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On the indicator plurality function

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Abstract. A solution of the conditional functional equation

$$f(x) \cdot f(y) \neq \underline{0} \implies f(x+y) = f(x) \cdot f(y),$$

for which there exists a number $r \in \mathbb{R}(1) \backslash \{1\}$ such that

$$f(rx) = f(x),$$

where $f : \mathbb{R}(n) := [0,\infty)^n \backslash \{\underline{0}\} \to \mathbb{R}(n)$, $\underline{0} := (0,\dots,0) \in \mathbb{R}^n$ and $x + y := (x_1 + y_1, \dots, x_n + y_n)$, $x \cdot y := (x_1 y_1, \dots, x_n y_n)$, $rx := (rx_1, \dots, rx_n)$, for $x = (x_1,\dots,x_n) \in \mathbb{R}(n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}(n)$, is called an indicator plurality function.

We study under which assumptions this function f must have its values in the set $0(n) := \{0,1\}^n \backslash \{\underline{0}\}$.

1. Introduction

F. S. Roberts, generalizing a description of the social choice, which was introduced by him in [5], [6], considers the following conditional equation

$$f(x) \cdot f(y) \neq \underline{0} \Rightarrow f(x+y) = f(x) \cdot f(y), \tag{1}$$

where $f : \mathbb{R}(n) := [0,\infty)^n \backslash \{\underline{0}\} \to \mathbb{R}(n)$, $\underline{0} := (0,\dots,0) \in \mathbb{R}^n$ and $x + y := (x_1+y_1,\dots,x_n+y_n)$, $x \cdot y := (x_1y_1,\dots,x_ny_n)$ for $x = (x_1,\dots,x_n) \in \mathbb{R}(n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}(n)$.

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In the paper [2] the following description of all solutions $f=(f_1,\dots,f_n)$ of this equation was given:

$$f_\nu(x)=\begin{cases}\exp a_\nu(x) & \text{for } x\in Z_\nu,\\ 0 & \text{for } x\in \mathbb{R}(n)\backslash Z_\nu,\end{cases}\tag{2}$$

where $a_\nu:\mathbb{R}^n\to\mathbb{R}$ is an additive function ($\nu=1,\dots,n$) and the sets Z_ν satisfy the conditions

$$Z_1\cup\dots\cup Z_n=\mathbb{R}(n),\tag{3}$$

$$ij\neq \underline{0}\Rightarrow Z_1^{i_1}\cap\dots\cap Z_n^{i_n}+Z_1^{j_1}\cap\dots\cap Z_n^{j_n}\subset Z_1^{i_1j_1}\cap\dots\cap Z_n^{i_nj_n},\tag{4}$$

where $i=(i_1,\dots,i_n)$, $j=(j_1,\dots,j_n)\in\{0,1\}^n$, $E_1+E_2=\{x+y:x\in E_1$ and $y\in E_2\}$ for $E_1,E_2\subset\mathbb{R}^n$ and, here and subsequently $E^1:=E$, $E^0:=\mathbb{R}(n)\backslash E$ for $E\subset\mathbb{R}(n)$.

For further reference we denote the whole above description (including (2), (3) and (4)) by (A).

Therefore, the sets Z_ν, (it is known that $Z_\nu\cup\{\underline{0}\}$ are cones over the field $\mathbb{Q}$ of rational numbers) and the additive functions $a_\nu:\mathbb{R}^n\to\mathbb{R}$ are the parameters giving the solution of the equation (1).

A solution of the equation (1) satisfying the additional condition

$$\exists r\in\mathbb{R}(1):[r\neq 1 \text{ and } \forall x\in\mathbb{R}(n): f(rx)=f(x)],\tag{5}$$

we call shortly an indicator plurality function.

Z. Moszner proved in [3] that every function $f:\mathbb{R}(n)\to\mathbb{R}(n)$ satisfying the condition (1) and the condition (5) with some r being an algebraic number must have values in the set $0(n):=\{0,1\}^n\backslash\{\underline{0}\}$. A. Bahyrycz in [1] showed that the above result holds for a transcendental number r for $n=1,2$. However, when $n\geq 3$, for every transcendental number r there exists an indicator plurality function having values outside the set $0(n)$.

From the description (A) of the solutions of the equation (1) it follows that an indicator plurality function has all its values in the set $0(n)$ if and only if all the additive functions a_ν are identically equal to zero.

We notice that condition (5) imposes on the functions a_ν and the sets Z_ν the conditions

$$rZ_\nu=Z_\nu,\tag{6}$$

$$a_\nu(rx)=a_\nu(x)\quad\text{for}\quad x\in Z_\nu.\tag{7}$$

Now the following question arises: Does there exist a necessary and sufficient condition imposed on the cones Z_ν satisfying (6) for which fulfilment of the condition (7) by the additive functions would imply their vanishing?

All linear spaces and all cones in this paper are considered over the field $\mathbb{Q}$, unless we assume differently.

First, we will prove the following

Lemma 1. *We assume that $Z \subset \mathbb{R}(n) \cup \{\underline{0}\}$ is a cone and for a $r \neq 1$: $rZ = Z$.*

Then the conditions

$$Z \subset (r-1) \operatorname{lin} Z \tag{8}$$

and

$$\begin{aligned} &\text{for every additive function } a : Z \to \mathbb{R} : \text{ if } a(rx) = a(x) \\ &\text{for } x \in Z, \text{ then } a = 0 \text{ on } Z \text{ are equivalent.} \end{aligned} \tag{9}$$

Proof. We can assume that $r > 1$.
((8) $\Longrightarrow$ (9)). Let B be a base of the space $\operatorname{lin} Z$, such that $B \subset Z$. Then every $x \in \operatorname{lin} Z$ has a representation $x = \sum_{i=1}^{k} q_i b_i$, where $k \in \mathbb{N}$ (the set of natural numbers), $q_i \in \mathbb{Q}$ and $b_i \in B$. We extend the additive function $a : Z \to \mathbb{R}$ to an additive function $\bar{a} : \operatorname{lin} Z \to \mathbb{R}$ in the standard way:

$$\bar{a}(x) = \sum_{i=1}^{k} q_i a(b_i) \quad \text{for} \quad x = \sum_{i=1}^{k} q_i b_i.$$

We will show that $\bar{a}(rx) = \bar{a}(x)$ for $x \in \operatorname{lin} Z$. For $x = \sum_{i=1}^{k} q_i b_i \in \operatorname{lin} Z$, by the definition of the function $\bar{a}$ and from the fact that $a(rz) = a(z)$ for $z \in Z$, we obtain

$$\bar{a}(x) = \sum_{i=1}^{k} q_i a(b_i) = \sum_{i=1}^{k} q_i a(rb_i).$$

Since $rb_i \in Z$ for every $i \in \{1, \ldots, k\}$, so $rb_i = \sum_{j=1}^{l} q_{ij} b_{ij}$, where $l \in \mathbb{N}$, $q_{ij} \in \mathbb{Q}$ and $b_{ij} \in B$, and $a(rb_i) = \sum_{j=1}^{l} q_{ij} a(b_{ij})$ and $rx = r(\sum_{i=1}^{k} q_i b_i) = \sum_{i=1}^{k} q_i r b_i = \sum_{i=1}^{k} q_i (\sum_{j=1}^{l} q_{ij} b_{ij}) = \sum_{i=1}^{k} \sum_{j=1}^{l} q_i q_{ij} b_{ij}$.

Therefore

$$\bar{a}(rx) = \sum_{i=1}^{k} \sum_{j=1}^{l} q_i q_{ij} a(b_{ij}) = \sum_{i=1}^{k} q_i \Bigg(\sum_{j=1}^{l} q_{ij} a(b_{ij}) \Bigg)$$
$$= \sum_{i=1}^{k} q_i a(rb_i) = \bar{a}(x),$$

which was to be proved.

For every $x \in Z$, also $\frac{1}{r-1}x \in \operatorname{lin} Z$ and

$$\bar{a}(x) + \bar{a}\left(\frac{1}{r-1}x\right) = \bar{a}\left(\frac{r}{r-1}x\right) = \bar{a}\left(\frac{1}{r-1}x\right),$$

hence $a(x) = \bar{a}(x) = 0$.

For the proof of the reverse implication ((9) $\Longrightarrow$ (8)) assume that Z is not contained in $(r-1)\operatorname{lin} Z$ and we will construct an additive function $a : Z \to \mathbb{R}$, such that $a(rx) = a(x)$ on Z and $a \neq 0$. We notice that the set $(r-1)\operatorname{lin} Z$ is a vector space.

From $x \in (r-1)\operatorname{lin} Z$ it follows that there exists $z \in \operatorname{lin} Z$ such that $x = (r-1)z$. Since $rz \in \operatorname{lin} Z$ and $-z \in \operatorname{lin} Z$, we have $x \in \operatorname{lin} Z$. From the above and from the fact that $Z \not\subset (r-1)\operatorname{lin} Z$ we conclude that $(r-1)\operatorname{lin} Z$ is a proper subspace of the space $\operatorname{lin} Z$. Let P be a complement of the space $(r-1)\operatorname{lin} Z$, i.e. $\operatorname{lin} Z = (r-1)\operatorname{lin} Z \oplus P$. The function $a(x_1 + x_2) = x_2$ for $x_1 \in (r-1)\operatorname{lin} Z$ and $x_2 \in P$ restricted to Z satisfies all the desired conditions. In particular, the condition $a(rx) = a(x)$ is fulfilled because $a((r-1)x) = 0$. $\square$

We notice that in the above lemma we cannot replace the condition (8) in the implication (9) $\Longrightarrow$ (8) by the condition $Z \subset (r-1)Z$.

Example. Define

$$Z = \Bigg\{ z \in \mathbb{R}(1) : z = \frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^n},$$

where $k \in \mathbb{N}$, $n \in (\mathbb{N} \cup \{0\}) \backslash \{1\}$, $q_i \in \mathbb{Q}$ and $\sum_{i=1}^{k} q_i > 0$, $z_i \in \mathbb{Z} \Bigg\} \cup \{0\}$, where $\mathbb{Z}$ is the set of entire numbers.

We will show that Z is a cone. Let $z, z^* \in Z$. Then

$$z = \frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^n} \quad \text{and} \quad z^* = \frac{\sum_{i=1}^{k} q_i^* r^{z_i}}{(r-1)^m},$$

where $k \in \mathbb{N}$, $n, m \in (\mathbb{N} \cup \{0\}) \setminus \{1\}$, $q_i, q_i^* \in \mathbb{Q}$, $\sum_{i=1}^{k} q_i > 0$, $\sum_{i=1}^{k} q_i^* > 0$, $z_i \in Z$.

We can assume that $m \leq n$. Then

$$z + z^* = \begin{cases} \dfrac{\sum_{i=1}^{k}(q_i + q_i^*) r^{z_i}}{(r-1)^n} & \text{for } m = n, \\[2ex] \dfrac{\sum_{i=1}^{k} q_i r^{z_i} + (r-1)^{n-m} \sum_{i=1}^{k} q_i^* r^{z_i}}{(r-1)^n} & \text{for } m < n. \end{cases}$$

We notice that $\sum_{i=1}^{k}(q_i + q_i^*) = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} q_i^* > 0$ for $m = n$ and $\sum_{i=1}^{k} q_i + 0 = \sum_{i=1}^{k} q_i > 0$ for $m < n$. Therefore $z + z^* \in Z$. It is easily seen that $qz \in Z$ for every $q \in \mathbb{Q}_+$ and $z \in Z$, hence Z is a cone and $rZ = Z$. Let $a : Z \to \mathbb{R}$ be an additive function, such that $a(rx) = a(x)$ for $x \in Z$. Let $z = \frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^n} \in Z$. Then $\frac{2r \sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}} \in Z$ and from the additivity of the function a we obtain

$$a\left(\frac{2r \sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}}\right) + a\left(\frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^n}\right) = a\left(\frac{(r^2+1) \sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}}\right).$$

Since

$$a\left(\frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}}\right) = a\left(\frac{r \sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}}\right) = a\left(\frac{r^2 \sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^{n+2}}\right),$$

we get

$$a\left(\frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^n}\right) = 0.$$

Therefore $a(z) = 0$ for every $z \in Z$. We notice that $1 \in Z$ ($k = 1$, $q_1 = 1$, $z_1 = 0$, $n = 0$) but $\frac{1}{r-1} \notin Z$, hence $Z \not\subset (r-1)Z$ and obviously $Z \subset (r-1) \operatorname{lin} Z$.

By Lemma 1 there follows immediately the following

Theorem 1. *The indicator plurality function f has its values only in the set $0(n)$ if and only if the sets Z_ν in its description (A) fulfilling the condition (6) satisfy the condition*

$$Z_\nu \subset (r-1) \operatorname{lin} Z_\nu \quad (\nu = 1, \dots, n)$$

with r occurring in (6).

We notice that for $n = 1$ we have $Z_1 = \mathbb{R}(1)$ and the condition (10) is obviously fulfilled.

This condition is also satisfied for $n = 2$. This follows from the fact that for $n = 2$ the sets $Z_1^0 \cup \{\underline{0}\}$ and $Z_2^0 \cup \{\underline{0}\}$ are also cones ([2] p. 179) and from the following

Lemma 2. *We assume that $Z \subset \mathbb{R}(n) \cup \{\underline{0}\}$ and $Z^0 \cup \{\underline{0}\}$ are cones. Then $Z \subset (r-1) \operatorname{lin} Z$ for every $r \in \mathbb{R} \backslash \{1\}$.*

Proof. It is suffices to show that $Z \subset (r-1) \operatorname{lin} Z$ for all $r > 1$.

We take $r > 1$ and $x \in Z \backslash \{\underline{0}\}$. We consider the cases:

1) $\frac{1}{r-1} x \in Z$. Then

$$x = (r-1) \frac{1}{r-1} x \in (r-1) Z \subset (r-1) \operatorname{lin} Z.$$

2) $\frac{1}{r-1} x \in Z^0$. We notice that for every $q \in \mathbb{Q}_+$ such that $1 < q < r$ we have $\frac{r-q}{r-1} x \in Z$. Indeed, if $\frac{r-q}{r-1} x \in Z^0$, then because $\frac{1}{r-1} x \in Z^0$ and $Z^0 \cup \{\underline{0}\}$ is a cone, we would have $\frac{q-1}{r-1} x \in Z^0$ and

$$\frac{r-q}{r-1} x + \frac{q-1}{r-1} x = \frac{r-q+q-1}{r-1} x = x \in Z^0,$$

which contradicts the fact that $x \in Z \backslash \{\underline{0}\}$. Since $\frac{r-q}{r-1} x \in Z$, $-\frac{r-q}{r-1} x \in \operatorname{lin} Z$ and

$$-\frac{r-q}{r-1} x + x = \frac{-r+q+r-1}{r-1} x = (q-1) \frac{1}{r-1} x \in \operatorname{lin} Z,$$

we have $\frac{1}{r-1} x \in \operatorname{lin} Z$, thus $x = (r-1) \frac{1}{r-1} x \in (r-1) \operatorname{lin} Z$. □

We notice that the converse of Lemma 2 is not true, even if we assume that Z is a cone satisfying the condition $rZ = Z$ and the condition $Z \subset (r-1) \operatorname{lin} Z$ is replaced by a stronger one: $Z \subset (r-1)Z$.

Example. We assume that $r > 1$ and consider the set $Z = \{(z, 0, \ldots, 0) \in \mathbb{R}(n) \cup \{\underline{0}\} : z = \frac{\sum_{i=1}^{k} q_i r^{z_i}}{(r-1)^l},$ where $l \in \mathbb{N} \cup \{0\},\ q_i \in \mathbb{Q},\ z_i \in \mathbb{Z}\}$.

It is easy to check that Z is a cone satisfying the condition $rZ = Z$. Since $\frac{1}{r-1} z \in Z$ for every $z \in Z$, the condition $Z \subset (r-1)Z$ is satisfied. Z^0 is not closed with respect to addition, because there exists $m \in \mathbb{N} \backslash \{1\}$ such that $\sqrt[m]{r} \notin \mathbb{Q}$ and then

$$(\sqrt[m]{r}, 0, \ldots, 0), (-\sqrt[m]{r} + [\sqrt[m]{r}] + 1, 0, \ldots, 0) \in Z^0$$

but

$$(\sqrt[m]{r},0,\dots,0)+(-\sqrt[m]{r}+[\,\sqrt[m]{r}\,]+1,0,\dots,0)=([\,\sqrt[m]{r}\,]+1,0,\dots,0)\in Z,$$

where $[\,\sqrt[m]{r}\,]$ is the greatest integer of $\sqrt[m]{r}$.

The condition (10) for $n=2$ follows also by the following

Lemma 3. *We assume that $Z\subset\mathbb{R}(2)\cup\{\underline{0}\}$ is a cone. Then a subspace* $\operatorname{lin} Z$ *is a vector space over* $\mathbb{R}$.

Proof. We consider the cases:

1) $Z=\{\underline{0}\}$. Then $\operatorname{lin} Z=\{\underline{0}\}$.
2) $Z\neq\{\underline{0}\}$. Only two subcases are possible:
   a) Z is contained in a line l given by $y=ax$ $(a\geq 0)$ or in $x=0$,
   b) Z is contained neither in a line $y=ax$ $(a\geq 0)$ nor in $x=0$.

Ad a) Then $\operatorname{lin} Z\subset l$. We will show that $l\subset\operatorname{lin} Z$. We assume that l has a representation $y=ax$ $(a\geq 0)$ [in case $x=0$ the proof runs analogously]. There exists $x_0>0$, such that $(x_0,ax_0)\in Z$ (because $Z\neq\{\underline{0}\}$ and it is contained in the line $y=ax$). We take an arbitrary $x>0$. Then we have $(x,ax)\in Z\subset\operatorname{lin} Z$ or $(x,ax)\in Z^0=\mathbb{R}(2)\backslash Z$. If $(x,ax)\in Z^0$, then we choose $q\in\mathbb{Q}_+$ such that $qx<x_0$. The pair $(x_0-qx,a(x_0-qx))$ belongs to $\mathbb{R}(2)$. We suppose that $(x_0-qx,a(x_0-qx))\in Z^0$. Then, since $Z^0\cup\{\underline{0}\}$ is a cone over $\mathbb{Q}$, we get $(qx,aqx)\in Z^0$ and

$$(x_0-qx,a(x_0-qx))+(qx,aqx)=(x_0,ax_0)\in Z^0$$

and this contradicts our assumption. Therefore $(x_0-qx,a(x_0-qx))\in Z$, hence $(qx-x_0,\ a(qx-x_0))\in\operatorname{lin} Z$ and since $(x_0,ax_0)\in Z$, we have $(qx,aqx)\in\operatorname{lin} Z$ and thus $(x,ax)\in\operatorname{lin} Z$, which was to be proved.

Ad b) There exist points $(x_1,y_1),(x_2,y_2)\in Z$ such that $x_1+x_2>0$ and the line containing these points has no representation $y=ax$ $(a\geq 0)$. We denote by l_i the line passing through the points $(0,0)$ and $(x_i,y_i)(i=1,2)$.

Similarly as in the proof of the case a) we get $\mathbb{R}^2=\{l_1,l_2\}\subset\operatorname{lin} Z$, where $\{l_1,l_2\}$ denotes a subspace of the space $\operatorname{lin} Z$ spanned by the lines l_1,l_2. □

It confirms the results from the paper [1] for $n=1,2$. For $n\geq 3$ the result of this paper (the existence of an indicator plurality function f having its values outside the set $0(n)$) proves that for every transcendental number $r \in \mathbb{R}(1)$ there exists an indicator plurality function (and so a cone) for which the conditions (10) are not saisfied.

In the example given in [1], such a cone is a set $D \cap [0, +\infty)$, where the set D is defined by a long construction on p. 27 in [1], because $1 \in D \cap [0, +\infty)$ and $\frac{1}{r-1} \notin \operatorname{lin}(D \cap [0, +\infty)) = D$.

For an algebraic number $r \neq 1$ the conditions (10) are satisfied, as it follows from the following

Lemma 4. *If $r \neq 1$ is an algebraic number and Z is a cone such that $rZ \subset Z$, then $Z \subset (r-1) \operatorname{lin} Z$.*

Proof. Since r is an algebraic number, there exists a polynomial of rational coefficients $p_1(x)$ such that $p_1(r) = 0$. Two cases are possible: $p_1(1) \neq 0$ or $p_1(1) = 0$. If $p_1(1) \neq 0$, then we put $p(x) = -\frac{p_1(x)}{p_1(1)}$. If $p_1(1) = 0$, then we consider a polynomial p_2 obtained by dividing $p_1(x)$ by $(x-1)^k$, where 1 is a root of order k of the polynomial $p_1(x)$ and we put $p(x) = -\frac{p_2(x)}{p_2(1)}$. For this polynomial $p(x) : p(1) = -1$ and $p(r) = 0$, therefore

$$p(x) = (x-1)[\alpha_m x^m + \cdots + \alpha_1 x + \alpha_0] - 1,$$

where $\alpha_j \in \mathbb{Q}$ for every $j \in \{0, \ldots, m\}$.

Since $p(r) = 0$, for $r \neq 1$ we obtain

$$\frac{1}{r-1} = \alpha_m r^m + \cdots + \alpha_1 r + \alpha_0.$$

Therefore for every $z \in Z$, where Z is a cone such that $rZ \subset Z$, we have $\frac{1}{r-1} z \in \operatorname{lin} Z$. □

We obtain in this way, via the theorem 1, another proof of the above mentioned results from the paper [3]. In the construction of the solution of equation (1) satisfying (5) with some transcedental number r, which was given in [1], the Axiom of Choice is used. Below we will show that one cannot give this construction without using non-measurable set.

We adopt the following definition (see [4]):

A function $f = (f_1, \ldots, f_n) : \mathbb{R}(n) \to \mathbb{R}(n)$ is called *nearly measurable* if for every $c \in \mathbb{R}(n)$ the sets $A_i(c) = \{tc \in \mathbb{R}(n) : f_i(tc) \neq 0\}$ for $i = 1, \ldots, n$ are Lebesgue linearly measurable.

We will show the

Theorem 2. *If a solution f of equation* (1)*, satisfying* (5) *with some $r \neq 1$, is nearly measurable, then it satisfies the conditions* (10)*, because $Z_\nu \cup \{\underline{0}\}$ for $\nu = 1, \dots, n$ are cones over $\mathbb{R}$.*

PROOF. Let us fix an arbitrary $c \in \mathbb{R}(n)$. By (3) and (4) we get

$$A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c) + A_1^{j_1}(c) \cap \dots \cap A_n^{j_n}(c) \subset A_1^{i_1 j_1}(c) \cap \dots \cap A_n^{i_n j_n}(c)$$

for every $i_1, \dots, i_n, j_1, \dots, j_n \in \{0, 1\}$ such that $(i_1 j_1, \dots, i_n j_n) \neq \underline{0}$ and $A_1^0(c) \cap \dots \cap A_n^0(c) = \emptyset$. The sets $A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c)$ are measurable and disjoint for different sequences of indices $i_1, \dots, i_n$, moreover the set of positive Lebesgue measure $D(c) := \{tc : t \in \mathbb{R}(1)\} = A_1(c) \cap \dots \cap A_n(c)$ is the sum of all such sets, thus there exists a sequence of indices $(i_1, \dots, i_n) \neq \underline{0}$ such that $A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c)$ is of positive measure.

We have

$$A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c) + A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c) \subset A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c),$$

therefore according to the theorem of STEINHAUS [7] there exists a segment of the half-line $D(c)$ having positive lenght and contained in $A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c)$. Since $A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c)$ is a cone

$$D(c) = A_1^{i_1}(c) \cap \dots \cap A_n^{i_n}(c) = Z_1^{i_1} \cap \dots \cap Z_n^{i_n} \cap D(c).$$

If $c \in Z_k$ then $i_k \neq 0$ and $Z_1^{i_1} \cap \dots \cap Z_n^{i_n} \cap D(c) \subset Z_k \cap D(c)$, therefore $Z_k \cap D(c) = D(c)$, hence $tc \in Z_k$ for every $t \in \mathbb{R}(1)$. Because $Z_k \cup \{\underline{0}\}$ is a cone, it follows that it is a cone over $\mathbb{R}$. □

Corollary. *A nearly measurable solution of the equation* (1) *satisfying* (5) *with some $r \neq 1$ must have its values in the set $0(n)$.*

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