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A Lie group structure on strict groups

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Abstract. The notion of strict groups encompasses many important infinite dimensional groups in differential geometry. It is shown that important strict groups carry the structure of a regular Lie group in the convenient setting. In the contact case this is related to the integrability of the Poisson algebra of a prequantizable Poisson manifold.

1. Introduction

Given a Lie groupoid $\Gamma \rightrightarrows M$ let $\mathrm{Bis}(\Gamma)$ be the group of all its bisections. Any subgroup $G \subset \mathrm{Bis}(\Gamma)$ is called a strict group. This concept adopts the C. Ehresmann's point of view to groups in geometry [6]. By considering the coarse groupoids it is apparent that all diffeomorphism groups are strict. The same is true for the space of smooth sections of a vector bundle, the space of connections of a trivial principal fibre bundle, and other important groups.

In this note we show that $\mathrm{Bis}(\Gamma)$ is a regular Lie group for every Lie groupoid Γ. A main ingredient of the proof is the statement that the space of all sections of a submersion is a convenient manifold. Next we make some comments on the contact case. In particular, we show that the group of all Legendrian bisections of a contact groupoid is a convenient Lie group, and we draw some conclusion concerning the integrability of the Poisson algebra of a prequantizable Poisson manifold.

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We will use the definition of a Lie group in the convenient setting of the infinite dimensional Lie theory due to A. Kriegl and P. Michor [12], and we follow methods used there. In the convenient setting the clue point is the idea of testing smoothness along smooth curves. A mapping between two possibly infinite dimensional manifolds is smooth if, by definition, it sends smooth curves to smooth curves. This concept is based on Boman's theorem which states that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is smooth whenever $f \circ c$ is smooth for any smooth curve $c : \mathbb{R} \to \mathbb{R}$. Furthermore, for modeling manifolds a special type of LCTVS is in use, namely a so-called convenient (for differential calculus) vector spaces which fulfil the Mackey completeness. Equivalently, E is convenient if any curve $c : \mathbb{R} \to E$ which is scalarwise smooth is smooth. This is also characterized by the fact that any smooth curve possesses an integral (antiderivative). Notice that all Fréchet spaces are convenient. The manifolds are defined by using open sets in the c^∞-topology on E, which is the final topology with respect to the space of smooth curves $C^\infty(\mathbb{R}, E)$. For Fréchet spaces the c^∞-topology coincides with the initial one.

Recall that a convenient Lie group G is called *regular* [15] if for $\mathfrak{g} = T_eG$ there exists a bijective evolution map

$$\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty((\mathbb{R}, 0), (G, e))$$

such that its evaluation at 1, $X \mapsto \mathrm{evol}^r_G(X)(1)$, is smooth. The inverse of evol^r_G is then called the right logarithmic derivative and denoted by δ^r_G.

An advantage of the convenient setting is that $\mathrm{Diff}(M)$ for M open is still a regular Lie group. Note that diffeomorphism groups on open manifolds cannot be described in terms of ILB-groups (H. Omori [16]). On the other hand, the identity component of $\mathrm{Diff}(M)$ in the convenient setting consists only of compactly supported diffeomorphisms. As in [12] also in our results there are no restrictive assumptions on Γ or M. It seems that [12] applied to strict groups give a unified method of introducing a Lie group structure for most remarkable transitive groups in differential geometry. In the nontransitive case this method is in general useless in view of a so-called "holonomic imperative", cf. [18]. It states that to any bisection of a groupoid over a foliated manifold is attached its holonomy class. Consequently, infinite dimensional regular Lie groups considered in [19] are modeled on the space of foliated 1-forms, rather than on the space of ordinary 1-forms.

Observe that often there is a closed subgroup $H \subset G$ and a Lie algebra $\mathfrak{h}$ such that smooth curves with values in $\mathfrak{h}$ are sent bijectively by evol^r_G to isotopies with values in H. However we would like to stress that in such a situation H possesses only a weak Lie subgroup structure (cf. [22]), and not the usual one. Such a situation could be also described in terms of diffeological Lie groups (J. M. Souriau [23]). The advantage of this setting is that Lie group structures are defined without using charts. However not all concepts and facts of the "usual" infinite dimensional Lie theory could be possible in such a framework.

The strict groups (e.g. the Lagrangian or Legendrian bisection groups) have been studied extensively by P. Dazord in [4] and [5] in terms of diffeological groups. We would like here to indicate that his results can be derived on the ground of usual Lie theory as well. For the case of symplectic groupoids, see [20]. Most of the presented results have been announced in [21].

Throughout all finite dimensional manifolds are assumed to be paracompact and C^∞-smooth, and all infinite dimensional manifolds are in the sense of [12].

2. The group of bisections and the main result

A groupoid is a small category all of whose arrows are invertible. We begin with the notation of a Lie groupoid following [2] (the "French notation" for groupoids, contrary to [14]).

A groupoid structure on a set Γ is given by two surjections (the source and target) $\alpha, \beta : \Gamma \to M \subset \Gamma$, by a multiplication $m : \Gamma_2 \to \Gamma$, where $\Gamma_2 = \{(x, y) \in \Gamma \times \Gamma : \alpha(x) = \beta(y)\}$, and by an inversion $i : \Gamma \to \Gamma$ such that the following axioms are fulfilled:

(Ass) If one of the products $m(x, m(y, z))$ and $m(m(x, y), z)$ is defined then so is the other and they are equal.

(Id) The products $m(\beta(x), x)$, $m(x, \alpha(x))$ are both defined and equal to x.

(Inv) $m(x, i(x))$ is defined and equal to $\beta(x)$, and $m(i(x), x)$ is defined and equal to $\alpha(x)$.

The elements of M are called unities. For simplicity we write $x.y$ for $m(x, y)$ and x^{-1} for $i(x)$. The symbol $\Gamma \rightrightarrows M$ will stand for the groupoid $(\Gamma, M, \alpha, \beta, m, i)$.

Next, a groupoid Γ is said to be a Lie groupoid if Γ is a smooth manifold (not necessarily separated), M is a separated submanifold, α and β are submersions, m is a smooth mapping, and i is a diffeomorphism. Notice that Γ is separated iff M is closed in Γ, cf. [2]. Moreover, the separatedness near M can be always assumed due to

Lemma 2.1 [2]. *Let N be a not necessarily separated manifold, and let $M \subset N$ be a separated submanifold. If there is a submersion $p : N \to M$, $p|_M = \mathrm{id}_M$, then there is a neighborhood U of M in N which is separated and with connected p-fibers. In case Γ is a Lie groupoid with $p = \beta$, U can be chosen symmetric (with respect to i) and such that M is closed in U.*

For $u \in M$ the set $\alpha(\beta^{-1}(u)) = \beta(\alpha^{-1}(u))$ is called the orbit at u. If Γ is α-connected (or, equivalently, β-connected), the family of orbits form a generalized foliation $\mathcal{F}_\Gamma$ of M; it is given by integrating the distribution $\alpha_* \ker T\beta|_M$. Γ is called transitive if it has only one orbit M.

For any subsets $A_1, A_2 \subset \Gamma$ we have the product

$$A_1.A_2 = \{x_1.x_2 \mid x_1 \in A_1,\ x_2 \in A_2,\ \alpha(x_1) = \beta(x_2)\}.$$

The family of all subsets of Γ with this product is a semi-group. By a bisection (or admissible section, cf. [14]) of a Lie groupoid Γ we mean a submanifold B of Γ such that $\alpha \mid B$ and $\beta \mid B$ are diffeomorphisms onto M. $\mathrm{Bis}(\Gamma)$, the set of all bisections, is exactly the group of all invertible elements of the above semigroup.

$\mathrm{Bis}(\Gamma)$ has natural left and right representations in Γ given by

$$\psi^l : \mathrm{Bis}(\Gamma) \ni B \mapsto \psi^l(B) := \{x \mapsto B.x\} \in \mathrm{Diff}(\Gamma),$$
$$\psi^r : \mathrm{Bis}(\Gamma) \ni B \mapsto \psi^r(B) := \{x \mapsto x.B\} \in \mathrm{Diff}(\Gamma).$$

Next there are the left and right representations in the unit space $(M, \mathcal{F}_\Gamma)$

$$\phi^l : \mathrm{Bis}(\Gamma) \ni B \mapsto \phi^l(B) := \beta \circ \psi^l(B)|_M \in \mathrm{Diff}(M, \mathcal{F}_\Gamma),$$
$$\phi^r : \mathrm{Bis}(\Gamma) \ni B \mapsto \phi^r(B) := \alpha \circ \psi^r(B)|_M \in \mathrm{Diff}(M, \mathcal{F}_\Gamma),$$

where $\mathrm{Diff}(M, \mathcal{F}_\Gamma)$ is the group of leaf preserving diffeomorphisms. $\mathrm{Bis}(\Gamma)_c$ will stand for the subgroup of all compactly controlled elements, that is all B such that $\phi^l(B)$, or equivalently $\phi^r(B)$, has compact support. In general, compactly controlled bisections need not have compact support, cf. Example 3 below.

Definition. A Lie algebroid over a manifold M is a triple $(E, [[\,,\,]], \rho)$ such that there is a vector bundle $E \to M$, $[[\,,\,]]$ is a Lie algebra bracket on the space $\mathrm{Sect}(E)$ of all smooth sections of E, and $\rho : E \to TM$ is a vector bundle morphism such that the following conditions are fulfilled:

(1) ρ induces $\tilde{\rho} : \mathrm{Sect}(E) \to \mathfrak{X}(M)$ which is a Lie algebra homomorphism;

(2) $[[\sigma_1, f\sigma_2]] = f[[\sigma_1, \sigma_2]] + \tilde{\rho}(\sigma_1)(f)\sigma_2, \quad \forall \sigma_1, \sigma_2 \in \mathrm{Sect}(E)$, $f \in C_c^\infty(M)$.

The map ρ is called an anchor.

Let us remind that a left-invariant vector field X on Γ is characterized by $T\beta(X) = 0$ and $Tl_x(X) = X$ for any left translation

$$l_x : \beta^{-1}(\alpha(x)) \ni y \mapsto x.y \in \beta^{-1}(\beta(x)).$$

Let $\mathfrak{X}_L(\Gamma)$ be the Lie algebra of all left-invariant vector fields on Γ. Then $\mathfrak{X}_L(\Gamma) \simeq \ker T\beta|_M$. Also $T\alpha|_{\mathfrak{X}_L(\Gamma)} : \mathfrak{X}_L(\Gamma) \to \mathfrak{X}(M)$ is well defined.

It is well known (J. Pradines [17]) that to any Lie groupoid $\Gamma \rightrightarrows M$ is assigned the associated Lie algebroid $\mathcal{A}(\Gamma)$, namely $\mathcal{A}(\Gamma) = (\mathcal{N}_\Gamma M, [[\,,\,]], T\alpha)$, where $[[\,,\,]]$ is a Lie algebra bracket on $\mathrm{Sect}(\ker T\beta|_M)$ introduced by means of the above identification and $\mathcal{N}_\Gamma M \simeq \ker T\beta|_M$. Hereafter $\mathcal{N}_N M$ stands for the normal bundle of M in N.

Our main result is the following

Theorem 2.2. *The groups* $\mathrm{Bis}(\Gamma)$ *and* $\mathrm{Bis}(\Gamma)_c$ *are regular Lie groups with the same Lie algebra* $\mathrm{Sect}_c(\mathcal{N}_\Gamma M)$.

Examples. 1. Lie groups coincide with Lie groupoids with a unique unity.

2. Another extreme example are manifolds: $\Gamma = M$.

3. If $\alpha = \beta$ then for all $u \in M$ the fiber $\alpha^{-1}(u)$ carries a Lie group structure. Any vector bundle is a Lie groupoid of this type, and its bisection is just a smooth section.

4. For any set M put $\Gamma = M \times M$, $\alpha((x, y)) = y$, $\beta((x, y)) = x$, $m((z, y), (y, x)) = (z, x)$ and $i((x, y)) = (y, x)$. We get the coarse groupoid with the space of units $M \simeq \Delta_M$. Notice that bisections of $\Gamma = M \times M$ coincide with with the diffeomorphisms on M.

5. Given a principal fiber bundle $P(M, \pi, G)$ one defines the equivalence relation $\sim$ on $P \times P$ by $(p_1, p_2) \sim (q_1, q_2)$ iff $\exists a \in G$ such that $p_i a = q_i$. Putting $\Gamma = P \times P/\sim$, $\alpha([(p_1, p_2)]) = \pi(p_2)$, $\beta([(p_1, p_2)]) = \pi(p_1)$, we get the gauge groupoid (with obvious m and i). Γ is identified with the set of equivariant bundle morphisms over id_M, and $\mathrm{Bis}(\Gamma)$ with the space of all connections on the trivial principal fibre bundle $M \times G$.

6. Assume that a Lie group G acts on a manifold M. Then we set: $\Gamma = G \times M$, $\alpha((g, x)) = g.x$, $\beta((g, x)) = x$, $(g', x').(g, x) = (g'g, x)$, and $(g, x)^{-1} = (g^{-1}, g.x)$. We say that Γ is a transformation groupoid. Here $\mathrm{Bis}(\Gamma) = \{\phi : C^\infty(M, G)$ such that $x \mapsto \phi(x).x$ is a diffeomorphism$\}$.

7. For any Lie groupoid Γ the tangent space
$T\Gamma = (T\Gamma \rightrightarrows TM, T\alpha, T\beta, \oplus, I)$ possesses a structure of Lie groupoid. Here the multiplication $\oplus$ is given by

$$X \oplus Y = \left(\frac{\mathrm{d}}{\mathrm{d}\, t}(x(t).y(t))\right)\bigg|_{t=0},$$

where $X = \frac{\mathrm{d}\, x}{\mathrm{d}\, t}|_{t=0}$, $Y = \frac{\mathrm{d}\, y}{\mathrm{d}\, t}|_{t=0}$, $\alpha(x(t)) = \beta(y(t))$, and the inversion $IX = \frac{\mathrm{d}\, x}{\mathrm{d}\, t} x(t)^{-1}|_{t=0}$ if $X = \frac{\mathrm{d}\, x}{\mathrm{d}\, t}|_{t=0}$. It is visible that $\mathrm{Bis}(T\Gamma) = \{TS,\ S \in \mathrm{Bis}(\Gamma)\}$.

3. Proof of Theorem 2.2

The proof follows the one for diffeomorphisms in [12], and makes use of tubular neighborhoods and the identification $\mathcal{N}_\Gamma M \simeq \ker T\beta|_M$. The topology of $\mathrm{Bis}(\Gamma)$ is the identification topology by charts of a Lie group structure of $\mathrm{Bis}(\Gamma)$. In particular, all bisections in the identity component are compactly controlled.

First we show that $C^\infty(M \leftarrow N)$, the space of all smooth sections of a surmersion $p : N \to M$, carries a manifold structure. We need the following fact [12]

Proposition 3.1. *Let $E \to M$ and $F \to M$ be two smooth vector bundles, let U be an open subset in E and $\mathrm{Sect}_c(U) = \{s \in \mathrm{Sect}_c(E) : s(M) \subset U\}$. If $\phi : U \to F$ is a smooth fiber respecting (nonlinear) map then $\phi_* : \mathrm{Sect}_c(U) \to \mathrm{Sect}_c(F)$ is smooth, where $\phi_*(s)(x) = \phi(s(x))$.*

Proposition 3.2. *Let $p : N \to M \subset N$ be an arbitrary surmersion such that $p|_M = \mathrm{id}_M$. The space $C^\infty(M \leftarrow N)$ equipped with the identification topology is a smooth separated manifold modeled on the spaces $\mathrm{Sect}_c(\mathcal{N}_N f(M))$, where $f \in C^\infty(M \leftarrow N)$.*

Proof. Let $\mathcal{F}_p$ be the foliation by p-fibers on a convex open separated neighborhood V of M in N (Lemma 2.1). The tangent bundle TV has the form $TV = T\mathcal{F}_p \oplus \mathcal{N}\mathcal{F}_p$, where $T\mathcal{F}_p$ (resp. $\mathcal{N}\mathcal{F}_p$) is the tangent (resp. normal) bundle of $\mathcal{F}_p$. Choose any linear connections ∇^1 in $T\mathcal{F}_p$ and ∇^2 in $\mathcal{N}\mathcal{F}_p$. Then we get the product connection ∇ in TV by $\nabla_X s = \nabla^1_X s_1 + \nabla^2_X s_2$, where $X \in \mathrm{Sect}(TV)$, $s = s_1 + s_2 \in \mathrm{Sect}(TV)$, $s_1 \in \mathrm{Sect}(T\mathcal{F}_p)$, and $s_2 \in \mathrm{Sect}(\mathcal{N}\mathcal{F}_p)$. The fibers of p are then totally geodesic submanifolds of V.

Now let $f \in C^\infty(M \leftarrow N)$. By applying Lemma 2.1 to $f \circ p$ and by choosing a connection ∇ as above but in a neighborhood of $f(M)$ rather than M, there exist a convex open separated neighborhood U_f of $f(M)$ in $\mathcal{N}_N f(M)$, a neighborhood V_f of $f(M)$ in N, and a diffeomorphism

$$\mu_f : \mathcal{N}_N f(M) \supset U_f \to V_f \subset N$$

such that $\mu_f(0_x) = x$ for all $x \in f(M)$, and μ_f is fiber preserving. Indeed, one can choose $\mu_f = \exp_\nabla |_{U_f}$.

In view of [12], the space $\mathrm{Sect}_c(\mathcal{N}_N f(M))$ is endowed with the inductive limit topology of the subspaces $\mathrm{Sect}_K(\mathcal{N}_N f(M))$, K running over compact subsets of $f(M)$, with the usual C^∞ topology.

For $f, g \in C^\infty(M \leftarrow N)$ we write $f \sim g$ if f and g agree off a compact subset. We define a chart $\phi_f : \mathcal{V}_f \to \mathrm{Sect}_c(\mathcal{N}_N f(M))$ where

$$\mathcal{V}_f := \{g \in C^\infty(M \leftarrow N) : \ f \sim g \text{ and } \mu_f^{-1} g(x) \in U_f, \ \forall x \in M\},$$

$$\phi_f(g)(x) := \mu_f^{-1} g(x). \tag{3.1}$$

Then ϕ_f maps bijectively $\mathcal{V}_f$ onto $\mathcal{U}_f := \{s \in \mathrm{Sect}_c(\mathcal{N}_N f(M)) : s(f(M)) \subset U_f\}$, an open subset of $\mathrm{Sect}_c(\mathcal{N}_N f(M))$ in the above topology.

For all $f, g \in C^\infty(M \leftarrow N)$ and $s \in \mathrm{Sect}_c(\mathcal{N}_N g(M))$

$$(\phi_f \circ \phi_g^{-1})(s) = (\mu_f^{-1} \circ \mu_g)_*(s). \tag{3.2}$$

In view of Proposition 3.1 with $E = \mathcal{N}_N g(M)$, $F = \mathcal{N}_N f(M)$ and $U = U_f \cap (\mu_f^{-1} \circ \mu_g)(U_g)$ the chart changing $\phi_f \circ \phi_g^{-1}$ are defined on open sets and smooth. So $\{(\mathcal{V}_f, \phi_f)\}$ is an atlas for $C^\infty(M \leftarrow N)$. The identification topology of this atlas is finer than the Whitney topology (cf. [12], 41.13) and, consequently, separated. The equality (3.2) implies that the smooth structure induced by this atlas is independent of the choice of a connection ∇ used in the definition of μ_f. Indeed, the choice of another such connection leads to an equivalent atlas. $\square$

Lemma 3.3. *A mapping $c : \mathbb{R} \to C^\infty(M \leftarrow N)$ is a smooth curve iff $\hat{c} : \mathbb{R} \times M \ni (t, x) \mapsto c(t)(x) \in N$ is smooth and the following condition is fulfilled: $(*)$ for any compact interval $[a, b] \subset \mathbb{R}$ there exists a compact subset $K \subset M$ such that for all $t \in [a, b]$ $c(t)$ stabilizes off K.*

Proof. In view of [12], 42.5, one has such a description of smooth curves for the manifold of all smooth mappings from M to N, $C^\infty(M, N)$. Since $C^\infty(M \leftarrow N)$ is a splitting submanifold of $C^\infty(M, N)$ this implies our assertion. $\square$

Proof of Theorem 2.2. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. The first part of the proof consists in showing that the group $\mathrm{Bis}(\Gamma)$ can be viewed as an open subset of $C^\infty(M \xleftarrow{\beta} \Gamma)$, the space of all sections of $\beta : \Gamma \to M$. By Proposition 3.2 the space $C^\infty(M \xleftarrow{\beta} \Gamma)$ is a manifold, so that this yields a Lie group structure on $\mathrm{Bis}(\Gamma)$ with the Lie algebra $\mathrm{Sect}_c(\mathcal{N}_\Gamma M)$.

For any $s \in C^\infty(M \xleftarrow{\beta} \Gamma)$ one has $\alpha \circ s \in C^\infty(M, M)$, and $\alpha \circ s \in \mathrm{Diff}(M)$ iff $s \in \mathrm{Bis}(\Gamma)$. Here we identify $\mathrm{Bis}(\Gamma)$ with a subset of $C^\infty(M \xleftarrow{\beta} \Gamma)$ by means of the embedding $B \mapsto \psi^r(B)|_M$. Let $c : \mathbb{R} \to C^\infty(M \xleftarrow{\beta} \Gamma)$ be a smooth curve with $\alpha(c(0))$ being a diffeomorphism. Then $t \mapsto \alpha(c(t))$ is a smooth curve in $C^\infty(M, M)$ and in view of the proof of Theorem 43.1 in [12] $\alpha(c(t))$ stays surjective and injective for t close to 0. It stays as well in the set of immersions by [12], 41.10. Thus $\mathrm{Bis}(\Gamma)$ is open in $C^\infty(M \xleftarrow{\beta} \Gamma)$.

We show that the multiplication and the inversion in $\mathrm{Bis}(\Gamma)$ are smooth. It suffices to prove that these mappings send smooth curves to smooth curves. We identify $B \in \mathrm{Bis}(\Gamma)$ with $\psi^r(B)|_M$.

Let $B, C : \mathbb{R} \to \mathrm{Bis}(\Gamma)$. Observe that for $u \in M$

$$\begin{aligned}\psi^r(B.C)(u) &= u.(B.C) = (u.B).(\alpha(u.B).C)\\ &= m(\psi^r(B)(u), \psi^r(C)(\alpha(\psi^r(B)(u)))).\end{aligned}$$

This gives the smoothness of multiplication due to Lemma 3.3 and the smoothness of m. To show the smoothness of inversion let $B : \mathbb{R} \to \mathrm{Bis}(\Gamma)$ and $\widehat{\psi^r(B)} : \mathbb{R} \times M \ni (t,u) \mapsto \psi^r(B)(t)(u) \in M$. We have then that $\widehat{\psi^r(B)}(t, \widehat{\psi^r(B^{-1})}(t,u)) = u$ for any $t \in \mathbb{R}$, $u \in M$. Therefore by the implicit function theorem for M $\widehat{\psi^r(B^{-1})}$ in smooth in the variables (t,u). In view of Lemma 3.3 B^{-1} is smooth. Thus the inversion in $\mathrm{Bis}(\Gamma)$ is smooth.

In the second part we prove the regularity of $\mathrm{Bis}(\Gamma)$. Let $X \in \mathrm{Sect}_c(\mathcal{N}_\Gamma M)$. There is a unique $\tilde{X} \in \mathfrak{X}_L(\Gamma)$ which extends X. Observe that the flow of a vector field tangent to the fibers of β exists uniquely since the fibres are separated. Let C_t be a flow of $\tilde{X}$. We wish to show that $\alpha \circ C_t|_M$ is a bijection for all t. Note that by definition $C_t \circ l_x = l_x \circ C_t$, $\forall t \in \mathbb{R}$, and $l_x^{-1} = l_{x^{-1}}$, so we can write

$$l_x^{-1} \circ C_{-t} = C_{-t} \circ l_x^{-1}. \tag{3.3}$$

The injectivity: Suppose $w = \alpha(C_t(u)) = \alpha(C_t(v))$, where $u, v \in M$, and put $x = C_t(u)$, $y = C_t(v)$. Then $l_x^{-1}(x) = l_y^{-1}(y) = w$. Hence for $z = C_{-t}(w)$ one gets by (3.3) the equalities $z = l_x^{-1}(C_{-t}(x)) = l_x^{-1}(u)$ and $z = l_y^{-1}(C_{-t}(y)) = l_y^{-1}(v)$. This yields $\alpha(z) = \alpha(u) = u$ and $\alpha(z) = \alpha(v) = v$, resp. Thus $u = v$ as required.

The surjectivity: Let $v \in M$. For $x = C_{-t}(v)$ we have $l_x^{-1}(v) = (C_t \circ l_x^{-1} \circ C_{-t})(v) = C_t(\alpha(x))$ in view of (3.3). Therefore $v = \alpha(l_x^{-1}(v)) = \alpha(C_t(\alpha(x)))$ which implies the surjectivity.

By considering time-dependent families from $\mathrm{Sect}_c(\mathcal{N}_\Gamma M)$ instead of its elements we get the bijection

$$\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)} : C^\infty(\mathbb{R}, \mathrm{Sect}_c(\mathcal{N}_\Gamma M)) \to C^\infty((\mathbb{R},0), (\mathrm{Bis}(\Gamma), e)).$$

By definition the evaluation of it at 1 is smooth. This shows that $\mathrm{Bis}(\Gamma)$ is regular. $\square$

Remark. In traditional settings of infinite dimensional analysis, e.g. [9], special and very technical implicit function theorems are needed to prove the smoothness of the group inversion. The above argument shows the usefullness and elegance of the convenient setting.

Let us describe $\delta^r_{\mathrm{Bis}(\Gamma)}$, the inverse of $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)}$. For $u \in M$ and $B \in \mathrm{Bis}(\Gamma)$ one has a diffeomorphism $\sigma_u^B : \beta^{-1}(u) \to \beta^{-1}(u)$ given by

$$\sigma_u^B(x) = x.\psi^r(B)(\alpha(x)).$$

Then clearly $\sigma_u^B(u) = \psi^r(B)(u)$. By gluing-up the tangent mappings $T\sigma_u^B$ of diffeomorphisms σ_u^B we get the canonical identification

$$\sigma^B : \ker T\beta \simeq \mathcal{N}_\Gamma M \simeq \mathcal{N}_\Gamma B. \tag{3.4}$$

We have also $T\psi^r(B) : TM \simeq TB$. By combining it with (3.4) we get $\tilde{\sigma}^B : T\Gamma|_M \simeq T\Gamma|_B$.

Given a smooth isotopy B_t in $\mathrm{Bis}(\Gamma)$ with $B_0 = M$ there is a unique time-dependent family of vector fields $\hat{X}_t$ along $\psi^r(B_t)(M)$ corresponding to B_t, i.e. for all $u \in M$

$$\hat{X}_t(\psi^r(B_t)(u)) = \frac{d}{ds}\psi^r(B_s)(u)|_{s=t}.$$

By definition, $\hat{X}_t$ are tangent to the fibers of β. Hence we get a unique smooth curve X_t in $\mathrm{Sect}_c(\ker T\beta|_M)$ such that $\tilde{\sigma}_*^{B_t} X_t = \hat{X}_t$. Then $\delta^r_{\mathrm{Bis}(\Gamma)}(B_t) = X_t$, and $\hat{X}_t = \tilde{X}_t$ on $\psi^r(B_t)(M)$.

4. The case of contact groupoids

It is also possible to endow with a Lie group structure two important strict groups which are not of the form $\mathrm{Bis}(\Gamma)$, i.e. are not equal to the group of all bisections of a Lie groupoid. In [20] this has been shown for the Lagrangian bisection group of a symplectic groupoid. Now we will deal with the Legendrian bisection group of a contact groupoid.

Definition. A Lie groupoid Γ endowed with a contact form θ (i.e. $\theta \wedge \mathrm{d}\,\theta^n \neq 0$, where $\dim \Gamma = 2n+1$) is called *contact* if there are smooth mappings $u, v : \Gamma \to \mathbb{R}_*$ (actually groupoid morphisms) such that the following conditions hold:

(i) $\theta(X_x \oplus Y_y) = u(x)\theta(Y_y) + v(y)\theta(X_x)$;

(ii) the inversion i is a conformal θ-morphism.

The operation $\oplus$ is defined in Example 7.

Proposition 4.1 below reveals how contact groupoids are connected with Jacobi structures. Let us recall that a Jacobi structure on a manifold M is a pair (Λ, E), where Λ is a bivector field on M, E is a vector field on M, and the equalities

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [E, \Lambda] = 0$$

are satisfied, where $[\,.\,,\,.\,]$ is the Schouten–Nijenhuis bracket. Specifically transitive odd-dimensional Jacobi manifolds coincide with contact manifolds. The importance of the Jacobi strucures comes from the fact that they correspond bijectively to Lie algebra brackets on $C^\infty(M)$ which are local, i.e.

$$\operatorname{supp}\{u,v\} \subset (\operatorname{supp}(u) \cap \operatorname{supp}(v)), \quad u,v \in C^\infty(M).$$

We have also the bundle homomorphism associated with Λ given by

$$\Lambda^\sharp : T^*M \to TM, \quad <\Lambda^\sharp\alpha, \beta> = \Lambda(\alpha,\beta),$$

for any $\alpha, \beta \in T^*M$. The distribution generated by $\Lambda^\sharp(T^*_xM)$ and E_x, $x \in M$, integrates to a generalized foliation (cf. [24]). This foliation is called *characteristic* and denoted by $\mathcal{F} = \mathcal{F}(\Lambda, E)$. It is well-known that any Jacobi structure induces a locally conformal symplectic (resp. contact) structure on each leaf of $\mathcal{F}(\Lambda, E)$ of even (resp. odd) dimension. If $E = 0$ then we get a Poisson structure, and each leaf of the characteristic foliation carries a symplectic structure.

Proposition 4.1 [11], [5]. *If $\Gamma = (\Gamma, \theta, u, v)$ be a contact groupoid over M then:*

(i) *the space of units M is canonically endowed with a Jacobi structure (Λ, E) whose characteristic foliation is equal to $\mathcal{F}_\Gamma$;*

(ii) *α (resp. β) is a conformal Jacobi u-morphism (resp. $(-v)$-morphism);*

(iii) *the inversion i is a contact anti-morphism (i.e. $i^*\theta = -\theta$), and M is a Legendrian submanifold.*

The symbol $(\Gamma, \theta) \rightrightarrows (M, \Lambda, E)$ will stand for Γ.

Recall that $S \subset M$, where (M, θ) is a contact manifold, is called a Legendrian submanifold if it is a maximal integral submanifold of the symplectic distribution $\ker(\theta)$. Clearly the set $\operatorname{Bis}(\Gamma, \theta)$ of all Legendrian bisections is a subgroup of $\operatorname{Bis}(\Gamma)$. This group has natural left and right representations in the contactomorphism group of Γ

$$\psi^l : \operatorname{Bis}(\Gamma, \theta) \ni C \mapsto \psi^l(C) = \{x \mapsto C.x\} \in \operatorname{Cont}(\Gamma, \theta),$$

$$\psi^r : \operatorname{Bis}(\Gamma, \theta) \ni C \mapsto \psi^r(C) = \{x \mapsto x.C\} \in \operatorname{Cont}(\Gamma, \theta).$$

The corresponding representations ϕ^l and ϕ^r take their values in $\operatorname{Diff}(M, \Lambda, E)$, the automorphism group of (M, Λ, E).

Example. Given two contact manifolds (M_i, θ_i), $i = 1, 2$, one defines a contact form θ on $M_1 \times M_2 \times \mathbb{R}_*$ by $\theta = \pi_1^* \theta_1 - t\pi_2^* \theta_2$, where π_i is the canonical projection on M_i. Now $f : M_1 \to M_2$ is a contact ϕ-diffeomorphism, $\phi \in C^\infty(M_1)$, if and only

$$\mathrm{graph}(f) := \{(x, f(x), \phi(x)) \mid x \in M_1\}$$

is a Legendrian submanifold of $M_1 \times M_2 \times \mathbb{R}_*$, cf. [10].

If (M, θ) is a contact manifold then $\Gamma = M \times M \times \mathbb{R}_*$ with $\hat{\theta} = \pi_1^* \theta - t\pi_2^* \theta$ is the *coarse contact groupoid*. Here the multiplication and the inversion are given by

$$(x_1, y_1, a_1).(x_2, y_2, a_2) = (x_1, y_2, a_1 a_2) \quad \text{iff} \quad y_1 = x_2,$$

$$(x, y, a)^{-1} = (y, x, a^{-1}).$$

Consequently, $\mathrm{Bis}(\Gamma, \theta) = \{\mathrm{graph}(f) : f \in \mathrm{Diff}(M, \theta)\}$.

It is important that the Lie algebroid associated with a contact groupoid assumes a special form.

Proposition 4.2. (i) *If (M, Λ, E) is a Jacobi manifold the one jet bundle $\pi : J^1(M, \mathbb{R}) \to M$ is endowed with a Lie algebroid structure such that the one jet mapping*

$$j : C^\infty(M) \ni f \mapsto (f, \mathrm{d}\, f) \in C^\infty(M \leftarrow J^1(M, \mathbb{R}))$$

is a bounded splitting Lie algebra monomorphism.

(ii) *The algebroid associated with a contact groupoid $(\Gamma, \theta) \rightrightarrows (M, \Lambda, E)$ is canonically isomorphic to the algebroid associated with (M, Λ, E).*

Note that $J^1(M, \mathbb{R}) \simeq T^*M \times \mathbb{R}$ is endowed with the canonical contact structure $\theta_M - \mathrm{d}\, t$, where θ_M is the canonical 1-form on T^*M. See [5] for the proof and more detailed exposition.

5. A Lie group structure on $\mathrm{Bis}(\Gamma, \theta)$

We begin with V. Lychagin's result [13] concerning the existence of canonical neighborhoods of Legendrian submanifold.

Lemma 5.1. *If S is a Legendrian submanifold of a contact manifold (M,θ) then there exist an open neighborhood U of S, an open neighborhood V of the zero section in $T^*S\times\mathbb{R}$, and a diffeomorphism $\xi:U\to V$ such that $\xi|_S=\mathrm{id}_S$ and $\xi^*(\theta_S-\mathrm{d}\,t)=\theta$.*

Theorem 5.2. *For any contact groupoid $(\Gamma,\theta)\rightrightarrows(M,\Lambda,E)$, the groups $\mathrm{Bis}(\Gamma,\theta)$ and $\mathrm{Bis}(\Gamma,\theta)_c$ (the subgroup of compactly controlled bisections) are regular splitting Lie subgroups of $\mathrm{Bis}(\Gamma)$, both with the Lie algebra $C^\infty_c(M)$.*

Remark. In [12], 43.13 it is shown that for a contact manifold (M,θ) the group $\mathrm{Diff}(M,\theta)$ is a regular Lie group. However it is not a splitting Lie subgroup of $\mathrm{Diff}(M)$, because the graphs of contact diffeomorphisms are contained in $M\times M\times\mathbb{R}$, and not in $M\times M$.

Proof. In view of Proposition 4.2 we use the identification $\mathcal{N}_\Gamma M\simeq T^*M\times\mathbb{R}\simeq J^1(M,\mathbb{R})$. We consider the β-fiber preserving chart (3.1) at $f=e$

$$\mu:J^1(M,\mathbb{R})\supset U\to V\subset\Gamma$$

with $\mu(0_u)=u$ for $u\in M$. Then on U we have two contact forms: $\theta_M-\mathrm{d}\,t$ and $\mu^*\theta$. By composing μ with the diffeomorphism ξ from Lemma 5.1 and possibly shrinking U and V we may have $\mu^*\theta=\theta_M-\mathrm{d}\,t$, but now μ no longer preserves the β-fibers.

Let $\mathcal{V}$ be a neighborhood of $e=M$ in $\mathrm{Bis}(\Gamma)$ consisting of all submanifolds $B\subset\Gamma$ such that $\psi^rB|_M:M\to\Gamma$ is compactly supported, $\psi^r(B)(M)\subset V$, and so small that $\mu^{-1}(B)$ is still the image of a β-section. We define a chart $\Phi:\mathrm{Bis}(\Gamma)\supset\mathcal{V}\to\mathcal{U}\subset\mathrm{Sect}_c(J^1(M,\mathbb{R}))$ at $e=M$ by

$$\Phi(B):=\mu^{-1}\circ\psi^r(B)|_M\circ(\pi\circ\mu^{-1}\circ\psi^r(B)|_M)^{-1}.$$

This gives a new chart of $\mathrm{Bis}(\Gamma)$ at $e=M$.

We write $\Phi(C)=(\Phi_1(C),\Phi_2(C))$ where $\Phi_1(C)\in C^\infty(M)$ and $\Phi_2(C)\in\Omega^1_c(M)$. Observe that the following statements are equivalent: (i) $C\in\mathrm{Bis}(\Gamma)$ is Legendrian; (ii) $\mu^{-1}(C)$ is Legendrian in $J^1(M,\mathbb{R})$; (iii) $\Phi(C)^*(\theta_M-\mathrm{d}\,t)=0$; (iv) $\Phi_2(C)=\mathrm{d}(\Phi_1(C))$. Since the mapping j in Proposition 4.2 is a bounded linear splitting embedding, we get that Φ_1 is a splitting submanifold chart (with values in $C^\infty_c(M)$) for $\mathrm{Bis}(\Gamma,\theta)$.

Next for arbitrary $C\in\mathrm{Bis}(\Gamma,\theta)$ we get a submanifold chart at C as follows: $\mathcal{V}_C:=\{B:B.C^{-1}\in\mathcal{V}\}$ and $\Phi_C(B):=\Phi(B.C^{-1})$. Thus $\mathrm{Bis}(\Gamma,\theta)$ is a splitted submanifold of $\mathrm{Bis}(\Gamma)$ and a Lie group.

To show the regularity of $\mathrm{Bis}(\Gamma,\theta)$ let us recall the following [12], 38.7. Let H be a topological Lie subgroup of a regular Lie group G. If there are an open neighborhood $U\subset G$ of e and a smooth mapping $p:U\to E$, where E is a convenient vector space, such that $p^{-1}(0)=U\cap H$ and p is constant on left cosets $Hg\cap U$, then H is regular. In our case one can use $U=\mathcal{V}$ and $p(C)=\Phi_2(C)-\mathrm{d}(\Phi_1(C))$. The regularity can be also checked directly as at the end of section 3. In particular, $\mathrm{evol}^r_{\mathrm{Bis}(\Gamma,\theta)}=\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)}|_{C^\infty_c(M)}$. □

Remark. One can consider as well contact groupoids in the wider sense, that is given by a distribution of hyperplanes $\mathcal{H}$. A Lie groupoid Γ equipped with $\mathcal{H}$ is contact if

(i) $X,Y\in\mathcal{H}\Rightarrow X\oplus Y\in\mathcal{H}$,

(ii) i preserves $\mathcal{H}$.

Then analogous statements remain true but the space of units is now endowed with a structure of Jacobi bundle. The group $\mathrm{Bis}(\Gamma,\theta)$ still admits a regular Lie group structure.

6. Integrability of prequantizable Poisson algebras

The third theorem of Lie asserts that any finite dimensional Lie algebra is actually the Lie algebra of a Lie group. Since a famous paper [7] it is well known that, in general, this theorem is no longer true in the infinite dimensional case. However there are several generalizations of this theorem, e.g. [1], [3], [8], [17]. Let us add that an abstract approach to the problem has been recently proposed in [22]. It appeals to the concept of weak Lie subgroups.

Let us recall that a Poisson manifold (M,Λ) is *prequantizable* if there exists a (global) symplectic groupoid $(\Gamma_0,\omega)\rightrightarrows(M,\Lambda)$ such that (Γ_0,ω) is prequantizable. If Γ is separated, the latter means that $[\omega]\in H^2(\Gamma_0,\mathbb{Z})$. Then there is a contact groupoid $(\Gamma,\theta)\rightrightarrows(M,\Lambda)$, called a natural contact groupoid over a prequantizable Poisson manifold (M,Λ), cf. [5]. In [5] P. Dazord gave conditions characterizing prequantizable Poisson manifolds: (M,Λ) is prequantizable iff (M,Λ) admits a contact groupoid with finite period.

If (M,Λ) is prequantizable, the exact sequence of Lie algebras

$$0\to\mathbb{R}\to C^\infty_c(M)\xrightarrow{\mathrm{d}}B\Omega^1_c(M)\to 0$$

can be integrated to the exact sequence of regular Lie groups

$$1 \to S^1 \to \mathrm{Bis}(\Gamma, \theta)_0 \to \mathrm{Bis}^{\mathrm{exact}}(\Gamma_0, \omega) \to 1.$$

Here $B\Omega^1_c(M)$ is the space of compactly supported exact 1-forms, while $\mathrm{Bis}^{\mathrm{exact}}(\Gamma_0, \omega)$ is the group of all exact Lagrangian bisections of the symplectic groupoid (Γ_0, ω), cf. [4], and $\mathrm{Bis}(\Gamma, \theta)_0$ is the identity component of $\mathrm{Bis}(\Gamma, \theta)$. This means that $\mathrm{Bis}(\Gamma, \theta)_0$ is a central extension of $\mathrm{Bis}^{\mathrm{exact}}(\Gamma_0, \omega)$. Therefore Theorem 5.2 can be reformulated as follows (see also [5]).

Theorem 6.1. *Any prequantizable Poisson algebra on a manifold M (i.e. a Lie algebra $(C^\infty_c(M), \{,\})$, where the bracket $\{,\}$ is defined by a prequantizable Poisson structure (M, Λ)) can be integrated to a regular Lie group $\mathrm{Bis}(\Gamma, \theta)_0$, where $(\Gamma, \theta) \rightrightarrows (M, \Lambda)$ is a natural contact groupoid over (M, Λ).*

Notice that a main result of [1] states that the Poisson algebra of any prequantizable symplectic manifold is integrable, and in [8] this was shown for locally conformal symplectic structures.

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