On transitivity of abnormality

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Abstract. Some soluble groups in which abnormality is a transitive relation are studying in this article. A new criterion of abnormality is also obtained.

R. W. CARTER [C] introduced the notion of an abnormal subgroup in connection with his famous investigation of nilpotent self-normalizing subgroups in soluble finite groups. According to definition a subgroup A is called *abnormal* in a group G (A *ab* G) if $g \in \langle A, A^g \rangle$ for each element $g \in G$. Maximal non-normal subgroups of arbitrary groups obviously are abnormal. In finite groups we can also mention the Carter subgroups and the normalizers of Sylow subgroups (see the surveys [BB, Section 6] and [VN]). The famous Tits example provides us with the non-trivial abnormal subgroup of the complete linear group $GL(n, K)$ over an arbitrary skew field K (see, for example, [BB, Section 1]). Finite groups with many abnormal subgroups have been studied in [F]. Infinite such groups have been investigated in [S1], [S2], [DeFKS] and others.

It follows immediately from the definition that abnormal subgroups are self-normalizing, and every subgroup containing an abnormal subgroup is also abnormal, in particular self normalizing. According to the survey [BB] the later can be used as a definition of the weak abnormality (*wab*). In the soluble case the weak abnormality coincides with the abnormality itself [BB, Section 6].

It is appropriate to mention that neither normality nor abnormality are not transitive relations. The groups, in which normality is transitive

(T-groups) and some their generalizations have been study by many authors ([BT], [G], [R1], [R2], [R3], [R4], [H], [HL], [KS1], [M1], [M2] and others). These works established the wide variety of interesting and important results placing T-groups in the main stream of study of the normal structure of groups (see, for example, the survey [R5]). As we will show (see below Corollary 12) in the soluble case the class of T-groups is a proper subclass of the class of TA-groups (groups, in which abnormality is transitive). From this point of view the question about the TA-groups seems very actual.

We will start with some preliminary results concerning with abnormal subgroups, and with their help we will obtain some new properties of such subgroups in some classes of soluble groups. Also we will prove a new criterion of abnormality in these classes. In finite groups the existence of an abnormal subgroup in a group is equivalent to non nilpotency of the group. Note that in the infinite groups the situation is totally different. It seems quite difficult to describe even soluble infinite groups with no abnormal subgroups. So any criterion of abnormality seems very valuable. Next, we will select a wide subclass of soluble TA-groups (weakly central extension) and will study it more specific. In particular, we will prove that in this class for the periodic groups the non-existence of abnormal subgroups is equivalent to hypercentrality. It is remarkable to note that this class contains the class of all soluble periodic T-groups.

Lemma 1 (see, for example, [BB]). *Let G be a group, F a subgroup of G. Then the following assertions hold.*

1. *If A ab G (A wab G), then A is (weakly) abnormal in any intermediate subgroup of G.*

2. *Let H be a subgroup of a group G and $N \trianglelefteq G$, $N \leq H$. Then H ab G (H wab G) if and only if H/N ab G/N (H/N wab G/N).*

Lemma 2 [DeFKS, Lemma 4]. *Let G be a (soluble) group and H its subgroup. Then the following statements are equivalent.*

1. *H wab G (H ab G).*

2. *$D = H[D, D]$ for every intermediate subgroup D for H.*

A subgroup H is called *contranormal* in a group G A cn G if $A^G = G$, where A^G is the normal closure of A in G.

Corollary 3. *Let G be a (soluble) group and H its subgroup. Then the following statements are equivalent.*

1. *H wab G (H ab G).*

2. *H cn D for every intermediate subgroup D for H.*

The following results clarify some details of the structure of weakly abnormal subgroups and could be very useful in their study.

Proposition 4. 1. *Let M be a weakly abnormal subgroup of a group G, $\langle x \rangle$ a cyclic p-subgroup of G such that $M \leq N_G(\langle x \rangle)$. If $[M, \langle x \rangle] \neq \langle x \rangle$, then $x \in M$.*

2. *Let M be a weakly abnormal subgroup of a group G, $\langle x \rangle$ a cyclic infinite subgroup of G and $M \leq N_G(\langle x \rangle)$. Then $x \in M\langle x^2 \rangle$.*

PROOF. 1. Consider the group $F = \langle x \rangle M$. Since M wab G and $F \geq M$, M wab F by Lemma 1. It follows from Lemma 2 that $F = [F, F]M = [\langle x \rangle M, \langle x \rangle M]M$. Consider the commutators $[x_1 m_1, x_2 m_2]$, $x_i \in \langle x \rangle$, $m_i \in M$, $i = 1, 2$, generate $[F, F]$. Using the well known commutator equalities $[ab, c] = [a, c]^b [b, c]$ and $[a, b]^{-1} = [b, a]$ we obtain that $[F, F] = [\langle x \rangle, M][M, M]$. Therefore $F = [F, F]M = [\langle x \rangle, M][M, M]M = [\langle x \rangle, M]M$. Since $[M, \langle x \rangle] \neq \langle x \rangle$ and $M \leq N_G(\langle x \rangle)$, $[M, \langle x \rangle] < \langle x \rangle$.

Let $|x| = p^\alpha$, and consider the quotient group $\bar{F} = F/\langle x^{p^{\alpha-1}} \rangle = \langle \bar{x} \rangle \bar{M}$. For this quotient group the following inequality is valid $[\bar{M}, \langle \bar{x} \rangle] \neq \langle \bar{x} \rangle$. However, $\bar{x}$, as an element of order p, belongs to the center of $\bar{F}$. Therefore $[\bar{F}, \bar{F}] = [\bar{M}, \bar{M}]$. It is possible only in the case where $\bar{x} \in \bar{M}$, i.e. $x \in M$.

2. We consider the quotient group $\bar{F} = F/\langle x^2 \rangle = \langle \bar{x} \rangle \bar{M}$. It is obvious that $\bar{F} = \langle \bar{x} \rangle \times \bar{M}$. Using the above arguments we can conclude that $x \in M\langle x^2 \rangle$. □

Corollary 5. 1. *Let M be a weakly abnormal subgroup of a group G, $\langle x \rangle$ a cyclic normal subgroup of G whose order is a power of prime. If $Z(G) \cap \langle x \rangle \neq 1$, then $x \in M$.*

2. *Let M be a weakly abnormal subgroup of a group G, $\langle x \rangle$ a cyclic infinite normal subgroup of G. Then $x \in M\langle x^2 \rangle$.*

The following result is a useful criterion of abnormality.

Proposition 6. *Let G be a soluble group, A an abelian normal subgroup of G having no central chief G-factors and defining the quotient group G/A with no proper abnormal subgroups. A subgroup B is abnormal in G if and only if B is a supplement to A in G.*

PROOF. 1. Let G be a soluble group, A its abelian normal subgroup having no central chief G-factors, and the quotient group G/A has no abnormal subgroups. Let B be a supplement to A in G, i.e. $G = AB$. For any subgroup $M \geq B$ we can write $M = A_M B$, where $A_M = A \cap M$. Since A is an abelian normal subgroup in G and $G = AM$, $A_M \trianglelefteq G$. Since A has no central chief factors, $[M, A_M] = A_M$. Indeed, if $K = [M, A_M] \neq A_M$, then A_M/K is a G-central factor, which contradicts the conditions of our proposition. So $[M, A_M] \leq [M, M]$ and $M = [M, M]B$. By Lemma 2, B is abnormal in G.

2. Let G be a soluble group, A its abelian normal subgroup having no central chief G-factors, and the quotient group G/A has no abnormal subgroups. Let B be an abnormal subgroup in G. The subgroup AB/A is abnormal in G/A, which is possible only if $G = AB$ (see Lemma 1/2). $\square$

Proposition 7. *Let G be a soluble group, A an abelian normal subgroup of G having no central chief G-factors and defining the quotient group G/A with no proper abnormal subgroups. Then G is a TA-group.*

PROOF. Let R ab M ab G. Consider the subgroup AM. Lemma 1/1 implies that AM ab G. Then AM/A ab G/A by Lemma 1/2. Since G/A has no proper abnormal subgroups, the latter is possible only if $G = AM$.

Consider the subgroup $A_M = A \cap M$. The subgroup $A_M R$ is abnormal in M by Lemma 1/1. The quotient group G/A has no abnormal subgroups. Therefore $G/A = AM/A \cong M/A_M$ also has no abnormal subgroups. It follows that $A_M R = M$, and $G = AM = A(A_M R) = AR$.

Let F be a subgroup of G containing R. Since $G = AR$, $F = A_F R$ where $A_F = (A \cap F) \trianglelefteq F$. Consider the subgroup $K = [A_F, R]$. It is clearly that K is a normal subgroup in G. If $K \neq A_F$, then K/A_F is a G-central factor in A, which contradicts the proposition conditions. This means that $K = A_F$. Then $A_F \leq [F, F]$ and $F = A_F R = [F, F]R$. By Lemma 2 R ab G. $\square$

Remind that a group is called *semiabelian* if it is generated by its cyclic normal subgroups. These groups play a very important role in the theory

of finite supersoluble groups (see [V]). A subgroup H is called a *weakly central* in a group G if H is generated by normal in G cyclic subgroups [S3]. This group G is called a weakly central extension with the kernel H. Periodic weakly central extensions with the hypercentral quotient group by the kernel were the subject of investigation in the article [S3]. Using above results and results of [S3] we shall show that weakly central extensions with a hypercentral quotient group by the kernel are TA-groups.

Since the group of automorphisms of a cyclic group is abelian, the following lemma is valid.

Lemma 8. *Let a group G be a weakly central extension with the kernel A. Then $[G', A] = 1$. In particular, $A \cap G' \le Z(G')$ and $A \cap G$ is abelian.*

Suppose G is a weakly central extension with the kernel A and $L(A)$ will denote the subgroup of G generated by all cyclic primary normal in G subgroups from A that intersect by identity with the center $Z(G)$ of G.

Lemma 9. *Let a group G be a weakly central extension with the kernel A. Then*

1. *$L(A)$ is an abelian group without involution and belongs to G';*

2. *$L(A) \cap Z(G) = 1$.*

PROOF. 1. trivially follows from Lemma 8.

2. Assume the contrary. Let $1 \ne a \in Z(G) \cap L(A)$, $a^p = 1$ for some prime p, and $a = x^\alpha y^\beta \ldots z^\gamma$, where $\langle x \rangle, \langle y \rangle, \ldots, \langle z \rangle$ are normal cyclic p-primary subgroups from the set of generators of $L(A)$, having identity intersections with the center of G. Since $a \in Z(G)$, $\langle a \rangle$ intersects by identity with any of such subgroups. Consider the quotient group $\bar{G} = G/\langle x \rangle$. We can consider $\bar{G}$ as a weakly central extension with the kernel $\bar{A} = A/\langle x \rangle$. Since $1 \ne a \in Z(G) \cap L(A)$, $a^p = 1$ and $\langle x \rangle \cap \langle a \rangle = 1$, it follows that $\langle \bar{a} \rangle = \langle a \rangle \times \langle x \rangle / \langle x \rangle$ is a subgroup of order p in the center $Z(\bar{G})$. Making all needed changes in the notations, we can come without loss of generality to the situation, in which $\bar{a} = \bar{y}^\beta \ldots \bar{z}^\gamma$, where $\bar{y}^\beta = \langle x \rangle y^\beta, \ldots, \bar{z}^\gamma = \langle x \rangle z^\gamma$, $\langle \bar{y} \rangle \cap Z(\bar{G}) = \cdots = \langle \bar{z} \rangle \cap Z(\bar{G}) = 1$. Indeed, for elements $x, y, \ldots, z$ there exist respective p'-elements (in particular, it could be some elements of infinite order) $c_x, c_y, \ldots, c_z$ such that $\langle [x, c_x] \rangle = \langle x \rangle$, $\langle [y, c_y] \rangle = \langle y \rangle$, $\ldots$, $\langle [z, c_z] \rangle = \langle z \rangle$, and the corresponding relations remain valid for the

images in the quotient groups. Passing to the quotient group $\bar{G}/\langle\bar{y}\rangle$, and so on, in a finite numbers of steps we arrive to the situation, in which the weakly central extension $\tilde{G}$ contains an element $\tilde{a} = \tilde{z}^\gamma$ such that $\langle\tilde{z}\rangle \cap Z(\tilde{G}) = 1$ and $1 \ne \tilde{a} \in \langle\tilde{z}\rangle \cap Z(\tilde{G})$. The contradiction we reached proves the lemma. $\square$

Lemma 10. *Let G be a periodic weakly central extension with the kernel A. If G/A has no weakly abnormal subgroups, then:*

1. for any weakly abnormal subgroup M of G we can write $G = L(A)M$;

2. $G/L(A)$ also has no proper weakly abnormal subgroups.

Moreover, in the soluble case a subgroup B is abnormal in G if and only if $G = L(A)B$.

PROOF. 1. Let M be a weakly abnormal subgroup in G. Since G/A has no weakly abnormal subgroups, Lemma 1 implies that $G = AM$. It follows from Corollary 5 and Proposition 6 that $G = L(A)M$.

2. In $G/L(A)$ there is no finite normal cyclic subgroups among the images of the generators of A, which intersect the center by identity. Corollary 5 and Proposition 6 imply that every weakly abnormal subgroup from G/L includes A/L. If $X/L(A)$ such a subgroup, it follows that X is weakly abnormal in G and $G = L(A)X$. So $G/L(A) = L(A)X/L(A)$, and $G/L(A)$ has no weakly abnormal proper subgroups.

Proposition 6 directly implies the last assertion. $\square$

Theorem. 1. *Let G be a periodic group, A a weakly central subgroup of G, G/A a soluble group without proper weakly abnormal subgroups. Then G is a TA-group.*

2. *Let G be a periodic group, A a weakly central subgroup of G, G/A a hypercentral group. If G has no weakly abnormal subgroups, then G is hypercentral.*

PROOF. 1. Note that G is obviously soluble. If N is weakly abnormal in M, and M is weakly abnormal in G, then since $G/L(A) = L(A)M/L(A) = M/(L(A) \cap M)$ has no proper abnormal subgroups (Lemma 10), $M = (L(A) \cap M)N$, so $G = L(A)M = L(A)N$.

If $K \ge N$, then $K = (L(A) \cap K)N$. Denote $L(A) \cap K = L_K$. If $Q = [N, L_K] \ne L_K$, then $Q \trianglelefteq N, Q \trianglelefteq L$, and therefore $Q \trianglelefteq G$. In the quotient group G/Q, the subgroup $L(A)/Q$ obviously coincides with

$L(G/Q)$, has a non-trivial intersection with the center, which is impossible by Lemma 10. Hence $Q = L_K \leq [K, K]$. This means that $K = [K, K]N$ and it is follows that N is contranormal in K and therefore, is abnormal in G by Corollary 3.

2. By Theorem 1 from [S3] the subgroup $L(A)$ is the hypercentral residual of G. Let $\langle x \rangle$ be a cyclic p-subgroup from $L(A)$ such that $[\langle x \rangle, \langle y \rangle] = \langle y \rangle$ for some element $x \in G$. Such elements exist by Lemma 9. Let P be a subgroup from $L(A)$ generated by all normal cyclic p-subgroups from $L(A)$ intersecting with $C_G(x)$ by identity. By Theorem 2 from [S3] $G = P \rtimes R$ where R is a complement to P in G. Hence $G = L(A)R$. By Lemma 10, R is abnormal in G. This means that $G = R$. We obtain a contradiction proved that $L(A) = 1$. $\hfill\square$

Simple examples show that the second part of the Theorem does not valid for non-periodic groups.

Corollary 11. *An arbitrary periodic group in which every subgroup of the derived subgroup is normal (KI-group) is a TA-group.*

Corollary 12. *A soluble periodic T-group is a TA-group.*

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