

Coincidence of correspondences in Kähler–Finsler manifolds

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Abstract. In this paper we present some results on coincidence of correspondences and fixed point properties in Kähler–Finsler manifolds. We follow the method of FRANKEL [6], which was generalized for the Finsler case by the author and L. KOZMA [8].

1. Introduction

Holomorphic correspondences are generalizations of holomorphic mappings as multivalued maps of a complex manifold ([4], [7]). Fixed points of correspondences of complex Kähler manifolds have been studied by T. FRANKEL [6]. He proved that for a Kähler manifold of positive sectional curvature a correspondence always has a fixed point (i.e. it intersects the diagonal of $N \times N$). The method of its proof, based upon the second variation formula of geodesics, proved effective in different situations ([1], [6]). L. Kozma and the present author generalized Frankel’s results on intersections of submanifolds for the case of Finsler manifolds [8]. In this paper the result of Frankel concerning correspondences are extended to the Finslerian case. We mention that we deduce results on coincidence of correspondences, while Frankel’s theorem refers only to fixed points of a correspondence. The proof follows the line of the original version of Frankel, however, at some points more elaborated arguments are needed due to the Finslerian context. Though S. S. CHERN says [5] that Finsler geometry is more natural than Riemannian geometry as a concept, the

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computational part of the subject requires much more effort. For the recent flourishing literature of Finsler geometry see [2], [3], [10], [11]. Our basic reference in this paper is [1].

2. Preliminaries

We recall some facts about Kähler–Finsler manifolds (see [1]).

Let M be a complex manifold of complex dimension n . The complexification $T_{\mathbb{C}}M$ of the real tangent bundle is decomposed as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ is the holomorphic tangent bundle over M and $T^{0,1}M$ is the conjugate of $T^{1,0}M$. $T^{1,0}M$ is also a complex manifold of $\dim_{\mathbb{C}} T^{1,0}M = n$. $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces of the complex structure J belonging to the eigenvalues i and $-i$, respectively.

A complex Finsler metric on a complex manifold is a continuous function $F : T^{1,0}M \rightarrow \mathbb{R}$ satisfying

- i) $G := F^2$ is smooth on $\widetilde{M} = TM \setminus \{\text{zero section}\}$,
- ii) $F(v) > 0$, $\forall v \in \widetilde{M}$,
- iii) $F(\mu_{\xi}(v)) = |\xi|F(v)$ for all $v \in T^{1,0}M$ and $\xi \in \mathbb{C}$.

Recall that $\mu_{\xi} : T^{1,0}M \rightarrow T^{1,0}M$ is given by $\mu_{\xi}(p, v) = (p, \xi v)$, $\forall (p, v) \in T^{1,0}M$ and $\xi \in \mathbb{C}$. F is called strongly pseudoconvex if the Levi matrix $(G_{\alpha\bar{\beta}})$ is positive definite on \widetilde{M} , where

$$G_{\alpha\bar{\beta}} = \frac{\partial G^2}{\partial v^{\alpha} \partial \bar{v}^{\beta}}.$$

The complex vertical bundle is

$$\mathcal{V}_{\mathbb{C}} = \ker d\pi \subset T_{\mathbb{C}}\widetilde{M}.$$

There is a canonical isomorphism $\iota_v : T_{\pi(v)}^{1,0} \rightarrow \mathcal{V}_v$. The complex radial vertical vector field $\iota : \widetilde{M} \rightarrow \mathcal{V}$ is defined by $\iota(v) = \iota_v(v) \forall v \in T^{1,0}\widetilde{M}$. The projection $d\pi$ commutes with J . It follows that we have the splitting $\mathcal{V}_{\mathbb{C}} = \mathcal{V}^{1,0} + \mathcal{V}^{0,1}$. The complex vertical bundle is $\mathcal{V} = \mathcal{V}^{1,0} = \ker d\pi \subset T^{1,0}\widetilde{M}$. The complex horizontal bundle is a complex subbundle of $T_{\mathbb{C}}\widetilde{M}$ which is

a direct summand of \mathcal{V} and is J -invariant. We have also the splitting $\mathcal{H}_{\mathbb{C}} = \mathcal{H}^{1,0} + \mathcal{H}^{0,1}$, and the complex horizontal bundle is $\mathcal{H} = \mathcal{H}^{1,0}$.

The complex horizontal map is a complex bundle map $\Theta : \mathcal{V}_{\mathbb{C}} \rightarrow T_{\mathbb{C}}$ which commutes with J and the conjugation and which satisfies the relation $(d\pi \circ \Theta)_v|_{\mathcal{V}^{1,0}} = \iota_v^{-1}|_{\mathcal{V}^{1,0}}$. The complex radial (Liouville) horizontal vector field is given by $\chi = \Theta \circ \iota$.

Then there exists a unique good vertical connection which makes the Hermitian structure $(G_{\alpha\bar{\beta}})$ in the vertical bundle \mathcal{V} parallel. It can be extended via the horizontal map to a complex linear connection on \widetilde{M} . This is called the complex Chern–Finsler connection ∇ .

The geodesics σ are characterized by the equation

$$\nabla_{T^H + \bar{T}^H} T^H = 0$$

where T^H is the horizontal lifting of the tangent vector $T = \dot{\sigma}$, and \bar{T} means the conjugate of T .

A vector field U along a curve σ is said to be parallel along σ iff

$$\nabla_{T^H + \bar{T}^H} U^H = 0.$$

The torsions θ , and τ of ∇ are defined as follows

$$\begin{aligned} \theta(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \mathfrak{X}(T^{1,0}\widetilde{M}) \\ \tau(X, \bar{Y}) &= \nabla_X \bar{Y} - \nabla_{\bar{Y}} X - [X, \bar{Y}], \quad \forall X, Y \in \mathfrak{X}(T^{1,0}\widetilde{M}). \end{aligned}$$

The curvature Ω is defined as usual. The holomorphic bisectional curvature is given as follows [1]:

$$R(T, U) = \langle \Omega(T^H + \bar{T}^H, U^H + \bar{U}^H) U^H T^H \rangle, \quad \forall T, U \in T^{1,0}M.$$

In the case of Chern–Finsler connection this takes the form

$$R(T, U) = \langle \Omega(T^H, \bar{U}^H) U^H, T^H \rangle - \langle \Omega(U^H, \bar{T}^H) U^H, T^H \rangle.$$

A strongly pseudoconvex Finsler metric F is called Kähler if its $(2, 0)$ torsion θ satisfies $\theta(H, \chi) = 0$, $\forall H \in \mathcal{H}$, where χ denotes the horizontal Liouville vector field and it is called strongly Kähler if its $(2, 0)$ torsion satisfies $\theta(H, K) = 0$, $\forall H, K \in \mathcal{H}$.

The horizontal (1, 1) torsion is defined by

$$\tau^{\mathcal{H}}(X, \bar{Y}) = \Theta(\tau(X, \bar{Y}))$$

where Θ is the horizontal map. The symmetric product $\langle\langle \cdot, \cdot \rangle\rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is locally given by

$$\langle\langle H, K \rangle\rangle_v = G_{\alpha\beta}(v)H^\alpha H^\beta, \quad \forall H, K \in \mathcal{H}_v, \quad v \in \widetilde{M}.$$

In the following the second variation formula will play a crucial role (see [1]).

Consider $F : T^{1,0}M \rightarrow \mathbb{R}$ be a Kähler Finsler metric on a complex manifold M . Take a geodesic $\sigma_0 : [a, b] \rightarrow M$ with $F(\dot{\sigma}) = 1$ and a regular variation $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ of σ_0 . Then it is known [1, p. 103],

$$\begin{aligned} \frac{d^2 l_\Sigma}{ds^2}(0) &= \operatorname{Re} \langle \nabla_{U^H + \bar{U}^H} U^H, T^H \rangle_{\dot{\sigma}} \Big|_a^b \\ &+ \int_a^b \left\{ \left\| \nabla_{T^H + \bar{T}^H} U^H \right\|_{\dot{\sigma}}^2 - \left| \frac{\partial}{\partial t} \operatorname{Re} \langle U^H, T^H \rangle_{\dot{\sigma}} \right|^2 \right. \\ &- \operatorname{Re} \left[\langle \Omega(T^H, \bar{U}^H) U^H, T^H \rangle_{\dot{\sigma}} - \langle \Omega(U^H, \bar{T}^H) U^H, T^H \rangle_{\dot{\sigma}} \right. \\ &\left. \left. + \langle \langle \tau^{\mathcal{H}}(U^H, \bar{T}^H), U^H \rangle \rangle_{\dot{\sigma}} - \langle \langle \tau^{\mathcal{H}}(T^H, \bar{U}^H), U^H \rangle \rangle_{\dot{\sigma}} \right] \right\} dt. \end{aligned}$$

3. Product of Kähler Finsler manifolds

In this section we construct the product of strongly Kähler Finsler manifolds.

Let $(M_1, F_1), (M_2, F_2)$ be two strongly Kähler Finsler manifolds with the Chern–Finsler connection. Consider the product manifold $M_1 \times M_2$ with the metric

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + F_2^2(v_2)} \quad \forall (v_1, v_2) \in TM_1 \times TM_2.$$

This is homogeneous, smooth and positive definite on $\widetilde{M}_1 \times \widetilde{M}_2$ because F_1, F_2 have these properties on $\widetilde{M}_1, \widetilde{M}_2$. The Levi matrix of F is positive definite on $\widetilde{M}_1 \times \widetilde{M}_2$ because it is of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A, B are the Levi matrix of F_1, F_2 .

Let $\mathcal{H}_1, \mathcal{H}_2$ be the horizontal bundle of the manifolds $(M_1, F_1), (M_2, F_2)$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

The metrics F_1, F_2 induce the Hermitian structures $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on the horizontal bundles. It follows that on the bundle $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ we have the Hermitian metric

$$\langle X + U, Y + V \rangle = \langle X, Y \rangle_1 + \langle U, V \rangle_2.$$

The Chern–Finsler connection of the product manifold is related to the Chern–Finsler connections of M_1 and M_2 as follows:

$$\nabla_{X+U}(Y + V) = \nabla_X Y + \nabla_U V, \quad \forall X, Y \in \mathfrak{X}(\mathcal{H}_1), \quad U, V \in \mathfrak{X}(\mathcal{H}_2).$$

From these relation follows that the product manifold is strongly Kähler if the manifolds M_1 and M_2 are. The holomorphic bisectional curvature of $M_1 \times M_2$ satisfies the relation:

$$R(X + U, Y + V) = R(X, Y) + R(U, V) \\ \forall X, Y \in T^{1,0}M_1, \quad \text{and} \quad U, V \in T^{1,0}M_2.$$

We have the isomorphism $o : T_{\mathbb{R}}M_1 \rightarrow T^{1,0}M_1$

$$\forall u \in T_{\mathbb{R}}M_1 \quad u_o = \frac{1}{2}(u - iJu).$$

Using the above isomorphism we can associate to F a function $F^o : T_{\mathbb{R}}M_1 \rightarrow \mathbb{R}^+$ by setting

$$\forall u \in T_{\mathbb{R}}M_1 \quad F^o(u) = F(u_o).$$

It is shown in [1, p. 114] that the geodesics of F and F^o are the same if F is Kähler.

Applying these facts we show that if $\sigma = (\alpha, \beta)$ is a geodesic for F , then α and β are geodesic for F_1 and F_2 , resp. In fact, α is also a geodesic for F^o , therefore, applying our result about geodesic on real warped product in [9] for $f \equiv 1$, α and β are geodesic for F_1^o and F_2^o , resp. It follows by [1, p. 114] again that α and β are geodesics for F_1 and F_2 resp.

4. The main result

A *holomorphic correspondence* of a complex manifold N of dimension n with itself is an n -dimensional complex analytic submanifold of $N \times N$. Two (holomorphic) correspondences V, W are said to have a *coincidence* iff $V \cap W \neq \emptyset$. A holomorphic correspondence $V \subset N \times N$ is called *transversal* if $T_{(p,q)}V \oplus T_{(p,q)}(\{p\} \times N) = T_{(p,q)}(N \times N)$ and $T_{(p,q)}V \oplus T_{(p,q)}(N \times \{q\}) = T_{(p,q)}(N \times N)$ hold for all $(p, q) \in V$. Since $T_{(p,q)}(\{p\} \times N)$ and $T_{(p,q)}(N \times \{q\})$ are orthogonal, it follows that any vector orthogonal to V at (p, q) cannot be tangent to $\{p\} \times N$ or $N \times \{q\}$.

A holomorphic map $f : N \rightarrow N$ gives rise to a correspondence, the graph $G(f)$ of f ; $G(f) = \{(p, f(p)) \mid p \in N\}$. $G(f)$ is a special type of correspondence since f is single valued. Let $\Delta = \{(p, p) \mid p \in N\}$ be the diagonal of $N \times N$. It is clear that a map f has a fixed point whenever $G(f)$ intersects the diagonal Δ . A correspondence will be said to have a fixed point if it intersects the diagonal.

The main result is the following

Theorem 1. *Two holomorphic compact correspondences V, W – one of them is transversal – of a connected strongly Kähler Finsler manifold N with positive holomorphic bisectional curvature have a coincidence.*

PROOF. The correspondences are complex analytic submanifolds V, W of $N \times N$. On the product manifold $N \times N$ we consider the metric $F : T^{1,0}N \times T^{1,0}N \rightarrow \mathbb{R}^+$ given by

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + F_1^2(v_2)} \quad \text{for } (v_1, v_2) \in T^{1,0}N \times T^{1,0}N.$$

We use the notations used in [1] and [8]. We take $M = N \times N$ and V, W are submanifolds of M .

We need only to show that V and W intersect. Suppose $V \cap W = \emptyset$. Then there exists a minimal geodesic $\sigma : [a, b] \rightarrow M$. Let $\sigma(a) \in V$, $\sigma(b) \in W$. σ is orthogonal to V and W in $\sigma(a)$ and $\sigma(b)$, respectively i.e. $\dot{\sigma}^H(a) \perp T_{\sigma(a)}^{(1,0)H}V$ and $\dot{\sigma}^H(b) \perp T_{\sigma(b)}^{(1,0)H}W$. According to the last argument of the previous section the geodesic has the form $\sigma = (\alpha, \beta) \in N \times N$ where both α and β geodesics. By the assumption of transversality of V or W we have $\dot{\alpha} \neq 0$ and $\dot{\beta} \neq 0$. Then it follows that F is smooth along σ .

We construct a regular variation $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ such that $\nabla_{T^H + \bar{T}^H} U^H = 0$. Denoting by $H_{\dot{\sigma}(b)} T^{1,0} M$ the horizontal lift of $T^{1,0} M$ to horizontal space in $\dot{\sigma}(b)$, let $P \subset H_{\dot{\sigma}(b)} T^{1,0} M$ be the parallel translation of $T_{\sigma(a)}^H(V)$ with respect to the Chern–Finsler connection along $\dot{\sigma}$ to the point $\dot{\sigma}(b)$. Considering the horizontal lifts to \widetilde{M} along $\dot{\sigma}$ we get

$$\begin{aligned} \dim_{\mathbb{C}}(P \cap T_{\sigma(b)}^H(W)) \\ = \dim_{\mathbb{C}} P + \dim_{\mathbb{C}}(T_{\sigma(b)}^H W) - \dim_{\mathbb{C}}(P + T_{\sigma(b)}^H W) \geq 1, \end{aligned}$$

for $\dim_{\mathbb{C}}(P + T_{\sigma(b)}^H W) \leq 2 \dim_{\mathbb{C}} N - 1$.

So we can choose a vector U^H in the intersection. Its parallel translation along $\dot{\sigma}$ will be denoted by U^H , too. Since U^H is orthogonal to $\dot{\sigma}$ at the end point, it remains orthogonal along the entire curve by the metrical property of the Chern–Finsler connection. We consider the regular variation of σ with transversal vector field U .

In this case the second variation formula reduces to the following form

$$\begin{aligned} \frac{d^2 l_{\Sigma}}{ds^2}(0) &= \operatorname{Re} \langle \nabla_{U^H + \bar{U}^H} U^H, T^H \rangle_{\dot{\sigma}} \Big|_a^b \\ &+ \int_a^b \left\{ \|\nabla_{T^H + \bar{T}^H} U^H\|_{\dot{\sigma}}^2 - \left| \frac{\partial}{\partial t} \operatorname{Re} \langle U^H, T^H \rangle_{\dot{\sigma}} \right|^2 - \operatorname{Re}[R_{\dot{\sigma}}(T, U)] \right\} dt \end{aligned}$$

because of Proposition 2.6.7 in [1, p. 120].

The first term of the integral is zero for U is parallel along $\dot{\sigma}$. Furthermore, U^H and T^H are orthogonal.

By the hypothesis on the holomorphic sectional curvature all the terms will be negative or zero except the first one at most.

In fact we have

$$\frac{d^2 l_{\Sigma}}{ds^2}(0) = \operatorname{Re} \langle \nabla_{U^H + \bar{U}^H} U^H, T^H \rangle_{\dot{\sigma}} \Big|_a^b - \int_a^b \operatorname{Re}[R_{\dot{\sigma}}(T, U)] dt.$$

The integral is positive because $R_{\dot{\sigma}}(T, U) = R_{\dot{\sigma}}(T_1, U_1) + R_{\dot{\sigma}}(T_2, U_2)$ where $T_1 = \dot{\alpha} \neq 0$ and $T_2 = \dot{\beta} \neq 0$ and U_1, U_2 are orthogonal to T_1, T_2 resp.

By the minimality of σ it follows that $\frac{d^2 l_{\Sigma}}{ds^2}(0) \geq 0$ for any transversal vector field U .

If we consider the variation belonging to the transversal vector JU^H , we show that the initial terms in the second variation cannot be positive in the same time (for the variations corresponding to U^H and JU^H respectively). This will give the contradiction.

Therefore we calculate $\nabla_{JU^H + \overline{JU^H}} JU^H$.

$$\nabla_{JU^H + \overline{JU^H}} JU^H = J(\nabla_{JU^H} U^H + \nabla_{\overline{JU^H}} U^H).$$

Using the torsion we have

$$\nabla_{JU^H} U^H = \nabla_{U^H} JU^H + [JU^H, U^H] + \theta(JU^H, U^H).$$

The last term $\theta(JU^H, U^H)$ vanishes because F is strongly Kähler Finsler metric and using again the Proposition 2.6.7 in [1, p. 120] it follows:

$$\begin{aligned} \nabla_{\overline{JU^H}} U^H &= \nabla_{U^H} \overline{JU^H} - [U^H, \overline{JU^H}] \\ &= J[\nabla_{\overline{U^H}} U^H + [U^H, \overline{U^H}]] - [U^H, \overline{JU^H}] \\ &= J\nabla_{U^H} U^H + J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}]. \end{aligned}$$

It follows now

$$\begin{aligned} \nabla_{JU^H + \overline{JU^H}} JU^H &= J(\nabla_{U^H} JU^H + [JU^H, U^H] + J\nabla_{\overline{U^H}} U^H \\ &\quad + J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}]) = -\nabla_{U^H + \overline{U^H}} U^H \\ &\quad + J[JU^H, U^H] - J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}]. \end{aligned}$$

Now V and W are complex manifolds, U^H is a horizontal lift, and tangent to \tilde{V} and \tilde{W} in $\dot{\sigma}(a)$ and $\dot{\sigma}(b)$ respectively. Since the horizontal space is a complex linear space, and we use the Chern–Finsler connection, all the brackets above are horizontal vectors, and are orthogonal to T^H in $\dot{\sigma}(a)$ and $\dot{\sigma}(b)$. So

$$\operatorname{Re}\langle \nabla_{JU^H + \overline{JU^H}} JU^H, T^H \rangle = -\operatorname{Re}\langle \nabla_{U^H + \overline{U^H}} U^H, T^H \rangle.$$

This means that $\frac{d^2 l_{\Sigma}}{ds^2}(0)$ cannot be non-negative for U and JU at the same time, which gives the contradiction. \square

5. Coincidence of mappings in Kähler Finsler manifolds

In this section we derive a theorem about coincidence of mappings in Kähler Finsler manifolds of positive holomorphic bisectional curvature. Let $f, g : N \rightarrow N$ be two holomorphic maps. We say that f and g have *coincidence* if $G(f) \cap G(g) \neq \emptyset$. A map $f : N \rightarrow N$ is called *biholomorphic* if f is a holomorphic diffeomorphism.

Theorem 2. *Let N be a compact strongly Kähler Finsler manifold of positive holomorphic bisectional curvature and $f, g : N \rightarrow N$ biholomorphic maps. There exists at least one point $p \in N$ such that $f(p) = g(p)$.*

PROOF. We consider the manifolds $V = G(f)$ and $W = G(g)$ resp. Both of these manifolds have complex dimension n , equal to the complex dimension of the manifold N . We apply now Theorem 1 for V, W and N and we obtain that $G(f) \cap G(g) \neq \emptyset$. \square

Corollary 3. *Let N be a compact strongly Kähler Finsler manifold of positive holomorphic bisectional curvature and $f : N \rightarrow N$ a holomorphic map. The map f has at least one fixed point.*

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