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Common fixed point theorems for single-valued and multi-valued mappings

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Abstract. In this paper, we prove some common fixed point theorems for single-valued and multi-valued mappings which extend, improve and unify a multitude of the corresponding results by Fisher [1]–[10], Fisher and Sessa [11], Jungck [12], Kim, Kim, Leem and Ume [14], Liu [15], Ohta and Nikaido [16] and others. At the same time, we correct errors for the results in [14], [16] and [18].

1. Introduction

Let (X,d) be a metric space and f, g be selfmappings of X. Let W and N denote the sets of nonnegative integers and positive integers, respectively. For $x,y \in X$ and $A,B \subset X$, we define some notations as follows:

$$O_f(x) = \{f^n x : n \in W\}, \quad O_f(x,y) = O_f(x) \cup O_f(y),$$

$$O_{f,g}(x) = \{f^n g^m x : n,\ m \in W\}, \quad O_{f,g}(x,y) = O_{f,g}(x) \cup O_{f,g}(y),$$

$$D(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\},$$

$$\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}, \quad \delta(A,A) = \delta(A),$$

$$H(A,B) = \max\big\{\sup\{D(a,B) : a \in A\}, \quad \sup\{D(A,b) : b \in B\}\big\},$$

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$$CB(X) = \{A : A \text{ is a nonempty bounded closed subset of } X\},$$
$$CL(X) = \{A : A \text{ is a nonempty closed subset of } X\},$$
$$B(X) = \{A : A \text{ is a nonempty bounded subset of } X\},$$
$$C_f = \{h : h : X \to X \text{ is a mapping satisfying } hf = fh\},$$
$$H_f = \Big\{h : h : X \to X \text{ is a mapping satisfying } h\Big(\bigcap_{n\in N} f^n(X)\Big) \subset \bigcap_{n\in N} f^n(X)\Big\}.$$

The mapping f is called a closed mapping if $y = fx$ whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} fx_n = y$ for some $x, y \in X$. For each $t \in [0, +\infty)$, $[t]$ denotes the largest integers not exceeding t. Let

$$\Phi = \{\phi : \phi : [0, +\infty) \to [0, +\infty) \text{ is upper semicontinuous and nondecreasing and } \phi(t) < t \text{ for } t > 0\}.$$

A number of generalizations of the well-known Banach contraction principle have received much attention in recent years. For instance, see [1]–[22]. Kim, Kim, Leem and Ume [14] considered the following conditions:
(1.1) there exists $m, n \in N$ and $r \in [0, 1)$ such that, for every $x, y \in X$,

$$d((fg)^m x, (fg)^n y) \le r\delta(O_{f,g}(x, y)),$$

(1.2) there exists $m, n \in W$ such that, for any distinct $x, y \in X$,

$$d((fg)^m x, (fg)^n y) < \delta(O_{f,g}(x, y)),$$

and established two common fixed point theorems. Rehman and Ahmad [18] extended the principle to multivalued mappings.

In this paper, we consider the following more general conditions (1.3) and (1.4) instead of (1.1) and (1.2), respectively:
(1.3) there exists $m, n, p, q \in N$ and $\phi \in \Phi$ such that, for any $x, y \in X$,

$$d(f^m g^n x, f^p g^q y) \le \phi(\delta(O_{f,g}(x, y))),$$

(1.4) there exists $m, n, p, q \in W$ with $m + p, n + q \in N$ such that, for any $x, y \in X$ with $f^m g^n x \ne f^p g^q y$,

$$d(f^m g^n x, f^p g^q y) < \delta\Big(\bigcup_{h\in H_{fg}} h(O_{f,g}(x, y))\Big),$$

and obtain common fixed point theorems. On the other hand, we point out that Theorem 2.4 of [14] is false and all the results of [18] are meaningless.

Lemma 1.1 [20]. *Let $\phi \in \Phi$. Then, for every $t > 0$, $\phi(t) < t$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$, where ϕ^n denotes the composition of ϕ with itself n times.*

Lemma 1.2. *Let f be a closed mapping from a compact metric space (X, d) into itself and $A = \bigcap_{n\in N} f^n(X)$. Then*

(i) $\{f^n : n \in W\} \subset C_f \subset H_f$;

(ii) *A is a nonempty compact subset of X;*

(iii) $A = f(A)$;

(iv) $\delta(f^n X) \downarrow \delta(A)$ *as* $n \to \infty$.

Proof. Let g be in C_f. Then

$$g(A) = g\Big(\bigcap_{n\in N} f^n(X)\Big) \subset \bigcap_{n\in N} g f^n(X) \subset \bigcap_{n\in N} f^n(X) = A,$$

which implies that $g \in H_f$ and so $C_f \subset H_f$. Obviously $\{f^n : n \in W\} \subset C_f$.

Assume that $\{x_n\}_{n\in N}$ is a sequence in X with $\lim_{n\to\infty} f x_n = a \in X$. The compactness of X ensures that there exists a subsequence $\{x_{n_k}\}_{k\in N}$ such that $\lim_{k\to\infty} x_{n_k} = t \in X$ and so the closedness of f implies that $a = ft$. Thus $f(X)$ is a closed subset of X. Since X is compact, so is $f(X)$. Similarly, we infer that $f^n(X)$ is compact for any $n \geq 2$. It is easy to see that A is a nonempty compact subset of X. It follows from (i) that $f(A) \subset A$. Conversely, for any $a \in A$ and $n \in N$, there exists $a_n \in f^{n-1}(X)$ with $f a_n = a$. From the compactness of X, we may (by selecting a subsequence, if necessary) assume that $\lim_{n\to\infty} a_n = t \in X$. In view of $\{a_k\}_{k\geq n+1} \subset f^n(X)$ and the compactness of $f^n(X)$, we immediately conclude that $t \in f^n(X)$ for all $n \in N$. That is, $t \in A$. Since $ft = a$, then $A \subset f(A)$. Therefore, we have $A = f(A)$. Since $\{\delta(f^n(X))\}_{n\in N}$ is nonincreasing and bounded in below, $\{\delta(f^n(X))\}_{n\in N}$ is convergent. By the compactness of $f^n(X)$, there exist $x_n, y_n \in f^n(X)$ such that $d(x_n, y_n) = \delta(f^n(X))$. Of course, we may extract subsequences $\{x_{n_k}\}_{k\in N}$, $\{y_{n_k}\}_{k\in N}$ of $\{x_n\}$, $\{y_n\}$ such that $x_{n_k} \to x$, $y_{n_k} \to y$ as $k \to \infty$, respectively. Note that $x_{n_i}, y_{n_i} \in f^{n_k}(X)$ and that $f^{n_k}(X)$ is closed for

$i \geq k \geq 1$. This implies that $x, y \in f^{n_k}(X)$ for $k \in N$. Consequently, $x, y \in \bigcap_{k\in N} f^{n_k}(X) = A$. It follows that

$$\delta(A) \leq \lim_{k\to\infty} \delta(f^{n_k}(X)) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = d(x, y) \leq \delta(A),$$

which implies that

$$\delta(A) = \lim_{k\to\infty} \delta(f^{n_k}(X)) = \lim_{n\to\infty} \delta(f^n(X)).$$

This completes the proof. □

2. Common fixed point theorems for single-valued mappings

Now, we give some common fixed point theorems for commuting single-valued mappings.

Theorem 2.1. *Let f, g be commuting mappings from a complete metric space (X, d) into itself and fg be closed. Assume that $O_{f,g}(x)$ is bounded for all $x \in X$ and (1.3) holds. Then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$. Moreover,*

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0, 1\}\} \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x)))$$

for all $i \in N$, where $k = \max\{m, n, p, q\}$.

Proof. For any $i, j, s, t, h \in W$, it follows from (1.3) that

$$\begin{aligned} &d(f^{i+k+s} g^{i+k+t} x, f^{i+k+j} g^{i+k+h} x) \\ &\quad\leq \phi(\delta(O_{f,g}(f^{i+k-m+s} g^{i+k-n+t} x, f^{i+k-p+j} g^{i+k-q+h} x))) \\ &\quad\leq \phi(\delta(O_{f,g}(f^{i+s} g^{i+t} x, f^{i+j} g^{i+h} x))) \\ &\quad\leq \phi(\delta(O_{f,g}((fg)^i x))), \end{aligned}$$

which implies that

$$\delta(O_{f,g}((fg)^{i+k} x)) \leq \phi(\delta(O_{f,g}((fg)^i x))) \tag{2.1}$$

for all $i \in W$. We now write $i = sk + t$ for some $s, t \in W$ with $t \leq k - 1$. (2.1) ensures that

$$\begin{aligned} \delta(O_{f,g}((fg)^i x)) &\leq \phi(\delta(O_{f,g}((fg)^{(s-1)k+t} x))) \\ &\leq \phi^2(\delta(O_{f,g}((fg)^{(s-2)k+t} x))) \\ &\leq \cdots \\ &\leq \phi^s(\delta(O_{f,g}((fg)^t x))) \\ &\leq \phi^s(\delta(O_{f,g}(x))). \end{aligned} \tag{2.2}$$

It follows from Lemma 1.1 and the boundedness of $O_{f,g}(x)$ and (2.2) that

$$\lim_{i\to\infty} \delta(O_{f,g}((fg)^i x))) = 0, \tag{2.3}$$

which means that $\{(fg)^i x\}_{i \in N}$ is a Cauchy sequence in X. By completeness of X, there exists $w \in X$ such that $\lim_{i\to\infty}(fg)^i x = w$. Note that

$$\begin{aligned} d((fg)^i f^a g^b x, w) &\leq d((fg)^i f^a g^b x, (fg)^i x) + d((fg)^i x, w) \\ &\leq \delta(O_{f,g}((fg)^i x)) + d((fg)^i x, w) \end{aligned}$$

for $a, b \in \{0, 1\}$. By (2.3) we have $\lim\limits_{i\to\infty} d((fg)^i f^a g^b x, w) = 0$ for $a, b \in \{0, 1\}$. This implies that

$$w = \lim_{i\to\infty} (fg)^i x = \lim_{i\to\infty} fg(fg)^i x. \tag{2.4}$$

Since fg is closed, we have $w = fgw$. For any $i, j, s, t \in W$, by (1.3), we have

$$\begin{aligned} d(f^i g^j w, f^s g^t w) &= d(f^{i+k} g^{j+k} w, f^{s+k} g^{t+k} w) \\ &\leq \phi(\delta(O_{f,g}(f^i g^j w, f^s g^t w))) \\ &\leq \phi(\delta(O_{f,g}(w))), \end{aligned}$$

which means that

$$\delta(O_{f,g}(w)) \leq \phi(\delta(O_{f,g}(w))).$$

From Lemma 1.1, we easily infer that $\delta(O_{f,g}(w)) = 0$. Therefore $w = fw = gw$, that is, the point w is a common fixed point of f and g. The uniqueness of the common fixed point w of f and g follows immediately from (1.3). For any $p \in W$ and $i \in N$, by (2.2), we have

$$\max\{d((fg)^i f^a g^b x, (fg)^{i+p} x) : a, b \in \{0,1\}\} \leq \delta(O_{f,g}((fg)^i x)) \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x))). \tag{2.5}$$

Letting p tend to infinity in (2.5), by (2.4), we have

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0,1\}\} \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x))).$$

This completes the proof. □

Taking $\phi(t) = rt$ in Theorem 2.1, we obtain the following:

Corollary 2.2. *Let f, g be commuting mappings from a complete metric sapce (X,d) into itself and fg be closed. Assume that $O_{f,g}(x)$ is bounded for all $x \in X$ and that there exist $m, n, p, q \in N$ and $r \in [0,1)$ such that*

$$d(f^m g^n x, f^p g^q y) \leq r\delta(O_{f,g}(x,y)) \tag{2.6}$$

for all $x, y \in X$. Then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0,1\}$. Moreover,

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0,1\}\} \leq r^{[\frac{i}{k}]}\delta(O_{f,g}(x))$$

for all $i \in N$, where $k = \max\{m, n, p, q\}$.

Remark 2.1. Corollary 2.1 with $m = n$ and $p = q$ extends, improves and unifies Theorem 1 of [8], Theorem 3 of [16] and Theorem 2.1 of [14].

Kim, Kim, Leem and Ume [14] and Ohta and Nikaido [16] proved the following theorems, respectively:

Theorem KKLU. *Let f, g be commuting mappings from a compact metric space (X,d) into itself and fg be closed. If (1.2) holds, then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i x = w$ for all $x \in X$.*

Theorem ON. *Let f be a continuous mappings from a compact metric space (X, d) into itself. Assume that there exists $k \in W$ such that*

$$d(f^k x, f^k y) < \delta(O_f(x, y)) \tag{2.7}$$

for all distinct $x, y \in X$. Then f has a unique fixed point $w \in X$ and $\lim_{n\to\infty} f^n x = w$ for all $x \in X$.

First we show, by an example, that (1.2) is not sufficient for the conclusions of Theorem KKLU.

Example 2.1. Let $X = \{0, 1\}$ with the usual metric d. Define $f, g : X \to X$ by $f(0) = g(1) = 1$ and $f(1) = g(0) = 0$. Then (X, d) is a compact metric space, f is continuous and $fg = gf = f$. The continuity of f ensures that f is closed. Taking $m = 1$ and $n = 2$, then we have

$$d(fgx, (fg)^2 y) = 0 < 1 = \delta(O_{f,g}(x, y))$$

for all distinct $x, y \in X$. Thus all the conditions of Theorem KKLU are satisfied. But f and g have no common fixed point in X.

Next we point out that Theorem ON is meaningless for $k = 0$. Suppose that $\delta(X) > 0$. Since X is compact, there exist $x, y \in X$ with $\delta(X) = d(x, y)$. For $k = 0$, by (2.7), we have

$$\delta(X) = d(x, y) < \delta(O_{f,g}(x, y)) \le \delta(X),$$

which is a contradiction. Thus X is a singleton for $k = 0$.

Now we establish the following result which is a correction of Theorem KKLU and Theorem ON.

Theorem 2.3. *Let f, g be commuting mappings from a compact metric space (X, d) into itself and gf be closed. If (1.4) holds, then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of H_{fg}. Moreover, $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$.*

Proof. Let $A = \bigcap_{i\in N}(fg)^i(X)$. It follows from Lemma 1.2 that A is a nonempty compact subset of X and tht $fg(A) = A$. In virtue of $f(A) \subset A$ and $g(A) \subset A$, we have $A = fg(A) = gf(A) \subset g(A) \subset A$ and so $A = g(A)$. Similarly $A = f(A)$. Thus $f^m g^n(A) = A = f^p g^q(A)$. We claim that A is a singleton. If not, then $\delta(A) > 0$. Obviously there exist $a, b, x, y \in A$ with $\delta(A) = d(a, b), a = f^m g^n x$ and $b = f^p g^q y$. Using (1.4) and Lemma 1.2, we obtain

$$\delta(A) = d(f^m g^n x, f^p g^q y) < \delta\left(\bigcup_{h \in H_{fg}} h(O_{f,g}(x, y))\right)$$
$$\leq \delta\left(\bigcup_{h \in H_{fg}} h(A)\right) \leq \delta(A),$$

which is a contradiction. Hence $A = \{w\}$ for some $w \in X$ and $fw = gw = w$. Suppose that u is also a common fixed point of f and g. Then $u = (fg)^n u$ for all $n \in N$ and hence $u \in A = \{w\}$, which means that $u = w$, that is, w is a unique common fixed point of f and g. It is easy to verify that w is a unique common fixed point of H_{fg}. Moreover, Lemma 1.2 ensures that

$$d((fg)^i f^a g^b x, w) \leq \delta((fg)^i X) \to \delta(A) = 0 \quad \text{as } i \to \infty,$$

where $a, b \in \{0, 1\}$. Consequently, $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for $a, b \in \{0, 1\}$. This completes the proof. □

From Lemma 1.2 and Theorem 2.3, we have the following:

Corollary 2.4. *Let f, g be commuting mappings from a compact metric space (X, d) into itself and gf be closed. Assume that there exist $m, n, p, q \in W$, $m + p$, $n + q \in N$ such that*

$$d(f^m g^n x, f^p g^q y) < \delta\left(\bigcup_{h \in C_{fg}} h(O_{f,g}(x, y))\right) \tag{2.8}$$

for all $x, y \in X$ with $f^m g^n x \neq f^p g^q y$. Then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of C_{fg}. Moreover, $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$.

Remark 2.2. Corollary 2.4 extends, improves and unifies Theorem 6 of [1], Theorem 4 of [2], Theorem 4 of [3], Theorem 9 of [4], Theorem 2 of [5], Theorem 4 of [7], Theorem 5 of [9], Theorem 2 of [10] and Theorem 4.2 of [12].

We provide some examples to demonstrate that the hypotheses of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are necessary.

Example 2.2. Let $X = [1, +\infty)$ with the usual metric d. Define mappings $f, g : X \to X$ by $fx = 2x$ and $gx = 3x$ for all $x \in X$, respectively. Then (X, d) is a complete metric space, $fg = gf$ and fg is closed. Furthermore, (1.3) and (2.6) are satisfied with $m = p = 1$, $n = q = 2$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. All of the conditions of Theorem 2.1 and Corollary 2.2 are satisfied except for the boundedness assumption, but f and g have no common fixed point in X.

Example 2.3. Let $X = [0, 1]$ with the usual metric d. Define mappings $f, g : X \to X$ by $fx = \frac{1}{4}(x^3 + 3)$ and $gx = (1 - x)^{\frac{1}{3}}$ for all $x \in X$, respectively. Then (X, d) is a compact metric space, $fgx = \frac{1}{4}(4 - x)$ is closed and $fg(1) = \frac{3}{4} \neq 0 = gf(1)$. Take $m = n = p = q = 1$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. It is easy to verify that f and g satisfy the following:

$$d(f^m g^n x, f^p g^q y) = \frac{1}{4}|x - y| \le \frac{1}{2}|x - y| \le \phi(\delta(O_{f,g}(x, y)))$$

for all $x, y \in X$ and

$$d(f^m g^n x, f^p g^q y) = \frac{1}{4}|x - y| < \frac{1}{2}|x - y| \le \delta(O_{f,g}(x, y))$$

for all $x, y \in X$ with $f^m g^n x \neq f^p g^q y$. Thus, the conditions of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are satisfied except for the commutativity assumption. But f and g however have no common fixed point in X.

Example 2.4. Let $X = [0, 1]$ with the usual metric d. Define mappings $f, g : X \to X$ by $f(0) = 1$, $fx = \frac{1}{3}x$ for $x \in (0, 1]$ and $g = f^2$. Then f and g are commuting and (X, d) is a compact metric space. Take $m = n = p = q = 1$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. Since $\lim_{i\to\infty} \frac{1}{i} = \lim_{i\to\infty} \frac{1}{27i} = \lim_{i\to\infty} fg(\frac{1}{i}) = 0$ and $0 \neq \frac{1}{9} = fg(0)$, so fg is not closed. For $x, y \in (0, 1]$, we have

$$d(fgx, fgy) = \frac{1}{27}|x - y| \le \frac{1}{6}|x - y| \le \frac{1}{2}\delta(O_{f,g}(x, y)).$$

For $x = 0, y \in [0, 1]$, we have

$$d(fgx, fgy) \le \frac{1}{9}\left|1 - \frac{1}{3}y\right| < \frac{1}{2} = \frac{1}{2}\delta(O_{f,g}(x, y)).$$

Thus, the conditions of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are satisfied except for the closedness assumption. But f and g have no common fixed point in X.

Theorem 2.5. *Let f, g be closed mappings from a compact metric space (X, d) into itself. Assume that there exist $m, n \in N$ such that*

$$d(f^m x, g^n y) < \delta\Big(\bigcup_{h\in H_f} h(O_f(x)), \bigcup_{t\in H_g} t(O_g(y))\Big) \tag{2.9}$$

for all $x, y \in X$ with $f^m x \neq g^n y$. Then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of H_f and H_g. Moreover, $\lim_{i\to\infty} f^i x = \lim_{i\to\infty} g^i x = w$ for all $x \in X$.

Proof. Put $A = \bigcap_{i\in N} f^i(X)$ and $B = \bigcap_{i\in N} g^i(X)$. In view of Lemma 1.2, we have $f(A) = A \neq \emptyset$, $g(B) = B \neq \emptyset$ and A and B are compact. Thus there exist $a, x \in A$ and $b, y \in B$ with $d(a, b) = \delta(A, B)$, $a = f^m x$ and $b = g^n y$. We assert that $\delta(A, B) = 0$. If not, by (2.9) and Lemma 1.2, we have

$$d(f^m x, g^n y) < \delta\Bigg(\bigcup_{h\in H_f} h(O_f(x)), \bigcup_{t\in H_g} t(O_g(y))\Bigg) \leq \delta(A, B).$$

Thus we have

$$\delta(A, B) = d(a, b) = d(f^m x, g^n y) < \delta(A, B),$$

which is a contradiction. Therefore $\delta(A, B) = 0$ and there is some $w \in X$ with $A = B = \{w\}$. It is clear that $fw = gw = w$. The rest of the proof is identical with the proof of Theorem 2.3. This completes the proof. □

Remark 2.3. Theorem 2.3 contains Theorem 2.5 of [15] as a special case.

3. Remarks on fixed point theorems of Rehman and Ahmad

In [18], Rehman and Ahmad proved the following:

Theorem RA1. *Let (X, d) be a complete metric space and $S, T : X \to CB(X)$ satisfy the following:*

$$H(Sx, Ty) \leq k\{D(x, Sx)D(y, Ty)\}^{\frac{1}{2}} \tag{3.1}$$

for all $x, y \in X$, where $k \in (0, 1)$. Then S and T have a unique common fixed point $w \in X$.

Corollary RA2. *Let (X,d) be a compact metric space and $S,T : X \to CL(X)$ satisfy* (3.1). *If S or T is continuous, then S or T has a fixed point in X.*

Theorem RA3. *Let (X,d) be a complete metric space and $S,T : X \to B(X)$ satisfy the following:*

$$\delta(Sx,Ty) \leq k\{H(x,Sx)H(y,Ty)\}^{\frac{1}{2}} \tag{3.2}$$

for all $x,y \in X$, where $k \in [0,1)$. Then S and T have a unique common fixed point $w \in X$.

Corollary RA4. *Let (X,d) be a compact metric space and $S,T : X \to CL(X)$ satisfy* (3.2). *If S or T is continuous, then S or T has a fixed point in X.*

Corollary RA5. *Let (X,d) be a compact metric space and $S,T : X \to B(X)$ satisfy* (3.2). *Then S and T have a unique common fixed point $u \in X$ and $Su = Tu = \{u\}$.*

Theorem RA6. *Let $\{T_n\}_{n\in N}$ be a sequence of self mappings of a complete metric space (X,d). If there exists a constant h such that, for all $x,y \in X$ and $i,j \in N$, $i,j \in N$*

$$d(T_ix,T_jy) \leq h\{d(x,T_ix)d(y,T_jy)\}^{\frac{1}{2}}, \tag{3.3}$$

for some $h \in (0,1)$, then $\{T_n\}_{n\in N}$ has a unique common fixed point $w \in X$.

Theorem RA7. *Let (X,d) and (X,d') be metric spaces satisfying the following:*

- (i) *$d(x,y) \leq d'(x,y)$ for all $x,y \in X$;*
- (ii) *X is complete with respect to d;*
- (iii) *$f,g : X \to X$ are self-mappings such that f is continuous with respect to d and, for all $x,y \in X$,*

$$d'(fx,fy) \leq h\{d'(x,fx)d'(y,gy)\}^{\frac{1}{2}} \tag{3.4}$$

for some $h \in [0,1)$. Then f and g have a unique common fixed point $w \in X$.

We assert, by the following results, that Theorems RA1, RA3, RA6, RA7 and Corollaries RA2$\sim$RA5 are meaningless.

Theorem 3.1. *Let (X, d), S, T and w be as in Theorem RA1. Then $w \in Sw = Tw = Sx = Tx$ for all $x \in X$.*

Proof. For any $x \in X$, by (3.1) and Theorem RA1, we have

$$H(Tx, Sw) \leq k\{D(x, Tx)D(w, Sw)\}^{\frac{1}{2}} = 0,$$

which implies that $Tx = Sw$. Similarly, $Sx = Tw$. Therefore $w \in Sw = Tw = Sx = Tx$ for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let (X, d) be a compact metric space and $S, T : X \to CL(X)$ satisfy (3.1). Then there exists $w \in X$ such that $w \in Sw = Tw = Sx = Tx$ for all $x \in X$.*

Proof. Since (X, d) is a compact metric space, $CL(X) = CB(X)$. Thus Corollary 3.2 follows from Theorem 3.1. □

Theorem 3.3. *Let (X, d), S and T be as in Theorem RA3. Then there exists $w \in X$ such that $Sx = Tx = \{w\}$ for all $x \in X$.*

Proof. It is easy to verify that (3.2) is equivalent to the following:

$$\delta(Sx, Ty) \leq k\{\delta(x, Sx)\delta(y, Ty)\}^{\frac{1}{2}} \tag{3.5}$$

for all $x, y \in X$, where $k \in [0, 1)$. We claim that there exists $w \in X$ such that $\delta(w, Sw)\delta(w, Tw) = 0$. Otherwise, $\delta(x, Sx)\delta(x, Tx) > 0$ for all $x \in X$. We consider the following two cases:

Case 1. Suppose that $k = 0$. (3.5) implies that $Sx = Ty$ for all $x, y \in X$. Take $x \in X$ and $y \in Sx$. Then $\delta(y, Ty) = \delta(y, Sx) = 0$, which is a contradiction.

Case 2. Suppose that $k \in (0, 1)$. Take $x_0 \in X$ and select a sequence $\{x_n\}_{n \in N}$ in X such that $x_{2n+1} \in Sx_{2n}$ for $n \in W$ and $x_{2n} \in Tx_{2n-1}$ for $n \in N$. (3.5) ensures that

$$\delta(x_{2n}, Sx_{2n}) \leq \delta(Sx_{2n}, Tx_{2n-1}) = k\{\delta(x_{2n}, Sx_{2n})\delta(x_{2n-1}, Tx_{2n-1})\}^{\frac{1}{2}},$$

which implies that

$$\delta(x_{2n}, Sx_{2n}) \leq r\delta(x_{2n-1}, Tx_{2n-1}),$$

where $r = k^2 \in (0,1)$. Similarly, $\delta(x_{2n-1}, Tx_{2n-1}) \le r\delta(x_{2n-2}, Sx_{2n-2})$. Set $M = \max\{\delta(x_2, Sx_2), \delta(x_1, Tx_1)\}$. For $m > n \ge 2$, we have

$$\begin{aligned}(3.6)\quad & \max\{\delta(x_{2n}, Sx_{2n}), \delta(x_{2n-1}, Tx_{2n-1})\} \\ & \qquad \le r^2 \max\{\delta(x_{2n-2}, Sx_{2n-2}), \delta(x_{2n-3}, Tx_{2n-3})\} \le r^{2(n-1)} M\end{aligned}$$

and

$$(3.7)\qquad d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} r^{i-2} M \le \frac{r^{n-2}}{1-r} M.$$

Since $r \in (0,1)$ and (X,d) is complete, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and so it converges to some $w \in X$. Using (3.5), we have

$$\begin{aligned}\delta(w, Sw) &\le d(w, x_{2n+2}) + \delta(Sw, x_{2n+2}) \\ &\le d(w, x_{2n+2}) + \delta(Sw, Tx_{2n+1}) \\ &\le d(w, x_{2n+2}) + k\{\delta(w, Sw)\delta(x_{2n+1}, Tx_{2n+1})\}^{\frac{1}{2}}.\end{aligned}$$

Letting n tend to infinity, by (3.6), (3.7) and boundedness of S, we immediately obtain $\delta(w, Sw) = 0$, which is also a contradiction.

Consequently, $\delta(w, Sw)\delta(w, Tw) = 0$ for some $w \in X$. We assume without loss of generality, that $\delta(w, Sw) = 0$, that is, $Sw = \{w\}$. Using (3.5), for any $x \in X$, we have

$$\delta(w, Tx) = \delta(Sw, Tx) \le k\{\delta(w, Sw)\delta(x, Tx)\}^{\frac{1}{2}} = 0,$$

which implies that $Tx = \{w\}$. Clearly $Tw = \{w\} = Tx = Sw$. On the other hand, by (3.5), we have

$$\delta(w, Sx) = \delta(Sx, Tw) \le k\{\delta(x, Sx)\delta(w, Tw)\}^{\frac{1}{2}} = 0.$$

Therefore, $Sx = \{w\} = Tx$. This completes the proof. □

Corollary 3.4. *Let (X,d) be a compact metric space and $S, T : X \to B(X)$ satisfy* (3.2). *Then there exists $w \in X$ such that $\{w\} = Sx = Tx$ for all $x \in X$.*

Proof. It follows from the compactness of X that $CL(X) \subset B(X)$. Thus Corollary 3.4 follows from Theorem 3.3. □

Theorem 3.5. *Let* (X, d), $\{T_n\}_{n\in N}$ *and* w *be as in Theorem RA6. Then* $T_n x = w$ *for all* $x \in X$ *and* $n \in N$.

Proof. Theorem RA6 ensures that $T_n w = w$ for all $n \in N$. By (3.2), for all $x \in X$ and $i, j \in N$, we have

$$d(w, T_j x) = d(T_i w, T_j x) \leq h\{d(w, T_i w) d(x, T_j x)\}^{\frac{1}{2}} = 0,$$

which implies that $T_j x = w$. This completes the proof. □

Theorem 3.6. *Let* $(X, d), (X, d'), f, g$ *and* w *be as in Theorem RA7. Then* $fx = gx = w$ *for all* $x \in X$.

Proof. In view of (3.3) and Theorem RA7, we have

$$d'(w, gx) = d'(fw, gx) \leq h\{d'(w, fw) d'(x, gx)\}^{\frac{1}{2}} = 0,$$

which implies that $w = gx$ for all $x \in X$. Similarly, $fx = w$ for all $x \in X$. This completes the proof. □

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