_{|h|\leq K} d_h^{(j)}(\Delta)e(hu)$. Then $|f_j^*(u) - \widetilde{f}_j(u)| \leq \frac{c}{\Delta K}$, and so

$$\left|\prod_{j=1}^{k} f_j^*(u_j) - \prod_{j=1}^{k} \widetilde{f}_j(u_j)\right| \leq \frac{ck}{\Delta K}, \tag{9}$$

since $0 \le f_j^*(u_j) \le 1$ holds for $u_j \in \mathbb{R}$. We obviously have $l(n) = \prod_{j=1}^k f_j(n\alpha_j)$. Let $l^*(n) := \prod_{j=1}^k f_j^*(n\alpha_j)$ and $\tilde{l}(n) := \prod_{j=1}^k \widetilde{f}_j(n\alpha_j)$. Let us observe that $f_j^*(u) = f_j(u)$ if $u \notin [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]$. Therefore $l(n) = l^*(n)$, except when $\{n\alpha_j\} \in [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]$ for some $j$. Furthermore $|l(n) - l^*(n)| \le 1$ always holds.

Let $S(x) := \sum_{n<x} g(n)\tilde{l}(n)$. We have

$$|M(x) - S(x)| \le \left| \sum_{n<x} g(n)(l(n) - \tilde{l}(n)) \right| \le \sum_{n<x} |l(n) - \tilde{l}(n)|$$
$$\le \sum_{n<x} |l(n) - l^*(n)| + \sum_{n<x} |l^*(n) - \tilde{l}(n)| = \sum_1 + \sum_2 .$$

From (9) we have that $\sum_2 \le \frac{ckx}{\Delta K}$. Furthermore,

$$\sum_1 \le \sum_{j=1}^k \sharp\{n \le x \mid \{\alpha_j n\} \in [u_j - \Delta, u_j + \Delta] \cup [v_j - \Delta, v_j + \Delta]\}$$

and by using that $\alpha_j n$ is uniformly distributed mod 1, we obtain that $\sum_1 \le c_1 k \Delta x$ with an absolute positive constant $c_1$ for every large $x$.

Let us observe furthermore that

$$S(x) = \sum_{h_1,\ldots,h_k} d(h_1,\ldots,h_k) S_{h_1,\ldots,h_k}(x)$$

where $h_1,\ldots,h_k$ run over the integers in $[-K, K]$,

$$d(h_1,\ldots,h_k) = \prod_{j=1}^k d_{h_j}^{(j)}(\Delta)$$

and

$$S_{h_1,\ldots,h_k}(x) = \sum_{n<x} g(n) e((h_1\alpha_1 + \cdots + h_k\alpha_k)n).$$

**Lemma 1.** *Assume that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{x} > 0.$$

*Then there are some integers* $h_1^*, \dots, h_k^*$ *such that*

$$\sum_{l=0}^{\infty} \sum_{c \in \mathbb{A}} \mathrm{Re}(1 - g(cq^l) e((h_1^*\alpha_1 + \cdots + h_k^*\alpha_k) cq^l)) \tag{10}$$

*is convergent.*

Proof of Lemma 1. The function $g(n)e((h_1\alpha_1 + \cdots + h_k\alpha_k)n)$ as a function of $n$ belongs to $\overline{\mathcal{M}}_q$. If (10) does not hold, then $\frac{|S_{h_1,\dots,h_k}(x)|}{x} \to 0$ $(x \to \infty)$ due to Delange's theorem [1], and so $\frac{|S(x)|}{x} \to 0$. Since $\frac{|M(x)|}{x} \le \frac{|S(x)|}{x} + \frac{|M(x) - S(x)|}{x}$, and the second term is less than $c_1 k\Delta + \frac{ck}{\Delta K}$, therefore

$$\limsup_{x \to \infty} \frac{|M(x)|}{x} \le c_1 k \Delta + \frac{ck}{\Delta K}. \tag{11}$$

This inequality holds for each $\Delta > 0$ and each $K > 0$. By letting $K \to \infty$, then $\Delta \to 0$, we obtain that

$$\limsup_{x \to \infty} \frac{|M(x)|}{x} = 0. \qquad \square$$

**Lemma 2.** *The relation*

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \mathrm{Re}(1 - g(bq^j) e((h_1\alpha_1 + \cdots + h_k\alpha_k) bq^j)) < \infty \tag{12}$$

*may hold at most for one collection of integers* $h_1, \dots, h_k$.

Proof of Lemma 2. The relation (12) is equivalent to

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \left\| \frac{\arg g(bq^j)}{2\pi} + (h_1\alpha_1 + \cdots + h_k\alpha_k) bq^j \right\|^2 < \infty.$$

Assume that (12) holds with $(h_1, \dots, h_k)$ as well as with $(h_1^*, \dots, h_k^*)$.

Let $\gamma = (h_1 - h_1^*)\alpha_1 + \cdots + (h_k - h_k^*)\alpha_k$. If $(h_1, \dots, h_k) \ne (h_1^*, \dots, h_k^*)$, then $\gamma$ is an irrational number, and

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}} \|\gamma b q^j\|^2 < \infty. \tag{13}$$

we shall see that (13) is impossible.

From (13) it follows that $\|\gamma q^j\| \to 0$ $(j \to \infty)$. Let $\gamma q^j = m_j + \delta_j$, where $\delta_j \in (-\frac{1}{2}, \frac{1}{2}]$, $m_j \in \mathbb{Z}$. Then $\|\delta_j\| = \|\gamma q^j\|$. Furthermore $\gamma q^{j+1} = q m_j + q\delta_j$, and so $\delta_{j+1} = q\delta_j$, $m_{j+1} = q m_j$ for every large $j$, and this contradicts to the fact that $\delta_j \neq 0$. The lemma is proved.

**Lenmma 3.** *Assume that* (12) *holds with* $(h_1, \dots, h_k)$. *Then*

$$\frac{M(x)}{x} = c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1,\dots,h_k}(x)}{x} + o_x(1).$$

Proof of Lemma 3. Repeating the argumentation of Lemma 1, we deduce that

$$\left| \frac{M(x)}{x} - d(h_1, \dots, h_k) \frac{S_{h_1,\dots,h_k}(x)}{x} \right| \le c_1 k \Delta + \frac{ck}{\Delta K},$$

whence

$$\left| \frac{M(x)}{x} - c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1,\dots,h_k}(x)}{x} \right|$$
$$\le c_1 k \Delta + \frac{ck}{\Delta K} + |d(h_1, \dots, h_k) - c_{h_1}^{(1)} \dots c_{h_k}^{(k)}|.$$

Then, by $K \to \infty$, and $\Delta \to 0$ we obtain that

$$\lim_{x\to\infty} \left| \frac{M(x)}{x} - c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \frac{S_{h_1,\dots,h_k}(x)}{x} \right| \to 0,$$

due to the fact that $d(h_1, \dots, h_k) \to c_{h_1}^{(1)} \dots c_{h_k}^{(k)}$ as $\Delta \to 0$.

Observe that $c_{h_1}^{(1)} \dots c_{h_k}^{(k)} \neq 0$.

From Lemma 3 we obtain that $\lim_{x\to\infty} \frac{M(x)}{x}$ exists if and only if $\lim_{x\to\infty} \frac{S_{h_1,\dots,h_k}(x)}{x}$ exists. Due to Delange's theorem [1] it exists and nonzero if and only if

$$\sum_{b\in\mathbb{A}} g(bq^j) e((h_1\alpha_1 + \dots + h_k\alpha_k) bq^j) \neq 0$$

for $j = 0, 1, \dots$, and

$$\sum_{j=0}^{\infty} \sum_{b\in\mathbb{A}} (1 - g(bq^j) e((h_1\alpha_1 + \dots + h_k\alpha_k) bq^j))$$

is convergent.

Hence Theorem 1 immediately follows.

## 3. On the distribution of $q$-additive functions

**Theorem 2.** *Let* $f \in \mathcal{A}_q$, $E(x) := \sharp\{n < x,\ n \in E\}$

$$F_x(y) := \frac{1}{E(x)} \sharp\{n < x,\ n \in E,\ f(n) < y\}.$$

*The limit* $\lim_{x\to\infty} F_x(y) = F(y)$ *exists for almost all* $y \in \mathbb{R}$*, where* $F$ *is a distribution function, if and only if the series*

$$\sum_{j=0}^{\infty} \sum_{b\in\mathbb{A}} f(bq^j) \tag{14}$$

and

$$\sum_{j=0}^{\infty} \sum_{b\in\mathbb{A}} f^2(bq^j) \tag{15}$$

*are convergent.*

Proof. Let $g_\tau(n) = e(\tau f(n))$, where $\tau \in \mathbb{R}$. Then $g_\tau(n) \in \overline{\mathcal{M}}_q$. Let $m_\tau(x) = \frac{1}{E(x)} \sum_{n<x} g_\tau(n) l(n)$. Assume first that (14), (15) are satisfied. Then $\sum_{j=0}^{\infty} \sum_{b\in\mathbb{A}} (1 - g_\tau(bq^j))$ is convergent, and by Theorem 1 we obtain that $m_\tau(x) \to m(\tau)$ $(x \to \infty)$, where $m(\tau) \neq 0$ in a neighborhood of 0, i.e. if $|\tau| < c$. Thus, by a wellknown theorem in probability theory we obtain that $F_x(y) \to F(y)$, the characteristic function of $F$ is $m(\tau)$.

Assume now that $\lim_{x\to\infty} F_x(y)$ exists. Then there exists $\lim_{x\to\infty} m_\tau(x) = m(\tau)$ in a suitable interval $|\tau| \leq c$. Applying Theorem 1, we obtain that

$$\sum_{l=0}^{\infty} \sum_{b\in\mathbb{A}} (1 - e(\tau f(bq^l) + (h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k) bq^l)) \tag{16}$$

is convergent, where $h_1(\tau), \ldots, h_k(\tau)$ are suitable integers. Consequently

$$\|\tau f(bq^l) + (h_1(\tau)\alpha_1 + \cdots + h_k(\tau)\alpha_k) bq^l\| \to 0,$$

and so

$$\|m\tau f(bq^l)+m(h_1(\tau)\alpha_1+\cdots+h_k(\tau)\alpha_k)bq^l\|\to 0,$$

for every $m\in\mathbb{N}$. Furthermore,

$$\|m\tau f(bq^l)+(h_1(m\tau)\alpha_1+\cdots+h_k(m\tau)\alpha_k)bq^l\|\to 0,$$

as $l\to\infty$.

Then, for every fixed $m\in\mathbb{N}$,

$$\sum_{l=0}^{\infty}\sum_{b\in\mathbb{A}}(1-e(m\tau f(bq^l)+m(h_1(\tau)\alpha_1+\cdots+h_k(\tau)\alpha_k)bq^l)) \tag{17}$$

is convergent as well. Applying (16) for $m\tau$ instead of $\tau$, we obtain that

$$\sum_{l=0}^{\infty}\sum_{b\in\mathbb{A}}(1-e(m\tau f(bq^l)+(h_1(m\tau)\alpha_1+\cdots+h_k(m\tau)\alpha_k)bq^l)), \tag{18}$$

(17) and (18) easily imply that

$$\sum_{l=0}^{\infty}\sum_{b\in\mathbb{A}}\|[(h_1(m\tau)-mh_1(\tau))\alpha_1+\cdots+(h_k(m\tau)-mh_k(\tau))\alpha_k]bq^l\|^2<\infty.$$

Applying the argument which was used in the proof of Lemma 2, and that $\alpha_1,\dots,\alpha_k$ are linearly independent, we obtain that $h_j(m\tau)=mh_j(\tau)$ $(j=1,\dots,k)$. Let now $K$ be fixed, $|K|\le c$. Then $h_j(K)=mh_j\left(\frac{K}{m}\right)$ holds for every $m=1,2,\dots$, and since $h_j\left(\frac{K}{m}\right)\in\mathbb{Z}$, therefore $m$ divides $h_j(K)$ for every $m$. Thus $h_j(K)=0$ $(j=1,\dots,k)$, $|K|\le c$. Consequently,

$$\sum_{l=0}^{\infty}\sum_{b\in\mathbb{A}}(1-e(\tau f(bq^l))) \tag{19}$$

is convergent for $|\tau|\le c$.

Hence one can deduce that (14), (15) are convergent.

## References

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K.–H. INDLEKOFER
FACULTY OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF PADERBORN
WARBURGER STRASSE 100
33098 PADERBORN
GERMANY

IMRE KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY I/D
H–1117 BUDAPEST
HUNGARY

Y.–W. LEE
FACULTY OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF PADERBORN
WARBURGER STRASSE 100
33098 PADERBORN
GERMANY

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