Publ. Math. Debrecen
61 / 3-4 (2002), 555–577

Conservative semisprays on Finsler manifolds

By CSABA VINCZE (Debrecen)

Abstract. In this paper we present a general theory of conservative torsion-free horizontal endomorphisms (nonlinear connections) on a Finsler manifold (M, E). Since their torsion vanishes these endomorphisms can always be written in the form

$$h = \frac{1}{2}\big(1 + [J, S]\big),$$

where S is a semispray on M. This means that all of problems can be formulated in terms of semisprays as well. For example their existence problem will be completely solved including a representative process to construct such kind of horizontal endomorphisms. Moreover, putting a positive definite two-dimensional Finsler manifold (or special types of Finsler manifolds such as Riemannian and Randers manifolds of dimension n) we characterize all of them under the condition $\operatorname{div} S = 0$.

1. Introduction

In their work [8] the authors constructed special Finsler connections on a Finsler manifold starting out from torsion-free conservative horizontal endomorphisms. Among others it was proved that for any conservative torsion-free horizontal endomorphism h there exists a unique Finsler connection (D, h) on M such that

(i) D is metrical;

(ii) the $(v)v$-torsion of D vanishes;

(iii) the $(h)h$-torsion of D vanishes.

Mathematics Subject Classification: 53C07, 53C60.
Key words and phrases: horizontal endomorphisms; Finsler manifolds, conservativity.
This paper was supported by FKFP (0184/2001), and OTKA T32058.

As we can see in [8] the rules of calculation with respect to these connections are formally the same as those with respect to the classical Cartan connection. Moreover, adding a further condition to ones above we can get it back yet. Of course this is not the only example of such kind of Finsler connections. It is well-known (see e.g. [9]) that the so-called *Wagner connections* have a similar character. Actually we can say that a Wagner connection is a "*Cartan connection with nonvanishing* $(h)h$*-torsion*", i.e. it is a *generalized Cartan connection*. A necessary and sufficient condition for a Wagner connection to coincide with the classical Cartan connection is just that the Wagner endomorphism arises from a semispray, i.e. its torsion vanishes. Due to an explicit relation between the (canonical) Barthel endomorphism of (M, E) and a Wagner endomorphism the problem of existence is solved. We can easily construct Wagner connections on a Finsler manifold starting out from a smooth function $\alpha \in C^\infty(M)$. However, apart from the Barthel endomorphism the analogous problem is open in case of conservative torsion-free horizontal endomorphisms.

The purpose of this paper are the following:

- to give a necessary and sufficient condition for a torsion-free horizontal endomorphism to be conservative;
- to give a process to construct such kind of horizontal endomorphisms;
- to characterize them in case of two-dimensional Finsler manifolds (or Riemannian and Randers manifolds of dimension n) under the condition $\operatorname{div} S = 0$.

(We emphasize again that these questions can also be formulated in terms of semisprays!)

1. Preliminaries

1.1. Throughout the paper we use the terminology and conventions described in [8] (see also [2], [3] and [9]). Now we briefly summarize the basic notation.

(i) M is an $n\,(>1)$-dimensional, C^∞, connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) $\Omega^k(M)$ ($k \in \mathbb{N}^+$) is the module of (scalar) k-forms on M, $\Omega^\circ(M) := C^\infty(M)$.

(v) $\Psi^k(M)$ ($k \in \mathbb{N}^+$) is the $C^\infty(M)$-module of vector k-forms on M, $\Psi^\circ(M) := \mathfrak{X}(M)$.

(vi) ι_X, $\mathcal{L}_X$ ($X \in \mathfrak{X}(M)$) and d are the *insertion operator*, the *Lie-derivative* (with respect to X) and the *exterior derivative*, respectively.

1.2. We shall apply some simple facts of the Frölicher–Nijenhuis calculus of vector-valued forms. Recall that if $K \in \Psi^1(M)$, $Y \in \mathfrak{X}(M)$ then their *Frölicher–Nijenhuis bracket* $[K, Y] \in \Psi^1(M)$ acts as follows:

$$[K,Y](X) = [K(X), Y] - K[X,Y] \quad (X \in \mathfrak{X}(M)). \tag{1}$$

For the derivation d_K induced by K we have:

$$d_K f = df \circ K \qquad (f \in C^\infty(M)). \tag{2}$$

1.3. *Vertical apparatus. Semispray, spray.* Let us consider the tangent bundle $\pi : TM \to M$. $\mathfrak{X}^{\mathrm{v}}(TM)$ denotes the $C^\infty(TM)$-module of vertical vector fields on TM, $C \in \mathfrak{X}^{\mathrm{v}}(TM)$, $J \in \Psi^1(TM)$ are the *Liouville vector field* and *vertical endomorphism*, respectively. We have:

$$\begin{cases} \operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^{\mathrm{v}}(TM),\ J^2 = 0, \\ [C, J] = -J \quad \text{(i.e. } J \text{ is homogeneous of degree 0)}, \\ d_J = d_J \circ \mathcal{L}_C - \mathcal{L}_C \circ d_J. \end{cases} \tag{3}$$

The vertical lift of a vector field $X \in \mathfrak{X}(M)$ is denoted by X^{v}.

Definition. A mapping $S : v \in TM \to S(v) \in T_vTM$ is said to be a *semispray* on M if it satisfies the conditions:

(Spr1) S is smooth on $\mathcal{T}M$,

(Spr2) $JS = C$.

A semispray is called a *spray* if it is homogeneous of degree 2, i.e.

(Spr3) $[C, S] = S$.

(Note that (Spr3) implies for any spray to be a vector field of class C^1 on TM.)

The *vertical* and *complete lifts* of a function $\alpha \in C^\infty(M)$ are given by

$$\alpha^{\mathrm{v}} := \alpha \circ \pi, \quad \alpha^c := S\alpha^{\mathrm{v}}, \tag{4}$$

where S is an arbitrary semispray on M, respectively. For any $X \in \mathfrak{X}(TM)$

$$J[JX, S] = JX. \tag{5}$$

(For a proof see [2], p. 295.)

Remark 1. In the sequel we shall consider forms over TM or $\mathcal{T}M$. *Differentiability* of *vector* (and *scalar*) *k-forms will be required only over* $\mathcal{T}M$, unless otherwise stated.

1.4. *Horizontal endomorphisms* ([2], [3] and see also [8])

Definition. A vector 1-form $h \in \Psi^1(TM)$ is said to be a *horizontal endomorphism* on M if the following conditions are satisfied:

(He1) h is smooth over $\mathcal{T}M$,

(He2) h is a projector, i.e. $h^2 = h$,

(He3) $\operatorname{Ker} h = \mathfrak{X}^{\mathrm{v}}(TM)$.

The *associated semispray* of h is defined by the formula

$$S_h := h(S), \tag{6}$$

where S is an arbitrary semispray on M. The *tension* of h is the vector 1-form

$$H := [h, C] \in \Psi^1(TM). \tag{7}$$

The vector 2-form

$$t := [J, h] \in \Psi^2(TM) \tag{8}$$

is said to be the *torsion* of h. If $H = 0$, then h is called *homogeneous.*

Remark 2. It is a well-known fact (see e.g. [2]) that a horizontal endomorphism h arises from a semispray, i.e. it has a form

$$h = \frac{1}{2}\big(1 + [J, S]\big), \tag{9}$$

if and only if its torsion vanishes.

J and h are obviously related as follows:

$$h \circ J = 0, \quad J \circ h = J \tag{10}$$

and, furthermore, any horizontal endomorphism h determines an *almost complex structure* $F \in \Psi^1(TM)$ ($F^2 = -1$, F is smooth on $\mathcal{T}M$) such that

$$F \circ J = h, \quad F \circ h = -J. \tag{11}$$

(For the details see e.g. [2].)

1.5. *Finsler manifolds*

Definition. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold* with *energy function* E if the following conditions are satisfied:

(F0) $\forall v \in \mathcal{T}M : E(v) > 0$, $E(0) = 0$,

(F1) E is of class C^1 on TM and smooth on $\mathcal{T}M$,

(F2) $C(E) = 2E$ (i.e. E is homogeneous of degree 2),

(F3) the *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is symplectic.

The mapping

$$\begin{cases} g : \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M) \times \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \\ (JX, JY) \to g(JX, JY) := \omega(JX, Y) \end{cases} \tag{12}$$

is a well-defined, nondegenerate symmetric bilinear form which is said to be the *Riemann-Finsler metric* of (M, E). The Finsler manifold is called *positive definite* if g is positive definite.

We have the following important identities:

$$\iota_C \omega = d_J E, \quad \mathcal{L}_C \omega = \omega \tag{13}$$

(i.e. the fundamental form ω is homogeneous of degree 1).

The fundamental lemma of *Finsler geometry* [2]. On a Finsler manifold (M, E) there is a unique horizontal endomorphism h such that

(B1) h is conservative, i.e. $d_h E = 0$,

(B2) h is homogeneous,

(B3) h is torsion-free, i.e. $t = 0$.

Explicitly,

$$h = \frac{1}{2}\big(1 + [J, S]\big), \tag{14}$$

where S is the *canonical spray* defined by the formula

$$\iota_S \omega = -dE. \tag{15}$$

h is called the *Barthel endomorphism* of the Finsler manifold (M, E).

Let h be an *arbitrary* horizontal endomorphism on M, $\nu := 1 - h$. The mapping

$$\begin{cases} g_h : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \\ (X, Y) \to g_h(X, Y) := g(JX, JY) + g(\nu X, \nu Y) \end{cases} \tag{16}$$

is a well-defined pseudo-Riemannian metric on $\mathcal{T}M$ which is said to be the *prolongation* of g *along* h.

Definition. The tensor field $\mathcal{C}$ satisfying the condition

$$\omega(\mathcal{C}(X, Y), Z) := \frac{1}{2}\mathcal{L}_{JX}(J^* g_h)(Y, Z) \quad \big(X, Y, Z \in \mathfrak{X}(\mathcal{T}M)\big)$$

is called the *first Cartan tensor* of the Finsler manifold.

Remark 3. It is easy to check that $\mathcal{C}$ is independent of the choice of h and

(i) it is semibasic,

(ii) its *lowered tensor*

$$\mathcal{C}_\flat(X, Y, Z) := g(\mathcal{C}(X, Y), JZ)$$

is totally symmetric,

(iii) $\mathcal{C}^\circ := \iota_S\, \mathcal{C} = 0$ (S is an arbitrary semispray on M).

Let a smooth function $\varphi : TM \to \mathbb{R}$ (or $\varphi : \mathcal{T}M \to \mathbb{R}$) be given. Since the fundamental form ω is symplectic, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ such that

$$\iota_{\operatorname{grad}\varphi}\omega = d\varphi; \tag{17}$$

this vector field is called the *gradient* of φ.

Lemma 1. *The gradient vector field of a vertical lift* $\alpha^{\mathrm{v}} := \alpha \circ \pi$ $(\alpha \in C^\infty(M))$ *has the following properties:*

(i) $\operatorname{grad} \alpha^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$, $[C, \operatorname{grad} \alpha^{\mathrm{v}}] = -\operatorname{grad} \alpha^{\mathrm{v}}$ *(i.e. it is homogeneous of degree 0),*

(ii) $\operatorname{grad} \alpha^{\mathrm{v}}(E) = \alpha^c$,

(iii) $\iota_{F \operatorname{grad} \alpha^{\mathrm{v}}} \mathcal{C} = -\frac{1}{2}[J, \operatorname{grad} \alpha^{\mathrm{v}}]$.

For a proof see [7] and [9]. □

Definition [4]. Let (M, E) be a Finsler manifold and consider the volume form

$$w := \frac{(-1)^{n(n+1)/2}}{n!} \omega^n$$

on $\mathcal{T}M$. The *divergence* of a vector field $X \in \mathfrak{X}(TM)$ is the function $\operatorname{div} X$ given by formula

$$(\operatorname{div} X) w = \mathcal{L}_X w. \tag{18}$$

Lemma 2 [7]. *For any function* $\varphi \in C^\infty(TM)$

$$\operatorname{div}(\operatorname{grad} \varphi) = 0; \qquad \operatorname{div} C = n. \tag{19}$$

2. Conservative vector fields on a Finsler manifold

In what follows, we will denote the canonical spray and the Barthel endomorphism of a Finsler manifold (M, E) by S_h and h, respectively. (It is well-known that the canonical spray is just the semispray associated with h.)

Definition. Let (M, E) be a Finsler manifold. A *horizontal endomorphism* $\widetilde{h}$ on M is said to be *conservative* (with respect to the energy function E) if

$$d_{\widetilde{h}} E = 0.$$

A *semispray* $\widetilde{S}$ is called *conservative* if the induced horizontal endomorphism

$$\widetilde{h} := \frac{1}{2}\big(1 + [J, \widetilde{S}]\big)$$

is conservative.

A *vector field* $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is *conservative* if the semispray

$$\widetilde{S} := S_h + V$$

is conservative:

Proposition 1. *A vector field $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is conservative if and only if*

$$\iota_V \omega = d_J(VE), \tag{20}$$

or in an equivalent form, a semispray $\widetilde{S}$ is conservative if and only if

$$\iota_{\widetilde{S}}\omega + dE = d_J(\widetilde{S}E).$$

Proof. For any vector field $X \in \mathfrak{X}(\mathcal{T}M)$,

$$\begin{aligned} i_V\omega(X) - d_J(VE)(X) &= \omega(V,X) - JX(VE) = dd_JE(V,X) - JX(VE) \\ &= V(JX(E)) - J[V,X](E) - JX(VE) \\ &= -[JX,V](E) + J[X,V](E) \overset{(1)}{=} -[J,V](X)(E). \end{aligned}$$

On the other hand if $\widetilde{S} := S_h + V$ and $\widetilde{h}$ is the induced horizontal endomorphism then we get:

$$\begin{aligned} d_{\widetilde{h}}E(X) \ &= \ \widetilde{h}(X)(E) = \frac{1}{2}(1 + [J,\widetilde{S}])(X)(E) \\ &\overset{(14)}{=} h(X)(E) + \frac{1}{2}[J,V](X)(E) \overset{(\mathrm{B1})}{=} \frac{1}{2}[J,V](X)(E), \end{aligned}$$

which implies our statement. □

Corollary 1. *If $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is a conservative vector field then the following homogeneity properties are valid:*

$$C(VE) = V(E); \quad \mathcal{L}_C(d_J(VE)) = 0; \quad [C,V] = -V. \tag{21}$$

Proof. From the hypothesis it follows that

$$0 \ = \ \frac{1}{2}\big(1 + [J, S_h + V]\big)(S_h)(E) = h(S_h)(E) + \frac{1}{2}[J,V](S_h)(E)$$

$$\overset{(B1)}{=} \frac{1}{2}[J,V](S_h)(E) \overset{(1)}{=} \frac{1}{2}[C,V](E) - \frac{1}{2}J[S_h,V](E)$$
$$\overset{(5)}{=} \frac{1}{2}[C,V](E) + \frac{1}{2}V(E).$$

So we get

$$\begin{aligned} 0 &= [C,V](E) + V(E) = C(VE) - V(CE) + V(E) \\ &\overset{(F2)}{=} C(VE) - V(E) \Rightarrow C(VE) = V(E). \end{aligned}$$

Consequently,

$$\mathcal{L}_C(d_J(VE)) \overset{(3)}{=} d_J\mathcal{L}_C(VE) - d_J(VE) = d_J(VE) - d_J(VE) = 0.$$

Finally,

$$\begin{aligned} \iota_{[C,V]}\,\omega &= \mathcal{L}_C\,\iota_V\,\omega - \iota_V\,\mathcal{L}_C\,\omega \overset{(13)}{=} \mathcal{L}_C\,\iota_V\,\omega - \iota_V\,\omega \\ &\overset{(20)}{=} \mathcal{L}_C(d_J(VE)) - \iota_V\,\omega = -\iota_V\,\omega \Rightarrow [C,V] = -V. \end{aligned}$$ □

Corollary 2. *Let $\widetilde{S}$ be a conservative semispray; $V := \widetilde{S} - S_h$. Then*

$$[C,\widetilde{S}] = \widetilde{S} - 2V,$$

i.e. the deviation of $\widetilde{S}$ is just $-2V$.

Proof. Using the homogeneity property

$$[C,S_h] = S_h$$

of the canonical spray, an easy calculation shows that

$$[C,\widetilde{S}] - \widetilde{S} = [C,V] - V \overset{(21)}{=} -2V.$$ □

Corollary 3. *Consider a torsion-free conservative horizontal endomorphism $\widetilde{h}$ given by the formula*

$$\widetilde{h} = \frac{1}{2}(1 + [J,\widetilde{S}]),$$

where $\widetilde{S}$ is a conservative semispray; $V := \widetilde{S} - S_h$.

Then

(i) *the associated semispray of $\widetilde{h}$ coincides with the canonical spray S_h,*

(ii) $\widetilde{H} = [J, V]$.

Proof. It is well-known (see [2]) that the associated semispray of $\widetilde{h}$ is just

$$\widetilde{S} + \frac{1}{2}\widetilde{S}^*,$$

where $\widetilde{S}^*$ denotes the deviation of $\widetilde{S}$. Compare this relation with Corollary 2 we get our first statement.

Using Grifone's decomposition formula (see also [2]) (i) implies that

$$\widetilde{h} = h + \frac{1}{2}\widetilde{H}.$$

On the other hand

$$\widetilde{h} = \frac{1}{2}(1 + [J, \widetilde{S}]) = \frac{1}{2}(1 + [J, S_h]) + \frac{1}{2}[J, V] = h + \frac{1}{2}[J, V],$$

and thus

$$\widetilde{H} = [J, V]. \qquad \square$$

Remark 4. The converse of the previous statement is obviously true: If $\widetilde{h}$ is a torsion-free horizontal endomorphism on M such that

(i) the associated semispray of $\widetilde{h}$ coincides with the canonical semispray S_h, i.e. $S_{\widetilde{h}} = S_h$,

(ii) the tension of $\widetilde{h}$ can be written in the form

$$\widetilde{H} = [J, V],$$

where $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is a conservative vector field, then $\widetilde{h}$ is also conservative.

(Use Grifone's decomposition formula to prove this observation!)

Theorem 1. *Let (M,E) be a Finsler manifold and suppose that the function $\varphi \in C^\infty(\mathcal{T}M)$ is homogeneous of degree 1, i.e. $C(\varphi) = \varphi$.*

Then the vector field $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ defined by the formula

$$\iota_V\,\omega = d_J\,\varphi \tag{22}$$

is conservative.

Proof. It is clear from the definition that V is a vertical vector field. Substituting the canonical spray S_h into (22) we get

$$\iota_V\,\omega(S_h) = \omega(V, S_h) = -\omega(S_h, V) = -\iota_{S_h}\omega(V) \overset{(15)}{=} V(E).$$

On the other hand

$$d_J\,\varphi(S_h) \overset{(2)}{=} J(S_h)(\varphi) \overset{(\mathrm{Spr2})}{=} C(\varphi) = \varphi,$$

using the homogeneity property of φ. This means that the vector field V satisfies the relation

$$\iota_V\,\omega = d_J(VE),$$

i.e., by Proposition 1, V is conservative. □

Definition. Let (M,E) be a Finsler manifold. The mapping

$$\Delta : C^\infty(\mathcal{T}M) \to C^\infty(\mathcal{T}M)$$

$$\varphi \to \Delta\varphi := -\operatorname{div}(J\operatorname{grad}\varphi)$$

is called the *Brickell operator* of the Finsler manifold (M,E).

(A nice application of the elliptic differential operator Δ can be found in Brickell's paper [1].)

Remark 5. Let $\left(U, (u^i)_{i=1}^n\right)$ be a chart on M and consider the induced chart

$$\left(\pi^{-1}(U),\ (x^i, y^i)_{i=1}^n\right); \quad x^i := u^i \circ \pi,$$

$$y^i : v \in \pi^{-1}(U) \to y^i(v) := v(u^i) \quad (1 \le i \le n)$$

of the tangent manifold TM. Then we get the following coordinate expression:

$$J\operatorname{grad}\varphi = -g^{ij}\frac{\partial\varphi}{\partial y^i}\frac{\partial}{\partial y^j}, \tag{23}$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) = \frac{\partial^2 E}{\partial y^i \partial y^j}; \quad (g^{ij}) = (g_{ij})^{-1}.$$

$$\Delta\varphi = g^{ij}\frac{\partial^2 \varphi}{\partial y^i \partial y^j}. \tag{24}$$

(The proof is an easy straightforward calculation based on the definitions and [4].)

Lemma 3. *Let a function $\varphi \in C^\infty(\mathcal{T}M)$ be given and suppose that φ is homogeneous of degree 0. Then the following assertions are equivalent:*

(i) $\varphi = \alpha^{\mathrm{v}}$ $\big(\alpha \in C^\infty(M)\big)$, *i.e.* φ *is a vertical lift;*

(ii) $\Delta\varphi \geq 0$ *(or $\Delta\varphi \leq 0$).*

Proof. Since φ is homogeneous of degree 0, it attains a maximum (or a minimum) on each fiber T_pM $(p \in M)$. So the nontrivial implication (ii) $\Longrightarrow$ (i) is a special case of Hopf's strong maximum principle (see [11], p. 26). □

Proposition 2. *If $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is a conservative vector field, then*

$$J\,\mathrm{grad}(VE) = -V \tag{25}$$

and, consequently,

$$\Delta(VE) = \mathrm{div}\,V. \tag{26}$$

Proof. For any vector field $X \in \mathfrak{X}(\mathcal{T}M)$,

$$\begin{aligned}\omega(J\,\mathrm{grad}(VE), X) &\overset{(12)}{=} g(J\,\mathrm{grad}(VE), JX) = g(JX, J\,\mathrm{grad}(VE)) \\ &\overset{(12)}{=} \omega(JX, \mathrm{grad}(VE)) = -\omega(\mathrm{grad}(VE), JX) \\ &\overset{(17)}{=} -JX(VE) = -d_J(VE)(X) \overset{(20)}{=} -\iota_V\omega(X),\end{aligned}$$

from which it follows that

$$J\,\mathrm{grad}(VE) = -V. \qquad \square$$

Proposition 3. *The normalized Liouville vector field $C_0 := L^{-1}C$, where the function L is defined by the formula $E = \frac{1}{2}L^2$, is conservative. Then*

$$\Delta(C_0 E) = (n-1)\frac{1}{L}.$$

(Note that the function L is uniquely determined up to the sign!)

Proof. From (F2) we get immediately the homogeneity property

$$C(L) = L,$$

i.e. L is homogeneous of degree 1. According to Theorem 1, this means that the vector field $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ defined by the formula

$$i_V\,\omega = d_J L$$

is conservative.

Since

$$\iota_C\,\omega \overset{(13)}{=} d_J E = d_J\left(\frac{1}{2}L^2\right) = L d_J L = L\,\iota_V\,\omega,$$

we get the relation

$$C = LV \Rightarrow V = \frac{1}{L}C.$$

On the other hand

$$\begin{aligned}\Delta(C_0 E) &\overset{(26)}{=} \operatorname{div} C_0 = \operatorname{div}\left(\frac{1}{L}C\right)\\ &\overset{(18)}{=} \frac{1}{L}\operatorname{div} C + C\left(\frac{1}{L}\right) \overset{\text{Lemma 2}}{=} n\frac{1}{L} - \frac{1}{L^2}C(L)\\ &= (n-1)\frac{1}{L}.\end{aligned}$$

□

Remark 6. Since the normalized Liouville vector field can be considered as a *canonical* object of a Finsler manifold (M, E) it seems to be actually the most important example of conservative vector fields. Moreover, by the help of C_0, we can construct further canonical ones on (M, E).

For example, an easy calculation shows that the torsion-free conservative horizontal endomorphism h_0 induced by the conservative semispray

$$S_0 := S_h + C_0$$

and the Barthel endomorphism are related as follows:

$$h_0 = h + \frac{1}{2L}\left(J - \frac{1}{L} d_J L \otimes C\right). \tag{27}$$

Denote $\mathcal{C}'$ and $\mathcal{C}'_0$ the second Cartan tensors belonging to h and h_0, respectively (for the definition see [8]). Then for any vector fields $X, Y \in \mathfrak{X}(\mathcal{T}M)$,

$$\begin{aligned} \mathcal{C}'_0(X,Y) = \mathcal{C}'(X,Y) &+ \frac{1}{2L}\left\{\mathcal{C}(X,Y) - \frac{1}{L} JX(L)JY - \frac{1}{L} JY(L)JX\right\} \\ &+ \frac{1}{L^2}\left[\frac{1}{L} JX(L)JY(L) - \frac{1}{2} dd_J L(JX,Y)\right] C. \end{aligned} \tag{28}$$

Using Theorems 4.3, 4.5 and 4.9 of our stimulating paper [8], pp. 46–51, these formulas allow one to describe some new (more or less canonical) Finsler connections on the Finsler manifold (M, E):

- A *Berwald-type Finsler connection* associated with h_0 (see 4.3. Theorem in [8], p. 47);
- A *Cartan-type Finsler connection* associated with h_0 (see 4.5. Theorem in [8], p. 47);
- A *Chern–Rund-type Finsler connection* associated with h_0 (see 4.9. Theorem in [8], p. 50);
- etc. (see for example a *Hashiguchi-type Finsler connection* associated with h_0; [6]).

(It seems to be an important application of our results to the theory of Finsler connections.)

By using Theorem 1, the following observations can be easily verified:

(i) For any vector field $X \in \mathfrak{X}(M)$ the vertical lift X^{v} is conservative. Moreover, the induced horizontal endomorphism coincides with the Barthel endomorphism.

(ii) For any function $\alpha \in C^\infty(M)$ the gradient vector field $\operatorname{grad}\alpha^{\mathrm{v}}$ is conservative. Then the induced horizontal endomorphism h_α and the Barthel endomorphism are related as follows

$$h_\alpha = h - \iota_{F \operatorname{grad} \alpha^{\mathrm{v}}} \mathcal{C}.$$

(Use Lemma 1 to derive this relation; see [10], pp. 25–26.)

(iii) The conservative vector fields form a $C^\infty(M)$-module by the "scalar" multiplication

$$C^\infty(M) \times \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M) \to \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M), \quad (\alpha, V) \to \alpha^{\mathrm{v}} V.$$

3. Conservative vector fields on special Finsler manifolds

We have seen (c.f. Remark 6/(ii)) that for any function $\alpha \in C^\infty(M)$ the gradient vector field $V := \operatorname{grad}\alpha^{\mathrm{v}}$ is conservative. Then, as an easy calculation shows, $\operatorname{div} V = 0$. In what follows we give some analogous results in case of special Finsler manifolds that it is essentially true vice-versa: if V is a conservative divergence-free vector field then it can be expressed (at least locally) as a linear combination of gradient vector fields with respect to the scalar multiplication introduced in Remark 6/(iii).

Theorem 2. *Let (M, E) be a two-dimensional, positive definite Finsler manifold and suppose that $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is conservative. Then the following assertions are equivalent:*

(i) $\operatorname{div} V = 0$;

(ii) *for any chart $\big(U, (u^i)_{i=1}^{2}\big)$ on M,*

$$V = \alpha_1^{\mathrm{v}} \operatorname{grad} x^1 + \alpha_2^{\mathrm{v}} \operatorname{grad} x^2$$

$$(x^i := u^i \circ \pi,\ \alpha_i \in C^\infty(M),\ 1 \leq i \leq 2).$$

Proof. First of all we recall that the divergence of a vertical vector field $JX \in \mathfrak{X}^{\mathrm{v}}(TM)$ can be calculated by the formula

$$\operatorname{div} JX = \widetilde{[J, JX]} + 2\widetilde{\mathcal{C}}(X), \tag{29}$$

where $\widetilde{[J, JX]}$ and $\widetilde{\mathcal{C}}$ are the semibasic trace of the vector 1-form $[J, JX]$ and $\mathcal{C}$, respectively.

(For the definitions and proof see [4].)

Using Lemma 1 and (29) we get immediately the implication (ii) $\Longrightarrow$ (i). It remains only to show that (i) $\Longrightarrow$ (ii) is also valid. Let $\left(U, (u^i)_{i=1}^{2}\right)$ be an arbitrarily fixed chart on M. Then we have the following local expressions for the vector field V:

$$V \upharpoonright \pi_0^{-1}(U) = V^i \frac{\partial}{\partial y^i} = \beta_i \operatorname{grad} x^i,$$

where $V^i = g^{ij}\beta_j$ $(1 \leq i \leq 2)$.

Formula (20) shows that

$$V^i g_{ij} = \frac{\partial}{\partial y^j}(VE)$$

and, consequently, $\beta_j = \frac{\partial}{\partial y^j}(VE)$ $(1 \leq j \leq 2)$.

Using this coordinate expression of the functions β_1 and β_2 the homogeneity properties (21) give rise to the relations

$$C(\beta_j) = y^i \frac{\partial}{\partial y^i}(\beta_j) = 0 \qquad (1 \leq j \leq 2); \tag{30}$$

$$y^i \frac{\partial}{\partial y^j}(\beta_i) = 0 \qquad (1 \leq j \leq 2); \tag{31}$$

$$\frac{\partial \beta_i}{\partial y^j} = \frac{\partial \beta_j}{\partial y^i} \qquad (1 \leq i, j \leq 2). \tag{32}$$

From the hypotesis $\operatorname{div} V = 0$ and (24), (26) we get that

$$\begin{aligned} 0 &= y^1 y^2 \left(g^{ij} \frac{\partial^2}{\partial y^i \partial y^j}(VE) \right) = y^1 y^2 \left(g^{ij} \frac{\partial}{\partial y^i}(\beta_j) \right) \\ &\overset{(32)}{=} y^1 y^2 \left(g^{11} \frac{\partial}{\partial y^1}(\beta_1) + 2g^{12} \frac{\partial}{\partial y^2}(\beta_1) + g^{22} \frac{\partial}{\partial y^2}(\beta_2) \right) \\ &\overset{(30),(31),(32)}{=} \Big(-(y^2)^2 g^{11} + 2y^1 y^2 g^{12} - (y^1)^2 g^{22} \Big) \frac{\partial}{\partial y^2}(\beta_1). \end{aligned}$$

Let $K \in \mathbb{R} \setminus \{0\}$ be an arbitrary *nonzero* real number and consider a tangent vector

$$v = v^1 \frac{\partial}{\partial u^1_p} + v^2 \frac{\partial}{\partial u^2_p} \in T_p M \setminus \{0\}$$

such that $v^2 = Kv^1$.

Suppose that

$$-(v^2)^2 g^{11}(v) + 2v^1v^2 g^{12}(v) - (v^1)^2 g^{22}(v) = 0, \text{ i.e.}$$

$$\pm g^{12}(v) = \frac{g^{11}(v)|K| + g^{22}(v)\frac{1}{|K|}}{2} \geq \sqrt{g^{11}(v)g^{22}(v)} \Rightarrow \det g^{ij}(v) \leq 0,$$

which is a contradiction to the condition of positive definiteness. Thus we get that

$$\frac{\partial}{\partial y^2}(\beta_1) \overset{(32)}{=} \frac{\partial}{\partial y^1}(\beta_2) = 0$$

holds everywhere except at the points of a set of measure zero (in case of $v^1 = 0$ or $v^2 = 0$). Since the functions β_1 and β_2 are smooth over $\pi_0^{-1}(U)$, this means that for *any* tangent vector $v \in \pi_0^{-1}(U)$

$$\frac{\partial}{\partial y_v^2}(\beta_1) = \frac{\partial}{\partial y_v^1}(\beta_2) = 0. \tag{33}$$

Using the homogeneity properties (30) of the functions β_i $(1 \leq i \leq 2)$ we can deduce the relations

$$\frac{\partial}{\partial y^1}(\beta_1) = \frac{\partial}{\partial y^2}(\beta_2) = 0 \tag{34}$$

in a similar manner. (33) and (34) imply the functions β_1 and β_2 to be vertical lifts:

$$\beta_i = \alpha_i \circ \pi \quad (\alpha_i \in C^\infty(M),\ 1 \leq i \leq 2). \qquad \square$$

Remark 7. (i) Since for any Finsler manifold the canonical spray S_h is divergence-free, i.e. $\operatorname{div} S_h = 0$, the conditions $\operatorname{div} V = 0$ and $\operatorname{div} \widetilde{S} = 0$ $(\widetilde{S} := S_h + V)$ are equivalent. Indeed,

$$\begin{aligned}\mathcal{L}_{S_h}\omega^n &= \iota_{S_h} d\omega^n + d\iota_{S_h}\omega^n \overset{(F3)}{=} d\iota_{S_h}\omega^n \\ &= nd(\iota_{S_h}\omega \wedge \omega^{n-1}) \overset{(15),\,(F3)}{=} 0,\end{aligned}$$

so we get the vanishing of $\operatorname{div} S_h$.

(ii) It can be easily seen from the proof of Theorem 2 that in case of $\operatorname{div} V = 0$, the function $V(E)$ is necessarily linear on each fiber T_pM ($p \in M$). Keeping our previous notations, for any chart $\big(U, (u^i)_{i=1}^{2}\big)$ on M,

$$V(E) \restriction \pi^{-1}(U) = y^1 \alpha_1^{\mathrm{v}} + y^2 \alpha_2^{\mathrm{v}}.$$

According to the implication (i) $\implies$ (ii) we can also say that the condition $\operatorname{div} V = 0$ determines a (locally) finitely generated nontrivial submodule of the module of conservative vector fields. (The trivial example of such a submodule is just the module $\mathfrak{X}(M)$; cf. Remark 6/(i), (iii).)

(iii) An alternative reasoning to prove the implication (i) $\implies$ (ii) can be realized as follows.

From the hypothesis $\operatorname{div} V = 0$ we get immediately that

$$\begin{aligned} 0 &\overset{(18)}{=} \mathcal{L}_V \omega^2 = \iota_V (d\omega^2) + d(\iota_V\, \omega^2) \overset{(\mathrm{F3})}{=} d(\iota_V\, \omega^2) \\ &= d(2\iota_V \omega \wedge \omega) \overset{(20)}{=} 2(dd_J\beta) \wedge \omega, \end{aligned}$$

where $\beta := V(E)$.

Using the vanishing of the form $(dd_J\beta) \wedge \omega$ it follows that for any vector field $X \in \mathfrak{X}(M)$,

$$\iota_{X^{\mathrm{v}}} \big[(dd_J\beta) \wedge \omega\big] = \iota_{X^h} \big[(dd_J\beta) \wedge \omega\big] = 0$$

and, consequently,

$$\begin{aligned} \iota_{X^{\mathrm{v}}} (dd_J\beta) \wedge \omega &= -(dd_J\beta) \wedge \iota_{X^{\mathrm{v}}} \omega, \\ \iota_{X^h} (dd_J\beta) \wedge \omega &= -(dd_J\beta) \wedge \iota_{X^h} \omega. \end{aligned}$$

By the help of these formulas it can be easily deduced that for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$(dd_J\beta)(X^{\mathrm{v}}, Y^h)\omega^2 = -2 dd_J\beta \wedge \iota_{X^{\mathrm{v}}}\, \omega \wedge \iota_{Y^h} \omega.$$

According to the tensorial character of our result we have:

$$(dd_J\beta)(JX, Y)\omega^2 = -2 dd_J\beta \wedge \iota_{JX}\, \omega \wedge \iota_Y\, \omega \qquad (X, Y \in \mathfrak{X}(TM)).$$

Let now $\left(U,(u^i)_{i=1}^{2}\right)$ be an arbitrary chart on M and, for brevity, let us set

$$\partial_k := \frac{\partial}{\partial y^k}, \quad \delta_k := \left(\frac{\partial}{\partial u^k}\right)^h, \quad \beta_k := \partial_k \beta \qquad (1 \leq k \leq 2).$$

Then

$$\begin{aligned} &dd_J\beta(J\,\mathrm{grad}\,\beta_k, \delta_k)\omega^2(\partial_1, \partial_2, \delta_1, \delta_2) \\ &\qquad = -2dd_J\beta(J\,\mathrm{grad}\,\beta_k, \delta_k)\det\left[(g_{ij})_{1\leq i,j\leq 2}\right]. \end{aligned}$$

On the other hand

$$\begin{aligned} dd_J\beta \wedge \iota_{J\,\mathrm{grad}\,\beta_k}\omega \wedge \iota_{\delta_k}\omega(\partial_1, \partial_2, \delta_1, \delta_2) = g_{k1}\Big((\partial_2\partial_2\beta)(\partial_1\partial_k\beta) \\ - (\partial_1\partial_2\beta)(\partial_2\partial_k\beta)\Big) + g_{k2}\Big((\partial_1\partial_1\beta)(\partial_2\partial_k\beta) - (\partial_1\partial_2\beta)(\partial_1\partial_k\beta)\Big). \end{aligned}$$

Since

$$\begin{aligned} dd_J\beta(J\,\mathrm{grad}\,\beta_k, \delta_k) &= (J\,\mathrm{grad}\,\beta_k)(\beta_k) = \omega(\mathrm{grad}\,\beta_k, J\,\mathrm{grad}\,\beta_k) \\ &= -g(J\,\mathrm{grad}\,\beta_k, J\,\mathrm{grad}\,\beta_k) = -\|J\,\mathrm{grad}\,\beta_k\|^2, \end{aligned}$$

we get that

$$-\det\left[(g_{ij})_{1\leq i,\,j\leq 2}\right] \cdot \sum_{k=1}^{2} \|J\,\mathrm{grad}\,\beta_k\|^2 = \det\left[(\partial_i\partial_j\beta)_{1\leq i,\,j\leq 2}\right](g_{11} + g_{22}).$$

Suppose that

$$\det\left[(\partial_i\partial_j\beta)_{1\leq i,\,j\leq 2}\right](v) \neq 0 \qquad (v \in T_pM).$$

This means that the mapping

$$\begin{aligned} (\beta_1, \beta_2) : T_pM &\to \mathbb{R}^2 \\ w &\to (\beta_1, \beta_2)(w) := (\beta_1(w), \beta_2(w)) \end{aligned}$$

is a local diffeomorphism at the "point" $v \in T_pM$, which contradicts the homogeneity property of the functions β_1, β_2. Indeed, these functions are obviously homogeneous of degree 0 (i.e. "constant along rays"); cf. Corollary 1.

The contradiction implies the vanishing of the vector fields $J\,\mathrm{grad}\,\beta_1$, $J\,\mathrm{grad}\,\beta_2$ and, consequently, it follows that β_1, β_2 are constant on each fiber T_pM $(p \in M)$. □

Corollary 4. *Suppose that (M, E) is a two-dimensional, positive definite Finsler manifold. If $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is a conservative vector field such that $\operatorname{div} V = 0$, then the induced horizontal endomorphism*

$$\widetilde{h} := \frac{1}{2}(1 + [J, \widetilde{S}]),$$

where $\widetilde{S} := S_h + V$, has the following simple form:

$$\widetilde{h} = h - \iota_{FV}\mathcal{C}.$$

(Cf. Remark 6/(ii).)

Definition. Let α be a Riemannian metric and β a (nonzero) 1-form on the manifold M. Consider the functions

$$(35)\qquad \begin{cases} L_\alpha : TM \to \mathbb{R}, & v \to L_\alpha(v) := [\alpha_{\pi(v)}(v, v)]^{1/2}; \\ \widetilde{\beta} : TM \to \mathbb{R}, & v \to \widetilde{\beta}(v) := \beta_{\pi(v)}(v); \\ L := L_\alpha + \widetilde{\beta}; & E := \dfrac{1}{2}L^2. \end{cases}$$

If

$$\|\widetilde{\beta}\| := \sup_{v \in \mathcal{T}M} \frac{\widetilde{\beta}(v)}{L_\alpha(v)} < 1,$$

then (M, E) is a Finsler manifold which is said to be the *Randers manifold* constructed from the Riemann manifold (M, α) by the perturbation with $\widetilde{\beta}$.

Remark 8. Consider a chart $\left(U, (u^i)_{i=1}^{n}\right)$ on M. If

$$\alpha_{ij} := \alpha\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \quad (\alpha^{ij}) := (\alpha_{ij})^{-1};$$

$$\beta_i := \beta\left(\frac{\partial}{\partial u^i}\right), \qquad \beta^i := \alpha^{ij}\beta_j,$$

$$\beta^\# \restriction U = \beta^i \frac{\partial}{\partial u^i}, \qquad b^2 := \|\beta^\#\|_\alpha^2 := \alpha(\beta^\#, \beta^\#)$$

then we have the following coordinate expression (see e.g. [5], p. 209):

$$(36)\quad g^{ij} = \frac{L_\alpha}{L}\big(\alpha^{ij}\circ\pi\big) - \frac{L_\alpha}{L^2}(y^i(\beta^j\circ\pi)+y^j(\beta^i\circ\pi)) + \frac{L_\alpha(b^2\circ\pi)+\widetilde{\beta}}{L^3}y^iy^j.$$

Theorem 3. *Let (M,E) be a positive definite Randers manifold of dimension n and suppose that $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is conservative. Then the following assertions are equivalent:*

(i) $\operatorname{div} V = 0$;

(ii) *for any chart $\big(U,(u^i)_{i=1}^{n}\big)$ on M,*

$$V \restriction \pi_0^{-1}(U) = \gamma_1^{\mathrm{v}}\operatorname{grad} x^1 + \cdots + \gamma_n^{\mathrm{v}}\operatorname{grad} x^n$$
$$(x^i := u^i\circ\pi,\ \gamma_i \in C^\infty(M),\ 1\le i\le n).$$

Proof. The implication (ii) $\implies$ (i) is trivial (cf. Theorem 2). To prove (i) $\implies$ (ii) first of all let

$$V \restriction \pi_0^{-1}(U) = \delta_1 \operatorname{grad} x^1 + \cdots + \delta_n \operatorname{grad} x^n,$$

where $\delta_i \in \mathcal{C}^\infty(\pi_0^{-1}(U))$, $1 \le i \le n$. In a similar manner as in the proof of Theorem 2, we get that for any indeces $i \in \{1,\dots,n\}$

$$(37)\qquad \delta_i = \frac{\partial}{\partial y^i}\ (VE).$$

(Note that the basic properties (30)–(32) of the coefficients are obviously not depend on the condition of dimensionality!)

The hypothesis $\operatorname{div} V = 0$ and (26) implies that

$$(38)\qquad \begin{aligned} 0 &= \Delta(VE) \overset{(24)}{=} g^{ij}\frac{\partial^2}{\partial y^i\partial y^j}(VE) \overset{(36),(21)}{=} \\ &= \frac{L_\alpha}{L}(\alpha^{ij}\circ\pi)\frac{\partial^2}{\partial y^i\partial y^j}(VE) \ \Rightarrow \\ 0 &= (\alpha^{ij}\circ\pi)\frac{\partial^2}{\partial y^i\partial y^j}(VE) = \Delta_\alpha(VE), \end{aligned}$$

where Δ_α denotes the Brickell operator of the Finsler (especially Riemann) manifold (M,E_α); $E_\alpha := \frac{1}{2}L_\alpha^2$.

Differentiating (38) by $\frac{\partial}{\partial y^k}$ it follows that

$$0 = \frac{\partial}{\partial y^k}(\Delta_\alpha(VE)) \overset{(37)}{=} \Delta_\alpha \delta_k.$$

Thus Lemma 3 implies the functions $\delta_1, \dots, \delta_k$ to be vertical lifts, i.e.

$$\delta_k = \gamma_k \circ \pi \qquad (\gamma_k \in C^\infty(M),\ 1 \leq k \leq n). \qquad \square$$

Corollary 5. *Let (M, E) be a positive definite Randers manifold and suppose that $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ is a conservative vector field such that $\operatorname{div} V = 0$. Then the induced horizontal endomorphism*

$$\widetilde{h} = \frac{1}{2}(1 + [J, \widetilde{S}]),$$

where $\widetilde{S} := S_h + V$, has the following simple form: $\widetilde{h} = h - \iota_{FV}\mathcal{C}$.

(Cf. Corollary 4.)

Corollary 6. *Suppose that (M, E) is a (positive definite) Riemann manifold and let $V \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M)$ be a conservative vector field.*

Then the following assertions are equivalent:

(i) $\operatorname{div} V = 0$;

(ii) $V = X^{\mathrm{v}}$ $(X \in \mathfrak{X}(M))$, *i.e. V is a vertical lift.*

Moreover, the induced horizontal endomorphism $\widetilde{h}$ coincides with the Barthel endomorphism.

Proof. Since a Riemann manifold (M, E) can be considered as a special Randers manifold with $\widetilde{\beta} \equiv 0$, Theorem 3 implies the equivalence

(iii) $\operatorname{div} V = 0$

(iv) $V \upharpoonright \pi_0^{-1}(U) = \gamma_1^{\mathrm{v}} \operatorname{grad} x^1 + \dots + \gamma_n^{\mathrm{v}} \operatorname{grad} x^n$ for an arbitrary chart $\big(U, (u^i)_{i=1}^{n}\big)$ on M.

Here, of course, all of vector fields $\operatorname{grad} x^i$ are vertical lifts, i.e.

$$\operatorname{grad} x^i = X_i^{\mathrm{v}} \qquad (X_i \in \mathfrak{X}(M),\ 1 \leq i \leq n).$$

(Note that $\operatorname{grad} x^i$ are just the vertical lifts of Riemannian gradients

$$\operatorname{grad}_R u^i \in \mathfrak{X}(U) \qquad (1 \leq i \leq n).)$$

Consequently,

$$V \restriction \pi_0^{-1}(U) = \gamma_1^{\mathrm{v}} X_1^{\mathrm{v}} + \dots + \gamma_n^{\mathrm{v}} X_n^{\mathrm{v}} = (\gamma_1 X_1 + \dots + \gamma_n X_n)^{\mathrm{v}}.$$

The relation $\widetilde{h} = h$ is trivial. □

References

[1] F. Brickell, A new proof of Deicke's theorem on homogeneous functions, *Proc. AMS* **16** (1965), 190–191.

[2] J. Grifone, Structure presque-tangente et connexions I, *Ann. Inst. Fourier, Grenoble* **22**, no. 1 (1972), 287–334.

[3] J. Grifone, Structure presque-tangente et connexions II, *Ann. Inst. Fourier, Grenoble* **22**, no. 3 (1972), 291–338.

[4] J. Grifone, Transformations infinitésimales conformes d' une variété finslerienne, *C. R. Acad. Sc. Paris, Sér. A* **280** (1975), 519–522.

[5] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press*, *Otsu, Japan*, 1986.

[6] J. Szilasi, Notable Finsler connections on a Finsler manifold, *Lecturas Matemáticas* **19** (1998), 7–34.

[7] J. Szilasi and Cs. Vincze, On conformal equivalence of Riemann-Finsler metrics, *Publ. Math. Debrecen* **52** (1998), 167–185.

[8] J. Szilasi and Cs. Vincze, A new look at Finsler connections and special Finsler manifolds, *www.emis.de/journals AMAPN* **16** (2000), 33–63.

[9] Cs. Vincze, On Wagner connections and Wagner manifolds, *Acta Math. Hungar.* **89** 1–2 (2000), 111–133.

[10] Cs. Vincze, On conformal equivalence of Riemann-Finsler metrics and special Finsler manifolds, Ph.D. dissertation, *University of Debrecen*, *Debrecen, Hungary*, 2000.

[11] K. Yano and S. Bochner, Curvature and Betti numbers, Annals of Mathematics Studies No. 32, *Princeton Univ. Press, Princeton, N. J.*, 1953.

CS. VINCZE
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: csvincze@math.klte.hu

(Received October 10, 2001; revised March 7, 2002)