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Static modules and Clifford theory for strongly graded rings

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. We use the concept of static module to obtain the direct (non-stable) Clifford correspondence for strongly graded rings in the case of simple and of indecomposable modules. These correspondences are compatible with induction and the main results of Clifford theory are easily obtained from here.

1. Introduction

Let G be a finite group, $R = \bigoplus_{x\in G} R_x$ a strongly G–graded ring, and consider a G–graded left R–module $M = \bigoplus_{x\in G} M_x$ which is finitely generated in R–mod. By definition, the x–suspension $M(x)$ of M is the G–graded R–module with $M(x)_y = M_{yx}$. Further, $E = \mathrm{End}_R(M)^{op}$ has a G–grading $E = \bigoplus_{x\in G} E_x$ where $E_x = \mathrm{Hom}_{R\text{-gr}}(M, M(x)) \simeq \mathrm{Hom}_{R_1}(M_y, M_{yx})$ for every $x, y \in G$. In this way M becomes a G–graded R, E–bimodule.

By a well-known result of E.C. Dade [5], [6], if M is gr–simple, the category (RgM)–mod of R–modules generated by M is equivalent with E–mod, via the functors $\mathrm{Hom}_R(M, -)$ and $M\otimes_E -$. Our objective is to obtain a direct Clifford correspondence in the case when M is gr–indecomposable. Let $J_{\mathrm{gr}}(E)$ be the graded Jacobson radical of E and let $D = E/J_{\mathrm{gr}}(E)$. Using the notion of static module and a graded version of Fitting's Lemma, we obtain the following theorem, which is the main result of the paper:

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Theorem. *Assume that M is gr–indecomposable of finite lenght in R–gr. Then the composite functor $D\otimes_E \mathrm{Hom}_R(M,-)$ induces an isomorphism between the Grothendieck groups associated to the category $(R|M)$*–mod *of R–modules which divide a finite direct sum of copies of M and to the category $(D|D)$*–mod *of finitely generated projective D–modules. This isomorphism is compatible with induction from subgroups, that is, for any subgroup H of G, the following diagram commutes to within natural equivalences of functors:*

$$\begin{array}{ccc} (R|M)\text{-mod} & \xrightarrow{D\otimes_E \mathrm{Hom}_R(M,-)} & (D|D)\text{-mod} \\ {\scriptstyle R\otimes_{R_H}-}\uparrow & & \uparrow{\scriptstyle D\otimes_{D_H}-} \\ (R_H|M_H)\text{-mod} & \xrightarrow{D_H\otimes_{E_H} \mathrm{Hom}_{R_H}(M_H,-)} & (D_H|D_H)\text{-mod} \end{array}$$

where $R_H=\bigoplus_{x\in H} R_x$ (and M_H, D_H are defined similarly).

Other Clifford type theorems are easy consequences of this theorem and the case of simple modules is also discussed.

2. Static modules and induction

Let A be a ring (associative with unit element), M a left A–module, and let $D=\mathrm{End}_A(M)^{op}$, so M is an A,D–bimodule.

1.1. Definition. a) An A–module V is M–static if the natural A–homomorphism $M\otimes_D \mathrm{Hom}_A(M,V)\to V$, $m\otimes f\mapsto mf=f(m)$ is an isomorphism.

b) A D–module W is called M–static if the natural D–homomorphism $W\to \mathrm{Hom}_A(M, M\otimes_D W)$, $w\mapsto f_w,\ f_w(m)=m\otimes w$ is an isomorphism.

Let $\mathcal{C}_0=(AsM)$–mod be the full subcategory of A–mod consisting of M–static A–modules, and (DsM)–mod the full subcategory of D–mod consisting of M–static D–modules.

For the following result see Alperin [1] and Nauman [15].

2.2. Theorem. a) *If M is finitely generated, then the A–module V is M–static iff there is an exact sequence $M^{(J)}\to M^{(I)}\to V\to 0$ such that the sequence $\mathrm{Hom}_A(M,M^{(J)})\to \mathrm{Hom}_A(M,M^{(I)})\to \mathrm{Hom}_A(M,V)\to 0$ is also exact.*

b) *There is an equivalence (AsM)*–mod $\rightleftharpoons$ *(DsM)*–mod , *given by the functors $\mathcal{H}=\mathrm{Hom}_A(M,-)$ and $\mathcal{T}=M\otimes_D -$.*

c) *If $\mathcal{C}$ is a full subcategory of (As,M)*–mod *and $\mathcal{H}(\mathcal{C})$ is the image of $\mathcal{C}$ under $\mathcal{H}$, then the restrictions of $\mathcal{H}$ and $\mathcal{T}$ give the equivalence $\mathcal{C}\rightleftharpoons\mathcal{H}(\mathcal{C})$.*

2.3. Remarks. a) M is an M–static A–module and D is an M–static D–module.

b) (AsM)–mod and (DsM)–mod are closed under direct summands and finite direct sums.

c) If M is finitely generated, then (AsM)–mod and (DsM)–mod are closed under arbitrary direct sums.

d) Let $\mathcal{C}_1 = (A|M)$–mod (respectively $\mathcal{C}_2 = (A\|M)$–mod) be the full subcategory of A–mod consisting of modules which divide a finite (respectively arbitrary) direct sum of copies of M. Then $\mathcal{C}_1 \subseteq (AsM)$–mod and $\mathcal{H}(\mathcal{C}_1)$ is the category of finitely generated projective D–modules. If M is finitely generated then $\mathcal{C}_2 \subseteq (AsM)$–mod and $\mathcal{H}(\mathcal{C}_2)$ is the category of projective D–modules.

e) Let $\mathcal{C}_3 = (AgM)$–mod $= \sigma[M]$ be the full subcategory of A–mod subgenerated by M. Assume that M is a finitely generated, self generator and projective object in (AgM)–mod. Then (AgM)–mod $= (AsM)$–mod and $\mathcal{H}(\mathcal{C}_3) = D$–mod.

This applies especially in the case when A is a G–graded ring and M is a gr–simple A–module (see [5], [6], [9]).

Consider now a unit-preserving homomorphism $A \to B$ of rings and let $N = B \otimes_A M$, $E = \mathrm{End}_B(N)^{op}$. Then the map $\varphi : D \to E$, defined by $\varphi(d)(b \otimes m) = b \otimes md = b \otimes d(m)$ is also a unit-preserving homomorphism of rings. Let $\mathcal{C}'_0 = (BsN)$–mod, $\mathcal{C}'_1 = (B|N)$–mod, $\mathcal{C}'_2 = (B\|N)$–mod and $\mathcal{C}_3 = (BgN)$–mod and consider the additive functors $B\otimes_A - : A$–mod $\to B$–mod and $E \otimes_D - : D$–mod $\to E$–mod.

The following lemma should be compared with [7], Proposition 3.10.

2.4. Lemma. *Suppose that if $V \in \mathcal{C}_i$ and $W \in \mathcal{H}(\mathcal{C}_i)$, then $B\otimes V \in \mathcal{C}'_i$ and $E \otimes_A W \in \mathcal{H}(\mathcal{C}'_i)$ $(i = 1, 2, 3)$. Then the following diagram commutes to within natural equivalences of functors:*

$$\begin{array}{ccc}
\mathcal{C}'_i & \xleftrightarrow{\ \mathrm{Hom}_B(N,-)\ } & \mathcal{H}(\mathcal{C}'_i) \\
{\scriptstyle B\otimes_A -}\big\uparrow & & \big\uparrow{\scriptstyle E\otimes_D -} \\
\mathcal{C}_i & \xleftrightarrow{\ \mathrm{Hom}_A(M,-)\ } & \mathcal{H}(\mathcal{C}_i)
\end{array}$$

The assumptions are fulfilled in the following situations:
a) $i = 1$*;*
b) $i = 2$ *and* M *is a finitely generated* A*–module;*
c) $i = 3$ *and* M *and* N *are finitely generated, self generator and projective in* (AgM)–mod *(respectively in* (BgN)–mod*).*

Proof. If $W \in \mathcal{H}(\mathcal{C}_i)$ then clearly $B \otimes_A (M \otimes_D W)$ and $N \otimes_E (E \otimes_D W)$ are naturally isomorphic B–modules, so the first assertion follows. The rest is also clear since the functors $B \otimes_A -$ and $E \otimes_D -$ are right exact and commute with direct sums.

2.5. Remarks. a) Assume that $N = B \otimes_A M$ is an M–static A–module. Then $\mathcal{C}'_0 = (BsN)$–mod is the category of B–modules which are M–statics as A–modules, $\mathcal{C}'_1$ (respectively $\mathcal{C}'_2$) is the category of B–modules which divide in A–mod a finite (respectively arbitrary) direct sum of copies of M.

Consequently, $\mathcal{H}(\mathcal{C}'_0)$ is the category of E–modules which are M–static when regarded as D–modules, $\mathcal{H}(\mathcal{C}'_1)$ (respectively $\mathcal{H}(\mathcal{C}'_2)$) is the category of E–modules which are finitely generated projective (respectively projective) as D–modules.

b) Let finally $\mathcal{C}'_4 = (BrM)$–mod be the category of B–modules U which have a presentation $B \otimes_A M^{(J)} \to B \otimes_A M^{(I)} \to U \to 0$ which splits in A–mod. Let $F = \operatorname{End}_A(N)^{op}$ so $E \subseteq F$. Then $\mathcal{C}'_4 \subseteq (BsN)$–mod and $\mathcal{H}(\mathcal{C}'_4)$ is the category of E–modules W for which $F \otimes_E W$ is projective in F–mod.

These facts are proved in [15] and in [1] and are related to the stable Clifford theory of [2].

3. Endomorphism rings of gr-indecomposable modules

We fix now a finite group G, a G–graded ring $R = \bigoplus_{x\in G} R_x$, and a finitely generated G–graded R–module $M = \bigoplus_{x\in G} M_x$. The x–suspension $M(x)$ of M is the G–graded R–module with $M(x)_y = M_{yx}$. Then we have that $E = \operatorname{End}_R(M)^{op} = \bigoplus_{x\in G} E_x$ is a G–graded ring with $E_x = \operatorname{Hom}_{R\text{-gr}}(M, M(x))$. Let

$$G(M) = \{x \in G \mid M \simeq M(x) \text{ in } R\text{-gr}\}$$

be the stabilizer of M. It is well-known that if M is gr–simple then $E = E_{G(M)} = \bigoplus_{x\in G(M)} E_x$ is a crossed–product of the skew–field $E_1 = \operatorname{End}_{R\text{-gr}}(M)$ with $G(M)$.

We shall consider the graded Jacobson radical $J_{\mathrm{gr}}(E)$ of E. It is also well-known that $J_{\mathrm{gr}}(E) \subseteq J(E)$.

We need the construction of the functor $R\bar{\otimes}_{R_H}-$ introduced in [5]. Let H be a subgroup of G and N an R_H–module. Then $R \otimes_{R_H} N$ is a G/H–graded R–module. Let $\operatorname{Soc}_H(R \otimes_{R_H} N)$ be the largest G/H–graded R–submodule of $R \otimes N$ with trivial H–component. This is called “the H–null socle” of $R \otimes N$. Then, by definition, $R\bar{\otimes}_{R_H} N = (R \otimes N)/\operatorname{Soc}_H(R \otimes N)$, which is also a G/H–graded R–module, and the functor $R\bar{\otimes}_{R_H}- : R_H\text{-mod} \to (G/H, R)$–gr can be defined and it is transitive.

3.1. Lemma. *If H is a subgroup of G, then $J_{\mathrm{gr}}(E_H) = J_{\mathrm{gr}}(E) \cap E_H$.*

Proof. Let $e \in J_{\mathrm{gr}}(E_H)$ and let $S = \bigoplus_{x\in G} S_x$ be a gr–simple left E–module. By the results of [7], for any $x \in G$, $S(x)_H = 0$ or $S(x)_H$ is a gr–simple R_H–module so $eS(x)_H = 0$. It follows that $eS = 0$ and consequently $e \in J_{\mathrm{gr}}(E)$.

Let now $e \in J_{\mathrm{gr}}(E)\cap E_H$ and let S be a gr–simple E_H–module. Again by [7], $S' = E\bar{\otimes}_{E_H} S$ is a gr–simple E–module with $S'_H \simeq S$. We have that $eS' = 0$ so $eS = 0$ and $e \in J_{\mathrm{gr}}(E_H)$.

We shall also need a graded version of Fitting's Lemma. Recall that if the G–graded R–module M has finite length in R–gr, then, by the structure off gr–simple modules, M has finite length in R–mod too.

3.2. Lemma. *Assume that M is gr–indecomposable and has finite length in R–gr. Then the following assertions hold:*

a) *Every homogeneous element of E is a unit or is nilpotent.*

b) *$E_1 = \mathrm{End}_{R\text{-gr}}(M)$ is a local ring and $D = E/J_{\mathrm{gr}}(E)$ is a $G(M)$–gr–field, that is, it is a crossed product of the skew–field $E_1/J(E_1) = (E/J_{\mathrm{gr}}(E))_1$ with $G(M)$.*

c) *We have that $J(E_1)E \subseteq J_{\mathrm{gr}}(E) = J(E_1)E_{G(M)} \otimes (\bigoplus_{x\notin G(M)} E_x)$ and D is naturally isomorphic to $D_1 \otimes_{E_1} E_{G(M)}$.*

Proof. a) Let $f \in E_x$, so $f : M \to M(x)$ is a grade-preserving R–homomorphism. It follows that $\mathrm{Ker}\, f$ and $f(N)$ are graded submodules of M for every graded submodule N of M. We prove that f is surjective if and only if it is injective.

Assume first that f is injective. We have the descending chain

$$M \supset f(M) \supset f^2(M) \supset \cdots \supset f^n(M) = f^{n+1}(M)$$

of graded submodules of M, so for every $u \in M$ there exists $v \in M$ such that $f^n(u) = f^{n+1}(v)$. It follows that $f(v) = u \in f(M)$ hence f is surjective.

Assume now that f is surjective. There exists $n \geq 1$ such that

$$0 \subset \mathrm{Ker}\, f \subset \cdots \subset \mathrm{Ker}\, f^n = \mathrm{Ker}\, f^{n+1}\,.$$

Let $u \in \mathrm{Ker}\, f$. Then there is $v \in M$ such that $u = f^n(v)$. But $f(u) = 0$ so $v \in \mathrm{Ker}\, f^{n+1} = \mathrm{Ker}\, f^n$, hence $u = 0$ and f is injective.

Now by the well-known argument we obtain that there is $n \geq 1$ such that $M \simeq f^n(M) \otimes \mathrm{Ker}\, f^n$ in R–gr. Since M is gr–indecomposable, we conclude that $f^n = 0$ or f is an isomorphism.

b) Let I be a maximal left ideal of E_1 and let $f \in I$ so $f^n = 0$ for some $n \geq 1$. Let $g \in E/I$. Then $Eg + I = E$ and $hg + f = 1$ for some $h \in E_1$. It follows that $hg(1+f+\cdots+f^{n-1}) = (1-f)(1+f+\cdots+f^{n-1}) = 1-f^n = 1$

hence g is a unit. This means that I is the set of all nonunits of E_1 so $J(E_1) = I$ and E_1 is local.

Let now I be a maximal graded left ideal of E and $f \in I$ a homogeneous element so $f^n = 0$ for some $n \geq 1$. Let $g \in E \setminus I$ be another homogeneous element. By the above argument we obtain that g is a unit. It follows that I is the graded ideal generated by the set of all homogeneous units of E, hence $J_{\mathrm{gr}}(E) = I$.

Clearly, if $x \in G/G(M)$ then $E_x \subseteq J_{\mathrm{gr}}(E)$. Therefore D is a crossed product of $E_1/J(E_1)$ with $G(M)$ since for $x \in G(M), E_x$ contains invertible elements.

c) We have that $J_{\mathrm{gr}}(E) \cap E_1 = J(E_1)$ and $J(E_1)E$ is a graded ideal of E, hence $J_{\mathrm{gr}}(E) \supseteq J(E_1)E$. Also, since $E_{G(M)}$ is a strongly graded ring,

$$J_{\mathrm{gr}}(E_{G(M)}) = J_{gr}(E) \cap E_{G(M)} = J(E_1)E_{G(M)} = E_{G(M)}J(E_1)\,,$$

and if $x \in G/G(M)$ then $E_x \subseteq J_{gr}(E)$, so

$$J_{\mathrm{gr}}(E) = J_{\mathrm{gr}}(E_{G(M)}) \oplus (\bigoplus_{x \notin G(M)} E_x) = J(E_1)E_{G(M)} \oplus (\bigoplus_{x \notin G(M)} E_x)\,.$$

Consider now the exact sequence $0 \to J(E_1) \to E_1 \to D_1 \to 0$ and apply the functor $- \otimes_{E_1} E_{G(M)}$. It follows that

$$D_1 \otimes_{E_1} E_{G(M)} \simeq E_1 \otimes_{E_1} E_{G(M)}/J(E_1) \otimes_{E_1} E_{G(M)} \simeq$$
$$\simeq E_{G(M)}/J(E_1)E_{G(M)} \simeq D\,. \qquad \square$$

As in the first section, let $(E|E)$–mod (respectively $(D|D)$–mod) be the category of finitely generated projective E–modules (respectively D–modules). Under the hypothesis of the above lemma, $J_{\mathrm{gr}}(E)$ is a nilpotent ideal. Since $J_{\mathrm{gr}}(E) \subseteq J(E)$, the idempotents of D can be lifted modulo $J_{\mathrm{gr}}(E)$. Also, if $P \in (E|E)$–mod then $P/J_{\mathrm{gr}}(E)P$ is naturally isomorphic to $D \otimes_E P$.

If $\mathcal{C}$ is a category of modules, we denote by $K(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$.

3.3. Proposition. *Assume that M is gr–indecomposable and has finite lenght. Then we have:*

a) *The addditive functor $D \otimes_E - : (E|E)\text{–mod} \to (D|D)\text{–mod}$ induces an isomorphism $K((E|E)\text{–mod}) \simeq K((D|D)\text{–mod})$ of groups.*

b) *If H is a subgroup of G then the following diagram commutes to within naturel equivalences of functors:*

$$\begin{array}{ccc} (E|E)\text{-mod} & \xrightarrow{D\otimes_E -} & (D|D)\text{-mod} \\ {\scriptstyle E\otimes_{E_H}-}\uparrow & & \uparrow{\scriptstyle D\otimes_{D_H}-} \\ (E_H|E_H)\text{-mod} & \xrightarrow{D_H\otimes_{E_H}-} & (D_H|D_H)\text{-mod} \end{array}$$

c) *If $H = G(M)$ then the additive functor $E\otimes_{E_H}$ — induces the isomorphism $K((E_H|E_H)\text{–mod}) \simeq K((E|E)\text{–mod})$.*

Proof. a) follows from [8], Proposition 22.15.

b) By Lemma 3.1, $D_H = (E/J_{\text{gr}}(E))_H = E_H/J_{\text{gr}}(E)_H = E_H/J_{\text{gr}}(E_H)$. Therefore, if $P \in (E_H|E_H)$–mod then we have the natural isomorphism of D–modules $D \otimes_{D_H} (D_H \otimes_{E_H} P) \simeq D \otimes_E (E \otimes_{E_H} P)$.

c) By Lemma 3.3, $D = D_{G(M)}$, so the assertion follows immediately from a) and b).

4. Direct Clifford theory for strongly graded rings

Assume that G is a finite group and $R = \bigoplus_{x\in G} R_x$ is a strongly G–graded ring. Further, let $M = \bigoplus_{x\in G} M_x$ be a gr–indecomposable R–module of finite lenght, so $M \simeq R \otimes_{R_1} M_x$ where M_1 is an R_1–module of finite lenght. Also $G(M) = \{x \in G \mid M_1 \simeq R_x \otimes_{R_1} M_1 \text{ in } R_1\text{-mod}\}$. As in the previous section let $E = \text{End}_R(M)^{op}$ and $D = D_{G(M)} = E/J_{\text{gr}}(E)$. If M_1 is simple then $D = E = E_{G(M)}$.

If M is not gr–simple let $\mathcal{C}_G = (R|M)$–mod and $\mathcal{D}_G = (D|D)$–mod and if M is gr–simple let $\mathcal{C}_G = (RgM)$–mod and $\mathcal{D}_G = D$–mod $= e$–mod. Then clearly $\mathcal{C}_H(R_H|M_H)$–mod and $\mathcal{D}_H = (D_H|D_H)$–mod, or $\mathcal{C}_H = (R_H g M_H)$–mod and $\mathcal{D}_H = E_H$–mod $= D_H$–mod, respectively.

We can now state the direct and two–step Clifford Theorem.

4.1. Theorem. *With the above notations, the following assertions hold:*

a) *The functor $D \otimes_E \text{Hom}_R(M, -)$ induces an isomorphism $K(\mathcal{C}_G) \simeq K(\mathcal{D}_G)$ of Grothendieck groups.*

b) *For every subgroup H of G, the following diagram commutes to within natural equivalences of functors:*

$$\begin{array}{ccc} \mathcal{C}_G & \xrightarrow{\mathcal{H}_G} & \mathcal{D}_G \\ {\scriptstyle R\otimes_{R_H}-}\uparrow & & \uparrow{\scriptstyle D\otimes_{D_H}-} \\ \mathcal{C}_H & \xrightarrow{\mathcal{H}_H} & \mathcal{D}_H \end{array}$$

c) *The functor* $R \otimes_{R_{G(M)}} - : \mathcal{C}_{G(M)} \to \mathcal{C}_G$ *induces an isomorphism of the Grothendieck groups* $K(\mathcal{C}_{G(M)})$ *and* $K(\mathcal{C}_G)$*; the inverse of this isomorphism is induced by the truncation functor* $(-)_{G(M)} : (G/G(M), R)\text{-gr} \to R_H\text{-mod}$*. In particular, every object of* $\mathcal{C}_G$ *is* $G/G(M)$*–gradable. Also, the following diagram commutes:*

d) *If* H *is a subgroup of* G *then* $H(M) = H \cap G(M)$ *and the following diagram commutes:*

$$\begin{array}{ccccc} K(\mathcal{C}_G) & \longleftrightarrow & K(\mathcal{C}_{G(M)}) & \xrightarrow{\mathcal{H}_{G(M)}} & K(\mathcal{D}_{G(M)}) \\ \uparrow & & \uparrow & & \uparrow \\ K(\mathcal{C}_H) & \longleftrightarrow & K(\mathcal{C}_{H(M)}) & \xrightarrow{\mathcal{H}_{H(M)}} & K(\mathcal{D}_{H(M)}) \end{array}$$

Proof. a) follows from Theorem 2.2, Remarks 2.3.d) and e) and Proposition 3.3.a.).

b) follows from Lemma 2.4 and Proposition 3.3.b).

c) is a consequence of a), b) and Lemma 3.2 and d) follows easily from a), b) and c).

4.2. Remarks. a) If M is gr–simple we do not need to pass to Grothendieck groups; in this case $\mathcal{H}_G, \mathcal{H}_H$ and $R \otimes_{R_{G(M)}} -$ are equivalences of categories.

b) If M is not gr–simple and $P \in (D|D)$–mod, then the corresponding object $N \in (R|M)$–mod may be obtained as follows: let $d \in D$ be a primitive idempotent such that $P \simeq Dd$ and let $e \in E$ be a primitive idempotent such that $d = e + J_{\text{gr}}(E)$. Then $N \simeq M \otimes_E Ee \simeq R \otimes_{R_{G(M)}} M_{G(M)}e$ since we clearly have that $e \in E_{G(M)}$.

The following result is an immediate consequence of Theorem 4.1.c).

4.3. Corollary. *Let* $M_{G(M)} = \bigoplus_{i=1}^s N_i$ *be a decomposition of the induced module* $M_{G(M)} = R_{G(M)} \bigoplus_{R_1} M_1$ *into indecomposable* $R_{G(M)}$*–modules corresponding to a decomposition* $D = \otimes_{i=1}^s P_i$ *of* D *into principal indecomposable modules. Then:*

a) $M \simeq \bigoplus_{i=1}^s R \otimes_{R_{G(M)}} N_i$ *is a decomposition of* M *into indecomposable* R*–modules and* $R \otimes N_i \simeq R \otimes N_j$ *implies* $N_i \simeq N_j$*.*

b) *If M is gr–simple then M is a semisimple R–module if and only if $E = E_{G(M)}$ is a semisimple ring.*

4.4. Corollary. *Let N be an R–module such that either $N \in (R|M)$–*mod *is an indecomposable R–module if M is not gr–simple, or $N \in (RgM)$*–mod *is a simple R–module if M is gr–simple. Then the following assertions hold:*

a) *${}_{R_1}N$ is a direct sum of R_1–modules conjugate to M_1 such that each type of isomorphism appears with the same multiplicity e.*

b) *There exists an indecomposable $R_{G(M)}$–submodule N' of N such that ${}_{R_1}N'$ is the sum of R_1–submodules of N isomorphic to M_1 and $N \simeq \otimes_{R_{G(M)}} N'$.*

c) *The multiplicity e equals* $\dim_{D_1}(D \otimes_{E_{G(M)}} \operatorname{Hom}_{R_{G(M)}}(M_{G(M)}, N'))$.

Proof. a), b) In the first case N has finite lenght in R_1–mod and the assumption of indecomposability implies that N divides M in R–mod, so by the Krull-Schmidt Theorem ${}_{R_1}N$ is a direct sum of conjugates of M_1. In the second case, since N is a simple R–module, it is a factor module of $M = R \otimes_{R_1} M_1$ which is a semisimple R_1–module having all the components conjugated to M_1.

By Theorem 4.1.c), there is an indecomposable in the first case, respectively simple in the second case $R_{G(M)}$–module U such that $N \simeq R \otimes_{R_{G(M)}} U$. It follows that ${}_{R_1}N \simeq \bigoplus_{i=1}^{t} R_{x_iG(M)} \otimes_{R_{G(M)}} U \simeq \bigoplus_{i=1}^{t} R_{x_i} \otimes_{R_1} U$ where $\{x_i \mid i = 1, \dots, t\}$ is a transversal for the left cosets of $G(M)$ in G. Clearly ${}_{R_1}U$ is homogeneous and by the uniqueness asserted in the Krull–Schmidt Theorem it is isomorphic to the sum of all R_1–submodules of N isomorphic to M_1, so $U \simeq N'$. Since $R_{x_i} \otimes_{R_1}$ – is an autoequivalence of R_1–mod, $R_{x_i} \otimes_{R_1} N'$ has the same number of components as N'.

c) We have that N' and $M_{G(M)}$ are finite direct sums of copies of M_1 and

$$\begin{aligned} \operatorname{Hom}_{R_H}(M_H, N') &\simeq \operatorname{Hom}_{R_H}(R_H \otimes_{R_1} M_1, N') \simeq \operatorname{Hom}_{R_1}(M_1, N') \\ &\simeq \operatorname{Hom}_{R_1}(M_1, eM_1) \simeq e \operatorname{Hom}_{R_1}(M_1, M_1) \simeq E_1^e \end{aligned}$$

in E_1–mod, where $H = G(M)$.

We also have that $D = D_H$ and

$$\begin{aligned} D \otimes_{E_H} \operatorname{Hom}_{R_H}(M_H, N') &\simeq (E_1/J(E_1) \otimes_{E_1} E) \otimes_E \operatorname{Hom}_{R_H}(M_H, N') \\ &\simeq E_1/J(E_1) \otimes_{E_1} \operatorname{Hom}_{R_1}(M_1, eM_1) \simeq D_1^e \end{aligned}$$

in D_1–mod.

4.5. Remarks. a) Any simple R–module N contains a simple R_1–submodule S, hence $N \in (RgR \otimes_{R_1} S)$–mod. Also, a simple (respectively indecomposable) R–module $N \in (RgM)$–mod (respectively $(R|M)$–mod) is just a simple (respectively indecomposable) object of (RgM)–mod (respectively $(R|M)$–mod).

b) We may also see the effect of conjugation on the functor $\mathcal{H}_G$. Let $y \in G$, $H = G(M)$, ${}^yH = yYy^{-1}$, and let N be an R–module as in Corollary 4.4. Then $\mathrm{End}_R(M(y))^{op} = {}^yE$ where $({}^yE)_x = E_{y^{-1}xy}$ for every $x \in G$, and the stabilizer of $M(y)$ is ${}^yH$ so ${}^yE_H = ({}^yE)_{{}^yH}$. With these notations, and using Lemma 3.2.c) and Fitting's Lemma (or Schur's Lemma if M_1 is simple), we obtain the isomorphisms

$$\begin{aligned}
&{}^yD\otimes_{{}^y(E_H)} \mathrm{Hom}_R(M(y), N)\\
&\qquad \simeq ({}^yD_1 \otimes_{{}^yE_1} {}^y(E_H)) \otimes_{{}^y(E_H)} \mathrm{Hom}_{R_H}(M(y)_{{}^yH}, N)\\
&\qquad \simeq D_1 \otimes_{E_1} \mathrm{Hom}_{R_1}(M_y, N)\\
&\qquad \simeq D_1 \otimes_{E_1} \mathrm{Hom}_{R_1}(M_y, \bigoplus_{i=1}^{t} R_{x_i} \otimes_{R_1} N')\\
&\qquad \simeq D_1 \otimes_{E_1} \mathrm{Hom}_{R_1}(M_y, R_y \otimes_{R_1} eM_1)\\
&\qquad \simeq eD_1 \otimes_{E_1} \mathrm{Hom}_{R_1}(M_y, M_y) \simeq D_1^e\,.
\end{aligned}$$

4.6. Corollary. (Dade) *The indecomposable R_1–module M_1 can be extended to an $R_{G(M)}$–module $M_1' \in \mathcal{C}_{G(M)}$ if and only if*

(i) *there exists a D–module P with $\dim_{D_1} P = 1$ in case M_1 is simple,*

(ii) *there exists a projective D–module P with $\dim_{D_1} P = 1$ in case M_1 is not simple.*

4.7. Remarks. a) D has a module P of dimension 1 over D_1 if and only if the group–extension $D_1^* \rightarrowtail hU(D) \twoheadrightarrow G(M)$ splits, and two such D–modules are isomorphic if and only if the corresponding splittings are D_1^*–conjugate. Since D_1 is a skew–field, P is a projective D–module if and only if it is D_1–projective or equivalently if and only if $J(D)P = 0$. This holds when $J(E_{G(M)}) = J_{gr}(E_{G(M)})$ or when $|G(M)|$ is invertible in E_1. In this case, if $\alpha : G(M) \to hU(D)$ is a splitting, then $P \simeq Dd$ where d is the idempotent of D given by $d = |G(M)|^{-1} \sum_{x\in G(M)} \alpha_x$ (see also [3], Corollary 2.14 and [4], Theorem 2.8).

b) Let K be a field of charastericitc $p > 0$, $N \rightarrowtail H \twoheadrightarrow G$ an extension of p–groups, and consider the trivial KN–module 1_N. This module is G–invariant and $\mathrm{End}_{KN}((1_N)^H)^{op} \simeq KG$. The trivial KG–module has dimension 1 over K and it is not a projective KG–module. It follows that the extension 1_H of 1_N to KH is not an N–projective KH–module.

The following corollary generalises [13], Theorem 3 and [10], Proposition 1.

4.8. Corollary. *Assume that $G(M)$ is a p–group and* $\operatorname{char} D_1 = p$. *Then M is a homogeneous (isotypic) R–module, and*

(i) *if M_1 is simple then all the simple modules of* (RgM)–mod *are isomorphic;*

(ii) *if M_1 is not simple then all the indecomposable R–modules of* $(R|M)$–mod *are isomorphic.*

Proof. By [12], Theorem 2, D is an Artinian ring such that $D/J(D)$ is simple, hence all the simple D–modules, respectively all projective indecomposable D–modules, are isomorphic, and the corollary follows from Theorem 4.1.

4.9. Remark. Assume that R is a finite dimensional K–algebra where $K \subseteq R_1$ is a perfect field of characteristic $p \geq 0$ and let M_1 be absolutely indecomposable, that is $D_1 = K$. Then there is $\alpha \in Z^2(G(M), K^*)$ such that $D \simeq K^\alpha G(M)$ as $G(M)$–graded K–algebras. If $G(M)$ is a p–group then D is a local ring and M and $M_{G(M)}$ are absolutely indecomposable modules. Also, every simple R–module $N \in (RgM)$–mod is a simple R_1–module and corresponds to the trivial $KG(M)$–module. Conversely, if K is algebraically closed and $K^\alpha G(M)$ is a local ring, then $G(M)$ is a p–group.

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