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Some remarks on the geometry of quasi-Banach spaces

F. ALBIAC (Pamplona) and C. LERÁNOZ (Pamplona)

Abstract. We study the definitions of strict p-convexity and uniform p-convexity and the relations between them for quasi-Banach spaces when $0 < p < 1$.

1. Introduction

The study of quasi-normed spaces arises naturally as a generalization of normed spaces by substituing the triangular inequality of the norm by a weaker condition. The geometrical meaning of that generalization is that whereas the unit ball of a normed space is a convex set, the unit ball of a quasi-normed space needs not be convex. From a topogical point of view, the topological vector spaces whose topology can be induced by a norm are those which have a convex bounded neighbourhood of zero, while the topological vector spaces whose topology can be induced by a quasi-norm are those which have a bounded neighbourhood of zero (see [3] and [7]). If that topology is complete, the normed spaces are called Banach spaces, and the quasi-normed spaces are called quasi-Banach spaces.

The geometry of the unit ball in Banach spaces has been widely studied and a great deal of information can be found in the mathematical literature. In this paper, we present some aspects of the geometry of the unit ball of those quasi-Banach spaces which are not Banach spaces. We will translate some concepts used to describe the characteristics of the unit ball of normed spaces into the setting of quasi-normed spaces and we will see to what extent we can generalize the properties relating those concepts.

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In Section 2, we define strict p-convexity of quasi-Banach spaces, for $0 < p < 1$, generalizing the classical concept of strict convexity of Banach spaces, and see some characterizations. We present the Lorentz sequence spaces $d(w,p)$ as an example of quasi-Banach spaces which are strictly p-convex but do not have an equivalent q-norm for any $0 < p < q \leq 1$.

In Section 3, we discuss the definition of uniform p-convexity ($0 < p < 1$) that best generalizes the classical concept of uniform convexity of Banach spaces.

We now introduce some definitions and notation we will be using in the sequel (see [3]).

Let X be a (real) vector space. A *quasi-norm* on X is a map $\|\cdot\| : X \longrightarrow [0,+\infty)$ satisfying

(i) $\|x\| > 0$ if $x \neq 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$, $\alpha \in \mathbb{R}$,

(iii) $\|x+y\| \leq C(\|x\| + \|y\|)$ for any $x, y \in X$, where $C \geq 1$ is a constant independent of x and y. If $C = 1$, $\|\cdot\|$ is a norm.

A quasi-norm is *p-subadditive* ($0 < p \leq 1$), and it is called *p-norm*, if

(iv) $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for any $x, y \in X$.

A quasi-norm clearly defines a metrizable locally bounded vector topology on X. If such topology is complete then we say that $(X, \|\cdot\|)$ is a *quasi-Banach space*. If the quasi-norm is also p-subadditive then X is a *p-Banach space*.

Given a vector space X and $0 < p \leq 1$, we will denote by $[x,y]_p$ the *p-segment with ending points* x, y:

$$[x,y]_p = \{\lambda x + \mu y : \lambda, \mu \geq 0,\ \lambda^p + \mu^p = 1\}.$$

A subset C of X is *p-convex* (where $0 < p \leq 1$) if given x, $y \in C$ and $0 \leq t, s \leq 1$ with $s^p + t^p = 1$, then $tx + sy \in C$, i.e. C is p-convex if it contains every p-segment with ending points in C.

We recall that if $S \subset X$ and $0 < p \leq 1$, then the *p-convex hull* of S is the smallest p-convex set that contains S and can be described as

$$p-\mathrm{co}(S) = \left\{ \sum_{k=1}^{N} \lambda_k x_k : \sum_{k=1}^{N} \lambda_k^p = 1,\ \lambda_k \geq 0,\ x_k \in S,\ k = 1, \ldots, N,\ N \in \mathbb{N} \right\}.$$

When $0 < p < 1$, the p-convex hull of a set S can be written as well as

$$p - \mathrm{co}(S) = \Big\{ \sum_{k=1}^{N} \lambda_k x_k : 0 < \sum_{k=1}^{N} \lambda_k^p \leq 1,\ \lambda_k \geq 0,\ x_k \in S,$$
$$k = 1, \ldots, N,\ N \in \mathbb{N} \Big\}.$$

Therefore, if $0 < p < 1$ any non empty, p-convex, closed set contains 0. That leads to interesting situations. For instance, a set formed by one single element $x \neq 0$ is not p-convex if $0 < p < 1$. In this case,

$$p - \mathrm{co}(\{x\}) = \{\lambda x : 0 < \lambda \leq 1\}.$$

Another particularity of the case $0 < p < 1$ is that $[x, y]_p$ needs not be a p-convex set: if x and y are not colinear then

$$p\text{−}\mathrm{co}[x, y]_p = \{\lambda x + \mu y : \lambda, \mu \geq 0,\ 0 < \lambda^p + \mu^p \leq 1\} = \bigcup_{z \in [x,y]_p} \{\lambda z : 0 < \lambda \leq 1\}.$$

2. p-convexity and strict p-convexity

Whereas the unit ball of a normed space is always a convex set, the unit ball B_X of a quasi-normed space X needs not be convex. There are even such examples as $X = L_p([0,1], dt)$, $0 < p < 1$, where the convex hull of B_X is the whole space. In fact, the unit ball B_X of a quasi-normed space X needs not be p-convex for any $0 < p \leq 1$ as the following example shows (see another example in [7], page 95). The set

$$B = \big\{(x, y) \in \mathbb{R}^2 : (|y| - 1) \leq (\log |x| - 1)^{-1} \leq 0\big\} \cup \big\{(0, y) : -1 \leq y \leq 1\big\}$$

is not p-convex for any $0 < p \leq 1$ but it is the unit ball of the quasi-Banach space $(\mathbb{R}^2, \mu_B)$, being μ_B the Minkowski functional of B:

$$\mu_B(x) = \inf\{t > 0 : x \in tB\}, \quad x \in \mathbb{R}^2.$$

Nevertheless, a theorem by Aoki ([1]) and Rolewicz ([6]) asserts that every quasi-norm is equivalent to a quasi-norm which is p-subadditive for some $p > 0$ (see Theorem 1.3 of [3] and Theorem 3.2.1 of [7]).

Definition. We say that a quasi-normed space $(X, \|\,.\,\|)$ is *p-convex*, $0 < p \leq 1$, if the unit ball $B_X = \{x \in X,\ \|x\| \leq 1\}$ is a p-convex set, equivalently, if the quasi-norm is p-subadditive:

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for any $x, y \in X$.

The following is an obvious remark.

Proposition 2.1. *If a quasi-normed space $(X, \|\,.\,\|)$ is p-convex for some $0 < p \leq 1$, then $(X, \|\,.\,\|)$ is r-convex for any $0 < r \leq p$.*

Let us recall that a normed space $(X, \|\,.\,\|)$ is said to be *strictly convex* if

$$\|x + y\| < \|x\| + \|y\|$$

whenever $x, y \in X$ are not colinear.

Strict convexity in a normed space $(X, \|\,.\,\|)$ can be characterized equivalently in different forms: via the extreme points of B_X or according to the behaviour of the middle points of segments ending in the unit sphere S_X of the space.

The classical definitions of strict convexity and extreme point can be easily generalized for quasi-normed spaces:

Definition. We say that a quasi-normed space $(X, \|\,.\,\|)$ is *strictly p-convex*, $0 < p < 1$, if

$$\|x + y\|^p < \|x\|^p + \|y\|^p$$

for any $x, y \in X$ different from 0_X.

Definition. Let $0 < p \leq 1$. If C is a closed, p-convex subset of X, a point $a \in C$ is *p-extreme* if it does not belong to any open p-segment with ending points in C, i.e., if $a = \lambda x_1 + \mu x_2$ with $x_1, x_2 \in C$ and $0 \leq \lambda, \mu \leq 1$, $\lambda^p + \mu^p = 1$ implies that either $\lambda = 0$ or $\mu = 0$.

The set of p-extreme points of C is denoted by $\partial_p C$.

It is well-known that a normed space $(X, \|\,.\,\|)$ is strictly convex if and only if the set of extreme points of B_X coincides with the unit sphere S_X or, equivalently, if and only if $\left\|\frac{x+y}{2}\right\| < 1$ for any different x, y such that $\|x\| = \|y\| = 1$. We will state a similar theorem in the more general setting of quasi-normed spaces.

Proposition 2.2. *Let $0 < p < 1$.*

(i) *A quasi-normed space $(X, \|\,.\,\|)$ is strictly p-convex if and only if $\partial_p B_X = S_X$.*

(ii) *If a quasi-normed space $(X, \|\,.\,\|)$ is strictly p-convex then*

$$\left\| \frac{x+y}{2^{1/p}} \right\| < 1,$$

for any $x, y \in X$ such that $\|x\| = \|y\| = 1$. The converse is not true.

Proof. (i) It is obvious that if $(X, \|\,.\,\|)$ is strictly p-convex then $\partial_p B_X = S_X$. Reciprocally, if $(X, \|\,.\,\|)$ is not strictly p-convex then there exist $x, y \in X$ different from 0_X such that

$$\|x+y\|^p = \|x\|^p + \|y\|^p.$$

We may assume that $\|x\| = 1$ and $\|y\| = \varepsilon \leq 1$; therefore, $\|x+y\| = \alpha = (1+\varepsilon^p)^{1/p}$. Now, we observe that

$$\frac{x+y}{\alpha} = \frac{1}{\alpha}x + \frac{\varepsilon}{\alpha}\frac{y}{\varepsilon}, \quad \left(\frac{1}{\alpha}\right)^p + \left(\frac{\varepsilon}{\alpha}\right)^p = 1,$$

so $\frac{x+y}{\alpha} \in S_X \setminus \partial_p B_X$.

(ii) It is obvious. A counterexample of the converse for $0 < p < 1$ is the quasi-Banach space $(\mathbb{R}^2, \mu_B)$ whose unit ball is a set $B = B_p \cup 1/2B_\infty \subset \mathbb{R}^2$ formed by the union of the unit ball B_p of $\ell_p^{(2)}$ with the $\ell_\infty^{(2)}$-ball of radius $1/2$. Given any $x = (x_1, x_2)$, $y = (y_1, y_2) \in B$, we check that $\mu_B(x+y) < 2^{1/p}$: if $x \in B_p$ and $y \in 1/2B_\infty$ then $\mu_B(x+y) \leq (|x_1+y_1|^p + |x_2+y_2|^p)^{1/p} \leq ((1/2)^p + (3/2)^p)^{1/p} < 2^{1/p}$ for all $0 < p < 1$; if $x, y \in 1/2B_\infty$ then $\mu_B(x+y) \leq 2\max(|x_1+y_1|, |x_2+y_2|) \leq 2 < 2^{1/p}$ for all $0 < p < 1$; if $x, y \in B_p$ and $x_1y_1 \neq 0$ or $x_2y_2 \neq 0$ then $\mu_B(x+y) \leq (|x_1+y_1|^p + |x_2+y_2|^p)^{1/p} < 2^{1/p}$ (see Lemma 2.4); and if $x, y \in B_p$ and $x_1y_1 = x_2y_2 = 0$ then $\mu_B(x+y) \leq \max(\mu_B(1,1), \mu_B(2,0)) = 2 < 2^{1/p}$ for all $0 < p < 1$. In this space,

$$\mu_B\left(\frac{x+y}{2^{1/p}}\right) < 1$$

for any $x, y \in \mathbb{R}^2$ such that $\mu_B(x) = \mu_B(y) = 1$, but it cannot be strictly p-convex because its unit sphere has points which are not p-extreme. $\square$

Strict p-convexity is an intermediate stage between p-convexity and q-convexity for $0 < p < q \leq 1$.

Proposition 2.3. *If a quasi-normed space $(X, \|\,.\,\|)$ is q-convex for some $0 < q \leq 1$, then $(X, \|\,.\,\|)$ is strictly p-convex for any $0 < p < q$.*

Proof. It is a straightforward consequence of the following lemma. $\square$

Lemma 2.4. *If $0 < p < 1$, then*

$$|a + b|^p \leq |a|^p + |b|^p$$

for any real numbers a and b. Furthermore, $|a+b|^p = |a|^p + |b|^p$ if and only if $ab = 0$.

Next, we see that the converse of Proposition 2.3 is not true: there are strictly p-convex quasi-Banach spaces whose quasi-norm is not q-subbaditive for any $p < q \leq 1$. The example is provided by the Lorentz sequence spaces.

For every $0 < p \leq 1$ and every non-increasing sequence of positive numbers $\omega = (\omega_n)_{n=1}^{\infty}$ so that $\omega_1 = 1$ we consider the *Lorentz space* $d(\omega, p)$ of all sequences of scalars $a = (a_n)_{n=1}^{\infty}$ for which

$$\|a\| = \sup_{\pi \in \Pi} \left(\sum_{n=1}^{\infty} |a_{\pi(n)}|^p \omega_n \right)^{1/p} < \infty,$$

the supremum being taken over the set Π of all permutations of the integers. It is known that $(d(\omega, p), \|\,.\,\|)$ is a p-Banach space.

Theorem 2.5. *If $0 < p < 1$ then $(d(\omega, p), \|\,.\,\|)$ is not q-convex for any $q > p$.*

If ω decreases then $(d(\omega, p), \|\,.\,\|)$ is strictly p-convex.

Proof. First, we check that $d(\omega, p)$ is not q-convex for any given $p < q \leq 1$. Let us take $x = (1, 0, 0, \dots)$ and $y = (b, b, 0, 0, \dots)$ so that

$\|y\| = 1$ (i.e., $b = \frac{1}{(1+w_2)^p}$). The points of the q-segment $[x, y]_q$ with ending points x and y are given by

$$\begin{aligned}\lambda(b, b, 0, 0, \dots) + (1 - \lambda^q)^{1/q}(1, 0, 0, \dots)\\ = (\lambda b + (1 - \lambda^q)^{1/q}, b\lambda, 0, 0, \dots), \quad \lambda \in [0, 1].\end{aligned}$$

For the closed unit ball to be q-convex it must contain all the q-segments with ending points in it, in particular it must be verified that

$$\begin{aligned}\|(\lambda b + (1 - \lambda^q)^{1/q}, b\lambda, 0, 0, \dots)\|\\ = \big((b\lambda + (1 - \lambda^q)^{1/q})^p + w_2\big(b\lambda\big)^p\big)^{1/p} \leq 1\end{aligned}$$

for all $\lambda \in [0, 1]$. Let us see that for any $p < q \leq 1$, there exists $\lambda = \lambda(q) \in (0, 1)$ (closed enough to 0) such that $f(\lambda) > 1$. Call $\lambda^q = t$, and

$$f(t) = (bt^{1/q} + (1 - t)^{1/q})^p + w_2 b^p t^{p/q}, \quad t \in [0, 1].$$

Then

$$f'(t) = \frac{p}{q}\big(bt^{1/q} + (1-t)^{1/q}\big)^{p-1}\big(bt^{\frac{1}{q}-1} - (1-t)^{\frac{1}{q}-1}\big) + \frac{p}{q} w_2 b^p t^{\frac{p}{q}-1}, \quad t \in (0, 1).$$

As $\lim_{t\to 0^+} f'(t) = +\infty$, it follows that there exists $\delta = \delta(q) > 0$ so that $t \in (0, \delta)$ implies $f'(t) > 1 > 0$. Then f is increasing in $[0, \delta]$. Now, $f(0) = 1$, so $f(t) > 1$ for every $t \in [0, \delta]$, or equivalently, $\|\lambda(b, b, 0, 0, \dots) + (1 - \lambda^q)^{1/q}(1, 0, 0, \dots)\| > 1$ if $\lambda \in [0, \delta^{1/p}]$.

Now, we see that if ω decreases then $d(\omega, p)$ is strictly p-convex. Let $x = (x_n)_{n=1}^\infty$, $y = (y_n)_{n=1}^\infty$ be any two elements in $d(\omega, p)$, and let $(|x_{\sigma(k)} + y_{\sigma(k)}|)_{k=1}^\infty$ be a decreasing rearrangement of the sequence $(|x_k + y_k|)_{k=1}^\infty$. Then,

$$\begin{aligned}\|x + y\|^p = \sum_{k=1}^\infty |x_{\sigma(k)} + y_{\sigma(k)}|^p \omega_k \overset{(1)}{\leq} \sum_{k=1}^\infty |x_{\sigma(k)}|^p \omega_k\\ + \sum_{k=1}^\infty |y_{\sigma(k)}|^p \omega_k \overset{(2)}{\leq} \|x\|^p + \|y\|^p.\end{aligned}$$

The equality $\|x + y\|^p = \|x\|^p + \|y\|^p$ holds if and only if the inequalities (1), (2) are actually equalities. By Lemma 2.4, the inequality (1) is an equality if and only if $x_{\sigma(k)}y_{\sigma(k)} = 0$ for all $k \in \mathbb{N}$; and, on the other hand, the inequality (2) is an equality if and only if both sequences $(|x_{\sigma(k)}|)_{k=1}^{\infty}$, $(|y_{\sigma(k)}|)_{k=1}^{\infty}$ are non increasing. Therefore, $\|x+y\|^p = \|x\|^p + \|y\|^p$ if and only if either $x = 0$ or $y = 0$.

Although the space $d(w,p)$ is infinite dimensional, the above proof also shows that for any $0 < \omega_2 < 1$, the 2-dimensional Lorentz spaces $(d((1,\omega_2),p), \|\,.\,\|)$ are strictly p-convex but they are not q-convex for any $q > p$. $\square$

The following question arises: Can the space $d(w,p)$ be endowed with an equivalent q-subadditive quasi-norm, for some $q > p$? As we will see, the answer depends on the sequence $w = (w_n)_{n=1}^{\infty}$. In order to deal with one of the cases, we need a lemma from [5].

Lemma 2.6 ([5], cf. [2]). *Let $(e_n)_{n=1}^{\infty}$ be the canonical basis in $d(w,p)$, $0 < p < 1$, $w \in c_0 \setminus \ell_1$. Then every normalized block basis*

$$u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i e_i, \quad n = 1, 2, \dots$$

such that $\lim_{i\to\infty} a_i = 0$ contains a subsequence $(u_{n_j})_{j=1}^{\infty}$ such that for some constant $C > 0$

$$\Big\| \sum_{j=1}^{\infty} \lambda_j u_{n_j} \Big\| \geq C \Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{1/p}$$

for any finitely nonzero scalars $(\lambda_j)_{j=1}^{\infty}$.

Theorem 2.7. *Let $0 < p < 1$. The quasi-norm on $d(w,p)$ is equivalent to a q-subadditive quasi-norm for some $p < q \leq 1$ if and only if $\omega = (\omega_n)_{n=1}^{\infty} \in \ell_1$. In that case, $d(w,p) \simeq \ell_\infty$.*

Proof. If $\omega = (\omega_n)_{n=1}^{\infty} \in \ell_1$ then it is easy to prove that $d(w,p) \simeq \ell_\infty$, so as a matter of fact $d(w,p)$ has an equivalent norm, which obviously is q-convex for all $p < q \leq 1$. It remains to see what happens in the other cases.

Case 1: If $\inf_n \omega_n > 0$ then we have $d(w,p) \simeq \ell_p$, which does not have an equivalent q-subadditive quasi-norm for any $p < q \leq 1$.

Case 2: If $(w_n)_{n=1}^{\infty} \in c_0 \setminus \ell_1$ then, by Lemma 2.6, there is a sequence $(u_n)_{n=1}^{\infty} \subset d(w,p)$ of elements of quasi-norm 1, and a constant C such that

$$CN^{1/p} \le \Big\| \sum_{n=1}^{N} u_n \Big\|$$

for all $N \in \mathbb{N}$. If the space $d(w,p)$ had an equivalent q-subadditive quasi-norm then there would exist a constant D so that

$$C^q N^{q/p} \le \|u_1 + u_2 + \cdots + u_N\|^q \le D \sum_{n=1}^{N} \|u_n\|^q = DN$$

for all $N \in \mathbb{N}$, but the previous inequality cannot be true for all $N \in \mathbb{N}$ if $q > p$. □

3. Uniform p-convexity

The aim of this section is to find the most appropriate definition of uniform p-convexity when $0 < p < 1$ in the sense that it should coincide with the concept of uniform convexity for $p = 1$ and it should be an intermediate property between strict p-convexity and q-convexity for $q > p$.

A Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if for every $\varepsilon > 0$ there is a number $\delta = \delta(\varepsilon)$ such that

$$\inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| \ge \varepsilon\right\} = \delta > 0.$$

If $(X, \|\cdot\|)$ is a p-Banach space and $0 < p < 1$, then for any $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| < \varepsilon$ it holds that

$$1 - \left\|\frac{x+y}{2^{1/p}}\right\| \ge 1 - \frac{(2^p + \varepsilon^p)^{1/p}}{2^{1/p}} \xrightarrow{\varepsilon \to 0} 1 - 2^{(p-1)/p} > 0.$$

Therefore, the obvious generalization of uniform convexity is the following:

Definition. Let $0 < p < 1$. A quasi-Banach space $(X, \|\,.\,\|)$ has the *p-midpoint property* if

$$\inf\left\{1 - \left\|\frac{x+y}{2^{1/p}}\right\| : \|x\| = \|y\| = 1\right\} = \delta > 0.$$

The reason for not calling the above property "uniform p-convexity" is that it does not imply strict p-convexity:

Theorem 3.1. *For every $0 < p < 1$, there are quasi-normed spaces that have the p-midpoint property but fail to be strictly p-convex, and there are strictly p-convex quasi-normed spaces that fail to have the p-midpoint property.*

PROOF. The counterexample in Proposition 2.2 is not strictly p-convex and satisfies that $\delta(x,y) = 1 - \mu_B\left(\frac{x+y}{2^{1/p}}\right) > 0$ for all $x, y \in \mathbb{R}^2$ such that $\mu_B(x) = \mu_B(y) = 1$. Since the unit sphere of $(\mathbb{R}^2, \mu_B)$ is compact and $\delta(x,y)$ is a continuous function, $(\mathbb{R}^2, \mu_B)$ is also an example of quasi-Banach space with the p-midpoint property that is not strictly p-convex.

A strictly p-convex quasi-Banach space which does not have the p-midpoint property is the space $\ell_1(\ell_{p_n}^n)$ where $(p_n)_{n=1}^{\infty}$ is a decreasing sequence of numbers converging to p. $\square$

In [8], ROLEWICZ gives a characterization of uniform convexity using the so-called "drops". The concept of drop has been generalized to p-Banach spaces in [4].

Definition ([4]). If C is a closed p-convex subset of a p-Banach space $(X, \|\,.\,\|)$ $(0 < p \leq 1)$, and $x \in X \setminus C$, the p-convex hull of $\{x\} \cup C$ is called *p-drop* and denoted by

$$\begin{aligned} D_p(x, C) &= p - \operatorname{co}(\{x\} \cup C) \\ &= \{sx + ty : y \in C,\ s, t \in [0,1],\ s^p + t^p = 1\}. \end{aligned}$$

If B_X is the closed unit ball of $(X, \|\,.\,\|)$ and $x \in X \setminus B_X$, we denote

$$R_p(x) = D_p(x, B_X) \setminus B_X.$$

Theorem 3.2 ([8]). *A Banach space $(X, \|\,.\,\|)$ is uniformly convex if and only if* $\operatorname{diam} R_1(a) \xrightarrow{\|a\| \to 1} 0$ *uniformly on $\|a\|$.*

Rolewicz' theorem asserts that uniform convexity can be characterized not only from the inside of the unit ball but also by some uniform behaviour measured from the outside of the ball, via the drops. That suggests the following definition:

Definition. Let $0 < p \leq 1$. A quasi-Banach space $(X, \|\cdot\|)$ is *uniformly p-convex* if $\operatorname{diam} R_p(a) \xrightarrow{\|a\|\to 1} 0$ uniformly on $\|a\|$.

Next, we check that uniform p-convexity is an intermediate property between strict p-convexity and q-convexity for $q > p$ as well as its relation with the p-midpoint property.

Theorem 3.3. *If $(X, \|\,.\,\|)$ is q-convex quasi-Banach space for some $p < q \leq 1$ then $(X, \|\,.\,\|)$ is uniformly p-convex. The converse is not true.*

Proof. For each $a \in X$ with $\|a\| > 1$, if $x_1, x_2 \in R_p(a)$ then

$$x_1 = (1 - s^p)^{1/p} a + s y_1, \qquad x_2 = (1 - t^p)^{1/p} a + s y_2,$$

for some $y_1, y_2 \in B_X$, $s, t \in [0, 1]$. Since

$$1 \leq \|(1 - s^p)^{1/p} a + s y_1\|^q \leq (1 - s^p)^{q/p} \|a\|^q + s^q$$

hence

$$1 \leq \frac{1 - s^q}{(1 - s^p)^{q/p}} \leq \|a\|^q.$$

It follows that $s \to 0$ when $\|a\| \to 1$. Analogously, $t \to 0$ when $\|a\| \to 1$. Therefore,

$$\begin{aligned}\|x_1 - x_2\|^q &= \|(1 - s^p)^{1/p} a + s y_1 - (1 - t^p)^{1/p} a - s y_2\|^q \\ &\leq \left|(1 - s^p)^{1/p} - (1 - t^p)^{1/p}\right|^q \|a\|^q + s^q + t^q \xrightarrow{\|a\|\to 1} 0.\end{aligned}$$

The 2-dimensional Lorentz spaces $d(\omega, p)$ with $\omega = (1, \omega_2)$, $0 < \omega_2 < 1$, are a counterexample of the converse. □

Theorem 3.4. *If $0 < p < 1$ and $(X, \|\,.\,\|)$ is uniformly p-convex then it is strictly p-convex. The converse is not true.*

Proof. First, we prove that $(X, \|\,.\,\|)$ is p-convex. Suppose it is not and take $a \in X$ with $\|a\| > 1$ such that $a \in p - \operatorname{co}(B_X)$. Then $-\lambda a, a \in R_p(-\lambda a)$ for all $\lambda > \|a\|^{-1}$ and $\operatorname{diam} R_p(-\lambda a) \geq \|\lambda a + a\| = (\lambda + 1)\|a\| > 1$ for all $\lambda > \|a\|^{-1}$.

Now, suppose that $(X, \|\,.\,\|)$ is not strictly p-convex. Then, there is a non p-extreme point in S_X, i.e. there exist $0 < \beta < \alpha < 1$, and $x, y \in S_X$ such that $\alpha^p + \beta^p = 1$ and $\|\alpha x + \beta y\| = 1$. Given $\varepsilon > 0$, denote

$$C = \left(\frac{\alpha^p}{(1 + \varepsilon)^p} + \beta^p\right)^{-1/p}, \quad \lambda = \frac{\alpha C}{1 + \varepsilon}, \quad \mu = (1 - \lambda^p)^{1/p} = \beta C.$$

These numbers satify that $\lambda^p + \mu^p = 1$ and, since $1 < C < 1 + \varepsilon$,

$$\frac{\alpha}{1+\varepsilon} < \lambda < \alpha, \qquad \beta < \mu < (1+\varepsilon)\beta.$$

Since $\|\lambda(1+\varepsilon)x+\mu y\| = C\|\alpha x+\beta y\| = C > 1$, we have that $\lambda(1+\varepsilon)x+\mu y \in R_p((1+\varepsilon)x)$, and

$$\big(\operatorname{diam} R_p((1+\varepsilon)x)\big)^p \geq \|\lambda(1+\varepsilon)x + \mu y - (1+\varepsilon)x\|^p$$
$$\geq (1-\lambda)^p(1+\varepsilon)^p - (1-\lambda^p) \xrightarrow{\varepsilon\to 0} (1-\alpha)^p - 1 + \alpha^p > 0.$$

A strictly p-convex quasi-Banach space which is not uniformly p-convex is the space $(\ell_1(\ell_{p_n}^n), \|.\|)$ where $(p_n)_{n=1}^\infty$ is a decreasing sequence of numbers converging to p. This space does not have the p-midpoint property which is a necessary condition for uniform p-convexity, as we see in the next theorem. □

Theorem 3.5. *If $0 < p < 1$ and $(X, \|\,.\,\|)$ is uniformly p-convex then it has the p-midpoint property. The converse is not true.*

Proof. For any $x, y \in S_X$, and any $\varepsilon > 0$,

$$\frac{x+y}{2^{1/p}} = \alpha_\varepsilon(\lambda_\varepsilon(1+\varepsilon)x + \mu_\varepsilon y)$$

with

$$\lambda_\varepsilon = \frac{1}{\alpha_\varepsilon 2^{1/p}(1+\varepsilon)}, \quad \mu_\varepsilon = \frac{1}{\alpha_\varepsilon 2^{1/p}}, \quad \alpha_\varepsilon = \left(\frac{(1+\varepsilon)^p+1}{2(1+\varepsilon)^p}\right)^{1/p}.$$

These numbers satisfy the following

$$\frac{1}{1+\varepsilon} < \alpha_\varepsilon < 1, \quad \frac{1}{2^{1/p}(1+\varepsilon)} < \lambda_\varepsilon < \frac{1}{2^{1/p}} < \mu_\varepsilon < \frac{1+\varepsilon}{2^{1/p}}, \quad \lambda_\varepsilon^p + \mu_\varepsilon^p = 1.$$

Therefore, $\lambda_\varepsilon(1+\varepsilon)x + \mu_\varepsilon y \in D_p((1+\varepsilon)x, B_X)$; and since

$$\|\lambda_\varepsilon(1+\varepsilon)x + \mu_\varepsilon y - (1+\varepsilon)x\|^p = \|(1-\lambda_\varepsilon)(1+\varepsilon)x - \mu_\varepsilon y\|^p$$
$$\geq (1-\lambda_\varepsilon)^p(1+\varepsilon)^p - \mu_\varepsilon^p$$
$$> \left(1 - \frac{1}{2^{1/p}}\right)^p (1+\varepsilon)^p - \frac{(1+\varepsilon)^p}{2} > \left(1 - \frac{1}{2^{1/p}}\right)^p - \frac{1}{2}$$

for all $\varepsilon > 0$, there must exist $\varepsilon_0 > 0$ such that $\|\lambda_\varepsilon(1+\varepsilon)x + \mu_\varepsilon y\| \leq 1$ for all $\varepsilon \leq \varepsilon_0$. Otherwise, $\operatorname{diam} R_p((1+\varepsilon)x) \overset{\varepsilon\to 0}{\nrightarrow} 0$. Then,

$$\left\|\frac{x+y}{2^{1/p}}\right\| = \alpha_{\varepsilon_0}\|\lambda_{\varepsilon_0}(1+\varepsilon_0)x + \mu_{\varepsilon_0}y\| \leq \alpha_{\varepsilon_0} < 1.$$

The counterexample in Proposition 2.2 is an example of quasi-Banach space with the p-midpoint property that is not strictly p-convex and, therefore, it is not uniformly p-convex. □

Finally, we see if spaces with the p-properties that we have studied can be renormed with an equivalent q-subadditive quasi-norm for some $p < q \leq 1$.

Theorem 3.6. *If $0 < p < 1$ and $(X, \|\,.\,\|)$ has the p-midpoint property then there exist $p < q \leq 1$ and a q-subadditive quasi-norm $\|\,.\,\|_q$ on X which is equivalent to $\|\,.\,\|$. In particular, uniformly p-convex quasi-Banach spaces can be renormed with an equivalent q-subadditive quasi-norm for some $p < q \leq 1$. The converse is not true.*

Proof. Let

$$\delta = \inf\left\{1 - \left\|\frac{x+y}{2^{1/p}}\right\| : \|x\| = \|y\| = 1\right\} > 0,$$

and fix $0 < \varepsilon < p\delta$. If $\|x\| = 1$ and $\|y\| \leq 1 - 2\varepsilon$, then

$$\left\|\frac{x+y}{2^{1/p}}\right\|^p \leq \frac{1 + (1-2\varepsilon)^p}{2} < \frac{2 - 2p\varepsilon}{2} = 1 - p\varepsilon.$$

If $\|x\| = 1$ and $1 - 2\varepsilon < \|y\| \leq 1$, then

$$\left\|\frac{x+y}{2^{1/p}}\right\|^p = \left\|\frac{x + \frac{y}{\|y\|}}{2^{1/p}} - \frac{y}{\|y\|}\frac{1-\|y\|}{2^{1/p}}\right\|^p \leq (1-\delta)^p + \varepsilon < 1 - p\delta + \varepsilon.$$

We have proved that there exists $0 < \gamma < 1$ such that

$$\|x+y\| \leq 2^{1/p}\gamma \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.

Now, take $p < q \le 1$ such that $2^{1/p}\gamma = 2^{1/q}$, by the theorem of AOKI and ROLEWICZ's (see Theorem 1.3 of [3] and Theorem 3.2.1 of [7]),

$$\|x\|_q = \inf\left\{ \Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} : \sum_{i=1}^{n} x_i = x,\, n \in \mathbb{N} \right\}$$

defines a q-subadditive quasi-norm on X which is equivalent to $\|\,.\,\|$.

Any finite dimensional ℓ_p space with $0 < p < 1$ is a counterexample of the converse. □

Theorem 3.7. *For every $0 < p < 1$, there are quasi-normed spaces that can be renormed with an equivalent norm (or 1-subadditive quasi-norm) but fail to be strictly p-convex, and there are strictly p-convex quasi-normed spaces that cannot be renormed with an equivalent q-subadditive quasi-norm for any $p < q \le 1$.*

PROOF. Any finite dimensional quasi-normed space can be renormed with an equivalent norm.

The space $(\ell_1(\ell_{p_n}^n), \|.\|)$, where $(p_n)_{n=1}^{\infty}$ is a decreasing sequence of numbers converging to p, is strictly p-convex but cannot be renormed with an equivalent q-subadditive quasi-norm for any $p < q \le 1$. □

References

[1] T. AOKI, Locally bounded linear topological spaces, *Proc. Imp. Acad. Tokyo* **18** (1942), 588–594.

[2] Z. ALTSHULER, P. CASAZZA and B. L. LIN, On symmetric basic sequences in Lorentz sequence spaces, *Israel J. Math.* **15** (1973), 140–155.

[3] N. J. KALTON, N. T. PECK and W. ROBERTS, An F-space sampler, London Math. Soc. Lecture Note Series 89, *Cambridge University Press*, 1984.

[4] D. N. KUTZAROVA and C. LERÁNOZ, On p-Drop Property, $0 < p < 1$, *Atti Sem. Fis. Univ. Modena* **XLII** (1994), 89–102.

[5] N. POPA, Basic sequences and subspaces in Lorentz spaces without local convexity, *Trans. Amer. Math. Soc.* **263** (1981), 431–456.

[6] S. ROLEWICZ, On a certain class of linear metric spaces, *Bull. Acad. Polon. Sci.* **5** (1957), 471–473.

[7] S. Rolewicz, Metric Linear Spaces, *D. Reidel Publishing Co. and PWN-Polish Scientific Publishers*, *Dordrecht–Warszawa*, 1985.

[8] S. Rolewicz, On drop property, *Studia Math.* **85** (1987), 27–35.

FERNANDO ALBIAC
DEPTO. DE MATEMÁTICA E INFORMÁTICA
UNIVERSIDAD PÚBLICA DE NAVARRA
31006 PAMPLONA
SPAIN
AND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI–COLUMBIA
COLUMBIA, MISSOURI 65211
USA

E-mail: albiac@math.missouri.edu

CAMINO LERÁNOZ
DEPTO. DE MATEMÁTICA E INFORMÁTICA
UNIVERSIDAD PÚBLICA DE NAVARRA
31006 PAMPLONA
SPAIN

E-mail: camino@unavarra.es

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