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On nilpotent loop rings and a problem of Goodaire

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Abstract. Let p be a prime, L a finite loop of p-power order and F a field of characteristic p. We show that the fundamental ideal of the loop ring FL is nilpotent if and only if the multiplication group of L is a p-group. We apply this theorem to answer a question of E. G. Goodaire.

1. Introduction

As in [2], we define loop rings as follows. Let $L = (L, \cdot)$ be a loop with unit 1 and R be a commutative associative ring with unity. The elements of the *loop ring* $A = RL$ are the formal (finite) sums $\sum_{g\in L} c_g g$; addition and multiplication are defined in the obvious way. The *augmentation map* $\epsilon : RL \to R$ is the map $\epsilon(\sum_{g\in L} c_g g) = \sum_{g\in L} c_g$; this map is clearly a surjective ring homomorphism. The kernel $\Delta(L)$ of ϵ is called the *fundamental ideal* of RL.

In general, it is of interest to study loops with weak associativity properties; the most important classes are the Moufang and Bol loops. The class of *right Bol loops* is characterized by the *right Bol identity*

$$(xy \cdot z)y = x(yz \cdot y), \tag{1}$$

while *Moufang loops* are loops satisfying (1) and its dual $x(y{\cdot}xz) = (x{\cdot}yx)z$ simultaneously.

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These forms of weak associativity are important in rings, as well. However, on the one hand, in the presence of R-linearity, the identities can take different forms which explains different terminology. For example, the Moufang identity in rings is equivalent with the fact that the ring associator $[x, y, z] = xy \cdot z - x \cdot yz$ alternates, such rings are therefore called *alternative rings*. Rings with the (right) Bol property are called *strongly right alternative rings*. On the other hand, weak associativity identities of a loop in general do not inherit to the loop ring. Recent investigations (for example [2], [4], [3]) are going on to characterize loops where this is yet the case. Another way to deal with this problem is to look for ideals of the loop ring such that the factor algebra belongs to a class with given weak associativity properties (see [6, Section 5]).

In this paper, we consider loop rings FL over a field F of characteristic $p > 0$ and assume that the order of L is a power of p. In Theorem 2.4, we show that the nilpotence of the fundamental ideal $\Delta(L) = \ker \epsilon$ of the loop ring FL is equivalent with the nilpotence of the multiplication group of L. We apply this result to give an affirmative answer to the following question of Goodaire [2]: Let L be an indecomposable Bol loop L with a unique non-identity associator and commutator and let F be a field of characteristic 2. Is it true that the fundamental ideal $\Delta(L)$ is nilpotent?

2. Nilpotent loop rings

Let A be an arbitrary algebra over the field F. We denote by L_x and R_x the left and right multiplication maps of A; these are clearly F-linear maps with

$$L_{x+y} = L_x + L_y \quad \text{and} \quad R_{x+y} = R_x + R_y.$$

By [7], we define the *associative multiplication algebra* $\mathfrak{M}(A)$ as the associative subalgebra of $\mathrm{End}(A)$, generated by $\{L_x, R_x \mid x \in A\}$. For any subset S of A, we write $\mathfrak{M}(A, S)$ for the subalgebra of $\mathfrak{M}(A)$, generated by $\{L_x, R_x \mid x \in S\}$.

We define the powers A^n of the algebra A inductively: We put $A^1 = A$ and $A^n = \sum_{i=1}^{n-1} A^i A^{n-i}$. The algebra A is nilpotent if $A^k = \{0\}$ for some positive integer k. By Schafer's theorem [7, Theorem II.2.4], the ideal I of A is nilpotent if and only if the associative algebra $\mathfrak{M}(A, I)$ is nilpotent.

If A is an algebra (ring) with unity 1, then we call an element $a \in A$ *unit* of A if elements $u, v \in A$ exist with $au = va = 1$. Observe that in contrast to [2], we do not require $u = v$. We denote by A^* the set of units of A.

If the algebra (ring) A has no unity, then one constructs the algebra $A_1 = F \oplus A$ with unity by a well-known method (adjunction of unity, see [7, p. 11]). For a nilpotent algebra A, we shall write $1 + A$ for the subset $\{1 + x \mid x \in A\}$ of A_1.

Lemma 2.1. *Let A be a nilpotent algebra over F. Then we have $A_1^* = \{c + x \mid c \in F^*,\ x \in A\}$. Moreover, A_1^* and $1 + A$ are loops and the isomorphy $A_1^* \cong F^* \times (1 + A)$ holds.*

Proof. We first show that $1 + A$ consists of units. For any element $a \in A$, $(1 + a)u = 1$ is equivalent with $L_{1+a}^{-1}(1) = u$, thus, it is sufficient to show that the linear map L_{1+a} of A_1 is invertible. Indeed, by the nilpotence of A, the restriction of L_a to A is nilpotent and by $L_a(A_1) \subseteq A$, L_a is nilpotent. This means that $L_{1+a} = 1 + L_a$ is invertible and $u = L_{1+a}^{-1}(1)$ is a right inverse of $1 + a$. Similarly, $v = R_{1+a}^{-1}(1)$ exists and it is a left inverse of $1 + a$.

For any $c \in F$, one has $(1+a)(c+A), (c+A)(1+a) \subseteq c+A$. This means $L_{1+a}(1+A) = L_{1+a}^{-1}(1+A) = 1+A$ and $R_{1+a}(1+A) = R_{1+a}^{-1}(1+A) = 1+A$, which implies that $1 + A$ is a loop.

Finally, the map

$$F^* \times (1 + A) \to \{c + x \mid c \in F^*,\ x \in A\}, \quad (c, 1 + x) \mapsto c + cx$$

is bijective and preserves product, hence $\{c + x \mid c \in F^*,\ x \in A\} \subseteq A_1^*$. The converse inclusion being trivial, the proof is done. □

Let L be a finite loop of order n and let us denote by λ_x and ρ_x the left and right translation maps in L, respectively. Let M be the *multiplication group*

$$M = \langle \lambda_x, \rho_x \mid x \in L \rangle$$

of L, and let us denote by M_1 the stabilizer subgroup of $1 \in L$. Clearly M is a finite transitive permutation group on L and $n = |M : M_1|$ holds. Moreover, M_1 cannot include a proper normal subgroup of M.

For a given field F, we denote by $M_n(F)$ the space of $n \times n$ matrices over F. We put FL for the loop ring of L over F. We define the representation $\pi : M \to GL(n, F)$ by permutation matrices, π can be extended to a homomorphism $FM \to M_n(F)$ of associative algebras in an obvious way. We denote the extension by π, as well. By definition, for the loop ring FL, one has $L_x = \pi(\lambda_x)$ and $R_x = \pi(\rho_x)$ for all $x \in L$, which implies $\mathfrak{M}(FL) = \pi(FM)$.

Lemma 2.2. *Let L be a finite loop of prime power order $n = p^m$ and let be F a field of characteristic $p > 0$. Then the following statements are equivalent.*

(i) *The multiplication group M of L is nilpotent.*

(ii) *The order of the group M is a power of p.*

(iii) *The fundamental ideal $\Delta(M)$ is nilpotent.*

Proof. The equivalence of (ii) and (iii) is known from [1] and (ii) $\Longrightarrow$ (i) is trivial. If we assume (i), then we have the decomposition $M = H \times S_p$, where S_p denotes the p-Sylow subgroup of M and H is the product of the p'-Sylow subgroups. Since $|L|$ is a p-power, we have $H \leq M_1$ and since M acts faithfully, we have $H = \{1\}$. This shows (i) $\Longrightarrow$ (ii). □

Lemma 2.3. *Let L and F be as in Lemma 2.2, let M denote the multiplication group of L and assume that the fundamental ideal $\Delta(L)$ is nilpotent. Then we have $\pi(g) \in 1 + \mathfrak{M}(FL, \Delta(L))$ for all $g \in M$. In particular, M is isomorphic to a subgroup of $1 + \mathfrak{M}(FL, \Delta(L))$.*

Proof. Since $\Delta(L)$ is a nilpotent ideal of the loop ring FL, by Schafer's theorem, $\mathfrak{M}(FL, \Delta(L))$ is a nilpotent associative algebra. This means that $1 + \mathfrak{M}(FL, \Delta(L))$ is a subgroup of $GL(n, L)$. Since π is a faithful representation of M, all we have to show is that the images of the generators of M are contained in $1 + \mathfrak{M}(FL, \Delta(L))$. For any $x \in L$, we have $\pi(\lambda_x) = L_x = 1 + L_{x-1} \in 1 + \mathfrak{M}(FL, \Delta(L))$. Similarly, $\pi(\rho_x) \in 1 + \mathfrak{M}(FL, \Delta(L))$, which finishes the proof. □

Theorem 2.4. *Let L be a finite loop of prime power order p^n and let F a field of characteristic $p > 0$. Then, the fundamental ideal $\Delta(L)$ is nilpotent if and only if the multiplication group M of L is nilpotent.*

Proof. By Schafer's theorem, $\Delta(L)$ is nilpotent if and only if the associative algebra $\mathfrak{M}(FL, \Delta(L))$ is nilpotent. By $\mathrm{chr}(F) = p$, this implies that the group $1+\mathfrak{M}(FL, \Delta(L))$ has an exponent of the form p^k. Assuming now the nilpotency of $\Delta(L)$, we obtain by Lemma 2.3 that M is nilpotent with exponent a power of p.

Conversely, if we assume the nilpotency of M, we have by Lemma 2.2, that $\Delta(M)$ is a nilpotent algebra. Then, the proof is done if we show

$$\mathfrak{M}(FL, \Delta(L)) \leq \pi(\Delta(M)).$$

Since $\{1-x \mid x \in L \setminus \{1\}\}$ is a basis of $\Delta(L)$, the associative multiplication algebra $\mathfrak{M}(FL, \Delta(L))$ is generated by the set

$$\{L_{1-x}, R_{1-x} \mid x \in L\}.$$

As we have already seen, $L_{1-x} = 1-\pi(\lambda_x) = \pi(1-\lambda_x)$, $R_{1-x} = 1-\pi(\rho_x) = \pi(1-\rho_x)$, all the generators of $\mathfrak{M}(FL, \Delta(L))$ are contained in $\pi(\Delta(M))$. Thus, $\mathfrak{M}(FL, \Delta(L))$ and $\Delta(L)$ are nilpotent. $\square$

3. Goodaire's problem

In the last section, we apply Theorem 2.4 for the class of finite indecomposable Bol loop with a unique associator and commutator element. The following fact gives the reason for the importance of this class: By [4], if L is a loop of this class and F is a field of characteristic 2, then FL is a right alternative algebra.

We recall that the property P is said to hold *universally* for a loop L, if every loop isotope of L possesses P. In this sense, the following concept are universal: the isomorphism classes of M, L', $Z(L)$, right Bol property.

We say that the Bol loop L is a 2-*loop*, if every element has 2-power order. In general, it is not known if this property is universal or not. By [5, Theorem 3.2], L is a finite *universal Bol* 2-*loop* if and only if $|L|$ is a power of 2.

We are now in a position to answer Goodaire's question.

Theorem 3.1. *Let L be a finite indecomposable Bol loop with a unique associator and commutator element and let F be a field of characteristic 2. Then, the fundamental ideal $\Delta(L)$ is nilpotent.*

Proof. The theorem follows immediately from Theorem 2.4 and the next lemma.

Lemma 3.2. *Let L be a finite indecomposable Bol loop with a unique non-identity associator and commutator element. Then the multiplication group M of L is a 2-group. In particular, the order of L is a power of 2.*

Proof. Let us denote by s the unique non-identity associator and commutator element of L. The fact that such element exists is equivalent to $\{1, s\} = L' \leq Z(L)$ (cf. [4, Lemma 3.2]). Clearly, this last property and the indecomposability are both universal, that is, any isotope of L

possesses them. By [2, Theorem 4.1], these properties imply that L is a 2-loop, hence a universal 2-loop. Then, by [5, Theorem 3.2], the order of L is a power of 2.

Let us now take an element $g \in M_1$. Obviously, $g(x) \equiv x \mod L'$, hence $g(x) = x$ or $g(x) = sx$ for any element $x \in L$. It is also true that $g(sx) = sg(x)$, which implies $g^2(x) = x$ and $g^2 = 1$. That means that M_1 is an elementary Abelian 2-group and $|M| = |L||M_1|$ is a power of 2. □

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