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Riemannian submersions and slant submanifolds

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Abstract. We study the relationship between slant submanifolds in both Complex and Contact Geometry through Riemannian submersions. We present some construction procedures to obtain slant submanifolds in the unit sphere and in a Stiefel manifold. We also generalize them by means of the Boothby–Wang fibration. Finally, we show some characterization theorems of three-dimensional slant submanifolds.

0. Introduction

The geometry of slant submanifolds has been increasingly studied since B.-Y. Chen defined slant immersions in complex manifolds as a natural generalization of both holomorphic and totally real immersions (see [7]). Later, a similar notion of slant submanifold was introduced in Contact Geometry, which is specially important for submanifolds tangent to the structure vector field of a contact metric manifold. The purpose of the present paper is to study the close relationship between both theories through Riemannian submersions.

In particular, we prove that, in some conditions, a submanifold of an almost Hermitian manifold is slant if and only if its lift by a Riemannian submersion is a slant submanifold of an almost contact metric manifold.

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We use this result as a method to find interesting examples. Examples of proper slant submanifolds of a Sasakian-space-form of constant ϕ-sectional curvature c have been given in [4], [10] ($c = -3$) and [6] ($c < -3$), but, until now, there were no examples in a Sasakian-space-form with $c > -3$. In fact, in this paper we exhibit a construction procedure to obtain examples of slant submanifolds in the unit sphere with its usual Sasakian structure ($c = 1$). Afterwards, we extend it in order to get ample examples in Sasakian-space-forms with constant ϕ-sectional curvature c, for any $c > -3$.

Moreover, we also construct examples of slant immersions into a Stiefel manifold and we generalize both procedures by using the Boothby–Wang fibration. Finally, we present some classifications of three-dimensional slant submanifolds of $\mathbb{R}^5$, by attending to their second fundamental form.

1. Preliminaries

In this section, we recall some basic formulas and definitions about slant submanifolds in both Complex and Contact Geometry, which we shall use later. For details and background on complex and contact manifolds, we refer to the standard references [1], [13].

A submanifold N of an almost Hermitian manifold $(\widetilde{N}, g, J)$ is said to be *slant* [7] if for each nonzero vector X tangent to N at p, the angle $\theta(X)$, $0 \leq \theta(X) \leq \pi/2$, between JX and T_pN is a constant, called the *slant angle* of the submanifold. In particular, holomorphic and totally real submanifolds appear as slant submanifolds with slant angle 0 and $\pi/2$, respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real. In the case where N is a Riemann surface and $\widetilde{N}$ is a Kaehler manifold, S. S. Chern and J. G. Wolfson introduced the notion of *Kaehler angle*, defined to be the angle between $J\partial/\partial x$ and $\partial/\partial y$, where $z = x + \sqrt{-1}y$ is a local complex coordinate on N [9]. It is clear that if N is a surface with constant Kaehler angle α, then it is a slant submanifold with slant angle θ satisfying $\theta = \alpha$ (resp. $\theta = \pi - \alpha$) when $\alpha \in [0, \pi/2]$ (resp. $\alpha \in (\pi/2, \pi]$).

Put $JX = PX + FX$, for any tangent vector field X, where PX (resp. FX) denotes the tangential (resp. normal) component of JX. Then, θ-slant submanifolds are characterized by the formula:

$$P^2 = -\cos^2\theta \, \mathrm{Id}\,.$$

A special type of proper slant submanifold is that of *Kaehlerian slant* submanifold, i.e., a proper slant submanifold satisfying $\nabla' P = 0$, where ∇' denotes the Levi–Civita connection on N. It is easy to show that a Kaehlerian slant submanifold is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $(\sec\theta)P$.

In a similar way, given a submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, G)$, it is said to be *slant* [4] if the angle $\theta(X)$ between ϕX and T_pM is a constant, which is independent of the choice of $p \in M$ and $X \in T_p(M) \setminus \mathrm{Span}(\xi_p)$. In particular, for $\theta = 0$ (resp. $\theta = \pi/2$) we obtain the invariant (resp. anti-invariant) submanifolds. Now, if we denote by TX (resp. NX) the tangential (resp. normal) component of ϕX, there is an equation which characterizes θ-slant submanifolds:

$$T^2 = -\cos^2\theta(\mathrm{Id} - \eta \otimes \xi).$$

In contact geometry, the similar notion to Kaehlerian slant submanifolds is given by proper θ-slant submanifolds satisfying

$$(\nabla_X T)Y = \cos^2\theta(g(X,Y)\xi - \eta(Y)X),$$

for any tangent vector fields X, Y, where ∇ denotes the Levi–Civita connection on M. This non-trivial fact is shown in [4]. Therefore, by following the complex case notation, we call such a submanifold a *Sasakian slant* submanifold. On the other hand, the possibility of obtaining an induced contact metric structure on a slant submanifold of a contact metric manifold is studied in [5].

2. Main results

Let $\widetilde{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, G) and $\widetilde{N}$ be a real $2m$-dimensional almost Hermitian manifold with structure (J, g). Let suppose that there exists a Riemannian submersion $\pi : \widetilde{M} \to \widetilde{N}$ satisfying the conditions:

i) The vertical subspace $\mathcal{V}_p$ of the submersion at $p \in \widetilde{M}$ is equal to the span of ξ_p,

ii) $\phi X^* = (JX)^*$,

for any vector field X on $\widetilde{N}$, where * denotes the horizontal lift with respect to π. In fact, since π is a Riemannian submersion, we also have

iii) $G(X^*, Y^*) = g(X, Y)$,

for any vector fields X, Y on $\widetilde{N}$.

Now, let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of $\widetilde{M}$ and N be an n-dimensional submanifold of $\widetilde{N}$. Throughout in this section we assume that the following diagram commutes

$$\begin{array}{ccc} M & \longrightarrow & \widetilde{M} \\ \big\downarrow & & \big\downarrow \pi \\ N & \longrightarrow & \widetilde{N} \end{array} \tag{2.1}$$

where M is the set of fibres over N.

Then, we state the following theorem:

Theorem 2.1. *In the above conditions, we have:*

(a) *M is θ-slant in $\widetilde{M}$ if and only if N is θ-slant in $\widetilde{N}$.*

Moreover, if $\widetilde{M}$ is a Sasakian manifold, we also have:

(b) *M is Sasakian θ-slant in $\widetilde{M}$ if and only if N is Kaehlerian θ-slant in $\widetilde{N}$.*

Proof. Statement (a) follows directly from i)–iii). In fact, in any almost contact metric manifold, $\phi\xi = 0$, and then, the condition of M being a slant submanifold is really related to its contact distribution, which is the horizontal subspace of the submersion at any point.

Now, suppose that $\widetilde{M}$ is a Sasakian manifold and denote by ∇ (resp. ∇') the Levi–Civita connection on M (resp. N). It follows from the well-known O'Neill equations of the submersion that

$$\nabla_{X^*} Y^* = (\nabla'_X Y)^* + \eta(\nabla_{X^*} Y^*)\xi, \quad \eta(\nabla_{X^*} Y^*) = -G(X^*, TY^*),$$

for any vector fields X, Y on $\widetilde{N}$ tangent to N. Then, we have

$$(\nabla_{X^*} T)Y^* = ((\nabla'_X P)Y)^* - G(X^*, T^2 Y^*)\xi, \quad (\nabla_\xi T)Y^* = 0,$$

which imply (b). $\square$

Notice that, in particular, statement (a) of Theorem 2.1 implies statements (3) and (4) of [13, Proposition 3.2, p. 459].

By using Theorem 2.1, we can show the following construction procedure for giving examples of proper slant submanifolds in the unit sphere.

Let $\pi : \mathbb{S}^{2m+1} \to \mathbb{CP}^m(4)$ be the well-known Hopf fibration, where $\mathbb{S}^{2m+1}$ (resp. $\mathbb{CP}^m(4)$) is endowed with its usual Sasakian (resp. Kaehlerian) structure. Given any isometric immersion $f : N \to \mathbb{CP}^m(4)$, then $M = \pi^{-1}(N)$ is a principal circle bundle over N with totally geodesic fibres and the lift $\hat{f} : M \to \mathbb{S}^{2m+1}$ of f is an isometric immersion such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\hat{f}} & \mathbb{S}^{2m+1} \\ \downarrow & & \downarrow \; \pi \\ N & \xrightarrow{f} & \mathbb{CP}^m(4). \end{array}$$

It follows from Theorem 2.1 that, in order to obtain a θ-slant submanifold of $\mathbb{S}^{2m+1}$, it is enough to consider a θ-slant submanifold of $\mathbb{CP}^m(4)$. For example, we could take the examples given in [11]. The above procedure was first pointed out by B.-Y. Chen and Y. Tazawa in [8].

We can also consider the lift of the Veronese sequence in order to get new examples of proper slant immersions into $\mathbb{S}^{2m+1}$. We recall that the Veronese sequence $\psi_0, \dots, \psi_m$ is defined, for any $p = 0, \dots, m$, by

$$\psi_p : \mathbb{S}^2 \to \mathbb{CP}^m : \psi_p[z_0, z_1] = [g_{p,0}(z_0/z_1), \dots, g_{p,m}(z_0/z_1)],$$

where $[z_0, z_1] \in \mathbb{CP}^1 = \mathbb{S}^2$, and

$$g_{p,j}(z) = \frac{p!}{(1+z\overline{z})^p} \sqrt{\binom{m}{j}} z^{j-p} \sum_k (-1)^k \binom{j}{p-k} \binom{m-j}{k} (z\overline{z})^k,$$

for any $j = 0, \dots, m$. It was shown in [2] that every ψ_p is a conformal minimal immersion with constant curvature and constant Kaehler angle α_p such that

$$\tan^2 \frac{\alpha_p}{2} = \frac{p(m-p+1)}{(p+1)(m-p)}.$$

By combining this procedure and a $\mathcal{D}$-homothetic deformation, we may also obtain the following theorem, similar to [6, Theorem 3.5]:

Theorem 2.2. *Let c be a constant with $c > -3$. Then, there exist proper slant submanifolds in a Sasakian-space-form with constant ϕ-sectional curvature c.*

Proof. First, we can choose a proper slant submanifold of $\mathbb{S}^{2m+1}$, given by the above construction procedure. We denote the usual Sasakian structure on $\mathbb{S}^{2m+1}$ by (ϕ, ξ, η, G). Then, for any $c > -3$, we consider the constant $a = 4/(c+3) > 0$ and the $\mathcal{D}$-homothetic deformation:

$$\widetilde{\phi} = \phi, \quad \widetilde{\xi} = \frac{1}{a}\xi, \quad \widetilde{\eta} = a\eta, \quad \widetilde{G} = aG + a(a-1)\eta \otimes \eta.$$

It was shown in [1] that $\mathbb{S}^{2m+1}$ with this structure is a Sasakian-space-form with constant ϕ-sectional curvature $(4/a) - 3 = c$.

Finally, it is easy to prove that a $\mathcal{D}$-homothetic deformation maps slant submanifolds into slant submanifolds. □

A more elaborate construction procedure for obtaining slant submanifolds in a certain almost contact metric manifold can be shown as follows. Let H be the closed connected subgroup in $\mathbb{S}^3 \times \mathbb{S}^3$ given by $H = \{(z, z) : z \in \mathbb{S}^1\}$ and consider the homogeneous space $(\mathbb{S}^3 \times \mathbb{S}^3)/H$. Since this is a compact simply connected 5-dimensional spin manifold with $H^2((\mathbb{S}^3 \times \mathbb{S}^3)/H; \mathbb{Z}) = \mathbb{Z}$, it follows from a classic result of Smale [12] that it is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$. On the other hand, if we denote by $V(2, 4)$ the Stiefel manifold of orthonormal 2-frames in 4-space, it is known that $V(2, 4)$ is diffeomorphic to $(\mathbb{S}^3 \times \mathbb{S}^3)/H$ and then, there is a diffeomorphism $f : V(2, 4) \to \mathbb{S}^2 \times \mathbb{S}^3$. Let $\widetilde{\pi} : \mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf fibration and put $F = (\mathrm{id} \times \widetilde{\pi}) \circ f$. Hence, $F : V(2, 4) \to Q_2$ is a submersion, where Q_2 denotes the complex quadric $\mathbb{S}^2 \times \mathbb{S}^2$. Now, put $\mathbb{S}^2_* = \mathbb{S}^2 \setminus \{(0, 0, 1)\}$ and let $E : \mathbb{S}^2_* \to \mathbb{C}$ be the corresponding stereographic projection, which preserves the complex structure of $\mathbb{C}$.

Then, $V(2, 4)_* \to Q_{2*} \to \mathbb{C}^2$ is a submersion, where Q_{2*} (resp. $V(2, 4)_*$) denotes the manifold $\mathbb{S}^2_* \times \mathbb{S}^2_*$ (resp. $F^{-1}(Q_{2*})$). It is clear that, if we consider on $\mathbb{C}^2$ its usual Kaehler structure, $V(2, 4)_*$ can be endowed with a natural almost contact metric structure such that $(E, E) \circ F|_{V(2,4)_*}$ is a Riemannian submersion satisfying the above stated conditions i)–ii). Hence, we obtain ample examples of slant surfaces in $V(2, 4)_*$ by considering the lifts of slant surfaces in $\mathbb{C}^2$ (see, for instance, [7]).

Moreover, we can give a generalization of the previous construction procedures. Let $\widetilde{M}$ be a $(2m + 1)$-dimensional compact regular contact manifold. According with a classical result of Boothby–Wang [3], one can see $\widetilde{M}$ as a circle bundle over a $2m$-dimensional compact symplectic manifold $\widetilde{N}$:

$$\pi : \widetilde{M} \longrightarrow \widetilde{N}.$$

Since $\widetilde{N}$ carries a global symplectic form Ω, there exist a Riemannian metric g and a tensor field J of type $(1,1)$ such that (g,J) is an almost Kaehler structure on $\widetilde{N}$ with Ω as its fundamental 2-form. Denote by η the contact form on $\widetilde{M}$ with $\mathrm{d}\,\eta = \pi^*\Omega$ and ξ its characteristic vector field and define a tensor field ϕ and a Riemannian metric G on $\widetilde{M}$ by $\phi X = (J\pi_* X)^*$ and $G = \pi^*(g) + \eta \otimes \eta$, respectively. Then, it can be proved that (ϕ, ξ, η, G) is a K-contact structure on $\widetilde{M}$ and $\pi : (\widetilde{M}, G) \to (\widetilde{N}, g)$ is a Riemannian submersion. Now, we have:

Theorem 2.3. *In the above conditions, let M be a submanifold of $\widetilde{M}$. Then, M is a $\mathbb{S}^1$-invariant θ-slant submanifold if and only if $M = \pi^{-1}(N)$, where N is a θ-slant submanifold of $\widetilde{N}$.*

Proof. Let N be a submanifold of $\widetilde{N}$ and denote by $M = \pi^{-1}(N)$. Then, M is a submanifold of $\widetilde{M}$ and the characteristic vector field ξ is tangent to M, in particular, M is $\mathbb{S}^1$-invariant.

The converse of the above stated fact also holds, that is, if M is a $\mathbb{S}^1$-invariant submanifold of $\widetilde{M}$, then ξ is tangent to M and there exists a submanifold N in $\widetilde{N}$ with $M = \pi^{-1}(N)$. Hence, the proof concludes by applying Theorem 2.1. $\square$

3. Some applications

We now proceed to show some applications of the above stated relationship between slant submanifolds and Riemannian submersions, by considering the differential map given by

$$\pi : \mathbb{R}^5 \longrightarrow \mathbb{C}^2; \quad (x_1, x_2, y_1, y_2, z) \longmapsto \frac{1}{2}(y_1, y_2, x_1, x_2).$$

It is easy to see that, if we have on $\mathbb{R}^5$ (resp. $\mathbb{C}^2$) its usual Sasakian (resp. Kaehlerian) structure, then π is a Riemannian submersion satisfying conditions i)–ii). Therefore, by using this submersion, we can obtain examples of slant submanifolds in $\mathbb{R}^5$ by taking the lifts of Examples 2.1, 2.3, 2.4

and 2.5 of [7]. Notice that those examples will be similar to Examples 3.7–3.10 of [4].

Now, suppose that we have a 3-dimensional submanifold M tangent to the structure vector field on $\mathbb{R}^5$ and a surface N in $\mathbb{C}^2$ satisfying diagram (2.1). Then, we have the following classification theorem:

Theorem 3.1. *In the above conditions, M is a 3-dimensional slant submanifold of $\mathbb{R}^5$ with parallel mean curvature vector if and only if M is one of the following submanifolds:*

(a) *a submanifold locally isometric to an open portion of the product of a plane circle and a circular cylinder.*

(b) *a submanifold locally isometric to an open portion of the product of a circular cylinder and $\mathbb{R}$.*

(c) *a minimal slant submanifold in $\mathbb{R}^5$.*

Moreover, if either case (a) *or case* (b) *occurs, then M is an anti-invariant submanifold.*

Proof. First, it is known that if the mean curvature vector of M is parallel then the mean curvature vector of N is also parallel, and that M is minimal if and only if N is minimal (see, for instance, [13, p. 462–463]). Hence, the proof of this theorem follows from Theorem 1.1 of [7, p. 50] and by taking into account that, if M is an anti-invariant submanifold, then $\eta(\nabla_{X^*}Y^*) = 0$, for any X, Y tangent to N, which means that, in this case, M is locally isometric to the Riemannian product of N and $\mathbb{R}$. □

In the same conditions, we can also classify the submanifold M attending to a particular behaviour of its second fundamental form σ:

Theorem 3.2. *M is a 3-dimensional slant submanifold of $\mathbb{R}^5$ satisfying*

$$(\nabla_X\sigma)(Y,Z) = G(Y,TX)NZ + G(Z,TX)NY \tag{3.1}$$

for any tangent vector fields X, Y, Z orthogonal to ξ, if and only if M is one of the following submanifolds:

(a) *a submanifold locally isometric to an open portion of the product of a plane circle and a circular cylinder.*

(b) *a submanifold locally isometric to an open portion of the product of a circular cylinder and $\mathbb{R}$.*

(c) *a lift by π of an open portion of a plane in $\mathbb{C}^2$.*

Moreover, if either case (*a*) *or case* (*b*) *occurs, then M is an anti-invariant submanifold.*

Proof. It follows from the O'Neill equations that

$$(\nabla_{X^*}\sigma)(Y^*,Z^*) = ((\nabla_X\sigma')(Y,Z))^* + G(Y^*,TX^*)NZ^* + G(Z^*,TX^*)NY^*,$$

for any X, Y, Z tangent to N, where σ' denotes the second fundamental form of N, and so, M satisfies (3.1) if and only if σ' is parallel.

Therefore, this proof works as that of Theorem 3.1, by applying now Theorem 1.2 of [7, p. 51]. □

A sufficient condition for a submanifold M, in the above conditions, to satisfy equation (3.1) is to be *totally contact geodesic*, i.e., such that

$$\sigma(X,Y) = \eta(X)\sigma(Y,\xi) + \eta(Y)\sigma(X,\xi),$$

for any tangent vector fields X and Y. In fact there are examples of totally contact geodesic slant submanifolds in $\mathbb{R}^5$ (see, for instance, Example 3.7 of [4]).

References

[1] D. E. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509, *Springer-Verlag*, *New York*, 1976.

[2] J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward, On conformal minimal immersions of $\mathbb{S}^2$ into $\mathbb{CP}^n$, *Math. Ann.* **279** (1988), 599–620.

[3] W. M. Boothby and H. C. Wang, On contact manifolds, *Ann. of Math.* **68** (1958), 721–734.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.* **42** (2000), 125–138.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Structure on a slant submanifold of a contact manifold, *Indian J. Pure Appl. Math.* **31** (2000), 857–864.

[6] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Existence and uniqueness theorem for slant immersions in Sasakian-space-forms, *Publ. Math. Debrecen* **58** no. 4 (2001), 559–574.

[7] B. Y. Chen, Geometry of Slant Submanifolds, *Katholieke Universiteit Leuven*, *Leuven*, 1990.

[8] B. Y. Chen and Y. Tazawa, Slant submanifolds of complex projective and complex hyperbolic spaces, *Glasgow Math. J.* **42** (2000), 439–454.

[9] S. S. Chern and J. G. Wolfson, Minimal surfaces by moving frames, *Amer. J. Math.* **105** (1983), 59–83.

[10] A. Lotta, Foliations of the Sasakian space $\mathbb{R}^{2n+1}$ by minimal slant submanifolds, *Acta Math. Hungar.* **84** (1999), 135–148.

[11] S. Maeda, Y. Ohnita and S. Udagawa, On slant immersions into Kähler manifolds, *Kodai Math. J.* **16** (1993), 205–219.

[12] S. Smale, On the structure of 5-manifolds, *Ann. of Math.* **75** (1962), 38–46.

[13] K. Yano and M. Kon, Structures on Manifolds, Series in Pure Mathematics, 3, *World Scientific*, *Singapore*, 1984.

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