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Solution of some functional equations involving symmetric means

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Abstract. This article concerns the study of functional equations of the form $f(M(x,y)) + f\left(\frac{xy}{M(x,y)}\right) = f(x) + f(y)$, where M is a given symmetric mean, and where f is unknown. The problem of finding general solutions of equations of this type was posed at the 37th International Symposium on Functional Equations in 1999 by Z. Daróczy and J. Matkowski. We present the general solution of this equation when M is the arithmetic mean or a root-power mean.

1. Introduction

This article deals mainly with the functional equation

$$f(x) + f(y) = f\left(\frac{x+y}{2}\right) + f\left(\frac{2xy}{x+y}\right), \tag{1}$$

for f mapping an open subinterval of the positive reals into the real numbers. The problem of finding all solutions of equation (1) was raised by Z. Daróczy [3] at the Thirty-seventh International Symposium on Functional Equations.

At the same symposium, J. Matkowski [4] posed the following related question. Let $I \subset \mathbb{R}_+ = (0, \infty)$ be an interval. Does there exist a symmetric mean $M : I^2 \to I$, different from the geometric mean, such

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that every function $f : I \to \mathbb{R}$ satisfying the functional equation

$$f\left(M(x,y)\right) + f\left(\frac{xy}{M(x,y)}\right) = f(x) + f(y), \quad x, y \in I,$$

must be of the form $f = L + c$ where L is a logarithmic function and c is a real constant? (A function $L : \mathbb{R}_+ \to \mathbb{R}$ is said to be logarithmic if it satisfies the Cauchy logarithmic functional equation $L(xy) = L(x) + L(y)$ for all $x, y \in \mathbb{R}_+$.)

In this article, we provide an affirmative answer to Matkowski's question in the case of the arithmetic mean (thus solving Daróczy's problem). We also show that there are other symmetric means for which the answer to Matkowski's question is positive. Lastly, we give the solutions of another generalization of (1).

2. Solution of (1)

For simplicity we assume that $I \subset \mathbb{R}_+$ is an open interval. (If I is half-open or closed, the endpoints can be handled easily after obtaining the solution on the interior of I.) Let $f : I \to \mathbb{R}$ be a solution of (1). We begin by writing (1) in the form

$$f(x) + f(y) - f\left(\frac{x+y}{2}\right) = f\left(\frac{2xy}{x+y}\right)$$

and defining $F : I^2 \to \mathbb{R}$ by

$$F(x,y) := f(x) + f(y) - f\left(\frac{x+y}{2}\right), \quad x, y \in I.$$

By definition, F satisfies for all $x, y, z, w \in I$

$$\begin{aligned} &F(x,y) + F(z,w) + F\left(\frac{x+y}{2}, \frac{z+w}{2}\right) \\ &= F(x,z) + F(y,w) + F\left(\frac{x+z}{2}, \frac{y+w}{2}\right). \end{aligned}$$

Applying this condition to the fourth term of (1), we see that

$$\begin{aligned}
&f\left(\frac{2xy}{x+y}\right)+f\left(\frac{2zw}{z+w}\right)+f\left(\frac{(x+y)(z+w)}{x+y+z+w}\right)\\
&=f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yw}{y+w}\right)+f\left(\frac{(x+z)(y+w)}{x+y+z+w}\right)
\end{aligned} \tag{2}$$

for all $x, y, z, w \in I$. Our plan is to substitute

$$y=\frac{xz(x+z)}{2x^2-xz+z^2}, \quad w=\frac{xz(x+z)}{x^2-xz+2z^2}$$

into (2). In order to do this, we must verify that such y, w will belong to I. That is the motivation for the following two lemmas.

Lemma 1. *Let $x, z \in I \subset \mathbb{R}_+$ and define maps $\lambda, \mu : I^2 \to \mathbb{R}_+$ by*

$$\lambda(x,z)=\frac{xz(x+z)}{2x^2-xz+z^2}, \quad \mu(x,z)=\frac{xz(x+z)}{x^2-xz+2z^2}, \quad x, z \in I. \tag{3}$$

Then $0<\lambda(x,z)\le z$ and $0<\mu(x,z)\le x$ for all $x, z \in I$. Hence we have $\lambda, \mu \in I$ if I has 0 as its left endpoint.

Proof. First, observe that the denominators of λ, μ as defined by (3) are positive, since both are greater than $(x-z)^2$ for $x, z > 0$. Thus we have $\lambda, \mu > 0$ for all $x, z > 0$.

Second, from $0 \le (x-z)^2$ we deduce also that $xz(x+z) \le z(2x^2 - xz + z^2)$ for $x, z > 0$. This proves $\lambda(x,z) \le z$. A parallel argument proves $\mu(x,z) \le x$. □

Next, we look more deeply into the case I has left endpoint $a > 0$.

Lemma 2. *Suppose $I = (a,b)$ or (a,∞) with $a > 0$. Then for all $x, z \in I$ we have:*

(i) *$\mu(x,z) > a$ for all $x \in (a,2a)$, $z < T(x)$, where $T : (a,2a) \to (a,\infty)$ is defined by*

$$T(x) := \frac{x\left(x+a+\sqrt{x^2+6ax-7a^2}\right)}{2(2a-x)}, \quad x \in (a,2a);$$

(ii) *$T(x) > x$ for all $x \in (a,2a)$, and $T(x) \to \infty$ as $x \to 2a^-$;*

(iii) *$\mu(x,z) > a$ for all $x \geq 2a$, $z > a$;*

(iv) *$\lambda(x,z) > a$ for all $x > a$, $z > B(x)$, where $B : (a,\infty) \to (a,\infty)$ is defined by*

$$B(x) := \frac{4ax}{x + a + \sqrt{x^2 + 10ax - 7a^2}}, \quad x > a;$$

(v) *$B = T^{-1}$;*

(vi) *$a < B(x) < \min(x, 2a)$ for all $x > a$, and $B(x) \to 2a^-$ as $x \to \infty$.*

PROOF. (i) Suppose $x, z > a > 0$. Then $\mu(x,z) > a$ is equivalent to

$$(2a - x)z^2 - x(x+a)z + ax^2 < 0. \tag{4}$$

If $x < 2a$, then we must have

$$\frac{x\left(x + a - \sqrt{x^2 + 6ax - 7a^2}\right)}{2(2a - x)} < z < \frac{x\left(x + a + \sqrt{x^2 + 6ax - 7a^2}\right)}{2(2a - x)}.$$

The second of these inequalities is $z < T(x)$. We claim that the first is automatically satisfied in this case. Indeed,

$$\frac{x\left(x + a - \sqrt{x^2 + 6ax - 7a^2}\right)}{2(2a - x)} = \frac{2ax}{x + a + \sqrt{x^2 + 6ax - 7a^2}},$$

which is easily seen to be less than a for all $x > a$.

(ii) If $x \in (a, 2a)$, then $T(x) > x$ is equivalent to $\sqrt{x^2 + 6ax - 7a^2} > 3(a - x)$, which is obviously true since the right side is negative. It is also obvious that $T(x) \to \infty$ as $x \to 2a^-$.

(iii) For all $x \geq 2a$, $z > a$, (4) is valid, since

$$(x - 2a)z^2 + x(x+a)z - ax^2 \geq x^2(z - a) + axz > axz > 0.$$

(iv) For $x, z > a$, we have $\lambda(x,z) > a$ if and only if

$$(x - a)z^2 + x(x+a)z - 2ax^2 > 0.$$

Since $x > a$, $z > 0$, this is equivalent to

$$z > \frac{-x(x+a) + x\sqrt{x^2 + 10ax - 7a^2}}{2(x - a)},$$

that is, $z > B(x)$.

(v) It is easy to see that $T(x)$ is strictly increasing on $(a, 2a)$, since $a > 0$. Thus T has an inverse function $T^{-1} : (a, \infty) \to (a, 2a)$. If $y = T^{-1}(x) \in (a, 2a)$ for $x \in (a, \infty)$, then

$$x = \frac{y\left(y + a + \sqrt{y^2 + 6ay - 7a^2}\right)}{2(2a - y)} = \frac{2ay}{y + a - \sqrt{y^2 + 6ay - 7a^2}},$$

or

$$x(y + a) - 2ay = x\sqrt{y^2 + 6ay - 7a^2}.$$

Since $x > a$, $x > y = T^{-1}(x)$, the equation above is equivalent to

$$(x - a)y^2 + x(x + a)y - 2ax^2 = 0.$$

Since $y > 0$, this in turn is equivalent to

$$y = \frac{-x(x + a) + x\sqrt{x^2 + 10ax - 7a^2}}{2(x - a)} = B(x).$$

That is, $T^{-1} = B$.

(vi) This follows immediately from (i), (ii), and (v), and the proof is complete. $\square$

Summarizing the results of the two lemmas, we have the following.

Proposition 3. *Let $I = (a, b)$ or (a, ∞) be an open subinterval of $\mathbb{R}_+$, define $\lambda, \mu : I^2 \to \mathbb{R}_+$ as in (3), and define maps B, T as in Lemma 2. If $a = 0$, define $\Omega = I^2$; if $a > 0$, define*

$$\Omega = I^2 \cap \big(\{(x, z) : x \in (a, 2a),\ z \in (B(x), T(x))\} \cup \{(x, z) : x \geq 2a,\ z > B(x)\}\big).$$

Then we have $\lambda, \mu \in I$ whenever $(x, z) \in \Omega$. Further, $(x, x) \in \Omega$ for all $x \in I$.

Proof. All but the last statement is obvious. For the last statement, note that $B(x) < x$ for all $x > a$ and $x < T(x)$ for all $x \in (a, 2a)$ by Lemma 2. $\square$

Now we are ready to proceed with the solution of (1).

Theorem 4. *Let I be an open subinterval of $\mathbb{R}_+$, and suppose $f : I \to \mathbb{R}$ satisfies* (1). *Then there exists a logarithmic function $L : \mathbb{R}_+ \to \mathbb{R}$ and a real constant c for which $f(x) = L(x) + c$, $x \in I$.*

Proof. At the start of this section, we saw that (1) implies (2). In the latter equation we now substitute

$$y = \frac{xz(x+z)}{2x^2 - xz + z^2}, \quad w = \frac{xz(x+z)}{x^2 - xz + 2z^2}.$$

By Proposition 3 we know that $y, w \in I$ whenever $(x, z) \in \Omega$. Simplifying (2) after these substitutions, we eventually arrive at

$$\begin{aligned} 2f\left(\frac{xz(x+z)}{x^2+z^2}\right) &= f\left(\frac{2xz(x+z)}{3x^2-2xz+3z^2}\right) \\ &\quad + f\left(\frac{xz(x+z)(3x^2-2xz+3z^2)}{2(x^2+z^2)^2}\right), \end{aligned} \tag{5}$$

for all $(x, z) \in \Omega$. Now define maps $p, q : \Omega \to I$ by

$$\begin{aligned} p(x,z) &= \frac{2xz(x+z)}{3x^2-2xz+3z^2}, \\ q(x,z) &= \frac{xz(x+z)(3x^2-2xz+3z^2)}{2(x^2+z^2)^2}, \quad (x,z) \in \Omega. \end{aligned} \tag{6}$$

We check that p, q are functionally independent by showing that the Jacobian of the transformation $(x, z) \mapsto (\log p, \log q)$ is nonzero. Indeed,

$$\begin{aligned} &\frac{\partial(\log p, \log q)}{\partial(x,z)} \\ &= \det \begin{bmatrix} \frac{1}{x} + \frac{1}{x+z} - \frac{6x-2z}{3x^2-2xz+3z^2} & \frac{1}{z} + \frac{1}{x+z} - \frac{-2x+6z}{3x^2-2xz+3z^2} \\ \frac{1}{x} + \frac{1}{x+z} + \frac{6x-2z}{3x^2-2xz+3z^2} - \frac{4x}{x^2+z^2} & \frac{1}{z} + \frac{1}{x+z} + \frac{-2x+6z}{3x^2-2xz+3z^2} - \frac{4z}{x^2+z^2} \end{bmatrix} \\ &= \det \begin{bmatrix} \frac{1}{x} + \frac{1}{x+z} - \frac{6x-2z}{3x^2-2xz+3z^2} & \frac{1}{z} - \frac{1}{x} + \frac{8(x-z)}{3x^2-2xz+3z^2} \\ \frac{1}{x} + \frac{1}{x+z} + \frac{6x-2z}{3x^2-2xz+3z^2} - \frac{4x}{x^2+z^2} & \frac{1}{z} - \frac{1}{x} - \frac{8(x-z)}{3x^2-2xz+3z^2} + \frac{4(x-z)}{x^2+z^2} \end{bmatrix}, \end{aligned}$$

where we have subtracted column 1 from column 2. Next, add row 1 to row 2 to get

$$\frac{\partial(\log p, \log q)}{\partial(x,z)} = \det \begin{bmatrix} \frac{1}{x} + \frac{1}{x+z} - \frac{6x-2z}{3x^2-2xz+3z^2} & \frac{1}{z} - \frac{1}{x} + \frac{8(x-z)}{3x^2-2xz+3z^2} \\ \frac{2}{x} + \frac{2}{x+z} - \frac{4x}{x^2+z^2} & \frac{2}{z} - \frac{2}{x} + \frac{4(x-z)}{x^2+z^2} \end{bmatrix}.$$

Thus we have

$$
\begin{aligned}
&x^2z(x+z)(x^2+z^2)(3x^2-2xz+3z^2)\frac{\partial(\log p,\log q)}{\partial(x,z)}\\
&\quad=\det\begin{bmatrix} -5x^2z+6xz^2+3z^3 & 3x^3+3x^2z-3xz^2-3z^3\\ -2x^2z+4xz^2+2z^3 & 2x^3+2x^2z-2xz^2-2z^3\end{bmatrix}\\
&\quad=\det\begin{bmatrix} -2x^2z & 0\\ -2x^2z+4xz^2+2z^3 & 2x^3+2x^2z-2xz^2-2z^3\end{bmatrix}.
\end{aligned}
$$

Therefore

$$\frac{\partial(\log p,\log q)}{\partial(x,z)}=\frac{-4(x^2-z^2)}{(x^2+z^2)(3x^2-2xz+3z^2)},$$

which is nonzero off the diagonal $x=z$. Therefore the transformation (6) maps Ω onto an open set $D=(p(\Omega),q(\Omega))\subset I^2$. Making the substitution (6) into (5), we have now

$$2f(\sqrt{pq})=f(p)+f(q),\quad (p,q)\in D.$$

This is a Jensen-type logarithmic equation which holds on the open set D, hence its solution is (see, e.g. [1], [2])

$$f(p)=L(p)+c,\quad p\in D_1\cup D_2\cup D_3, \tag{7}$$

for some logarithmic function $L:\mathbb{R}_+\to\mathbb{R}$ and a real constant c, where

$$
\begin{aligned}
&D_1=\{p:(p,q)\in D\text{ for some }q\}, \quad D_2=\{q:(p,q)\in D\text{ for some }p\},\\
&D_3=\{\sqrt{pq}:(p,q)\in D\}.
\end{aligned}
$$

Referring back to our proposition, we recall that $(x,x)\in\Omega$ for all $x\in I$. By (6) we also have

$$p(x,x)=q(x,x)=x,\quad x\in I.$$

Hence $D_1=D_2=D_3=I$, and (7) gives the desired result. Since any function of the form (7) satisfies (1), this completes the proof. $\square$

Remark 5. Note that we have also proved that (2) has the same general solution as (1). Therefore the two equations are equivalent.

3. Some generalizations

We turn next to Matkowski's question. Our theorem provides a positive answer in the case of the arithmetic mean (and also for the harmonic mean). Now we extend that result to a whole class of means, the root-power means. Let $I \subset \mathbb{R}_+$ be an interval, and let $p \neq 0$ be a real number. Define the symmetric mean $M_p : I^2 \to I$ by

$$M_p(x, y) = \left(\frac{x^p + y^p}{2} \right)^{1/p}.$$

For $p = 0$ we define M_0 to be the geometric mean

$$M_0(x, y) = \sqrt{xy},$$

which is the limit as p tends to 0 of $M_p(x, y)$. We shall find the general form of $f : I \to \mathbb{R}$ fulfilling the functional equation

$$f\left(M_p(x, y)\right) + f\left(\frac{xy}{M_p(x, y)} \right) = f(x) + f(y), \quad x, y \in I. \tag{8}$$

For $p = 0$ it is well known that the solution of (8) is logarithmic plus a constant. We show now that the same is true for $p \neq 0$.

Theorem 6. *Let $I \subset \mathbb{R}_+$ be an open interval, and let p be a real number. Then every function $f : I \to \mathbb{R}$ satisfying the functional equation (8) must be of the form $f = L + c$ where L is a logarithmic function and c is a real constant. Thus for any two real numbers p_1, p_2 the functional equations (8) with $p = p_1$ and $p = p_2$ are equivalent.*

Proof. For $p = 0$ the statement is trivial. Now let $p \neq 0$. Substitute $s = x^p$, $t = y^p$ in (8) and write the equation as

$$f\left(\left(\frac{s + t}{2} \right)^{1/p} \right) + f\left(\left(\frac{2st}{s + t} \right)^{1/p} \right) = f(s^{1/p}) + f(t^{1/p}).$$

Let $J := \{x^p : x \in I\}$, and define $g : J \to \mathbb{R}$ by

$$g(s) := f(s^{1/p}), \quad s \in J.$$

Thus we see that $J \subset \mathbb{R}_+$ is an open interval and that g satisfies (1) on J. Hence, by Theorem 4 we conclude that there is a logarithmic function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ and a real constant c for which

$$g(s) = L_1(s) + c, \quad s \in J.$$

This means that f has the form

$$f(x) = L_1(x^p) + c = pL_1(x) + c, \quad x \in I.$$

Defining $L := pL_1$, we have proved the asserted claim. □

Finally, we present the solution of another generalization of (1), and some applications.

Theorem 7. *Let I be an open interval, and let $\theta : I \to J$ be a continuous and strictly monotonic function mapping I onto an open interval $J \subset \mathbb{R}_+$. Then every function $f : I \to \mathbb{R}$ satisfying the functional equation*

$$f(x) + f(y) = f\left(\theta^{-1}\left[\frac{\theta(x)+\theta(y)}{2}\right]\right) + f\left(\theta^{-1}\left[\frac{2\theta(x)\theta(y)}{\theta(x)+\theta(y)}\right]\right) \tag{9}$$

must be of the form

$$f(x) = L[\theta(x)] + c, \quad x \in I,$$

where $L : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function and c is a real constant.

Proof. Let $s = \theta(x)$, $t = \theta(y) \in J$. Then the map $f \circ \theta^{-1} : J \to \mathbb{R}$ satisfies (1). Therefore by Theorem 4 we have

$$f \circ \theta^{-1} = L + c$$

on J, for some logarithmic function L and real constant c, and this completes the proof. □

Example 8. Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$f(x) + f(y) = f\left(\frac{x+y+2xy}{2+x+y}\right) + f\left(\frac{2xy}{x+y}\right), \quad x, y \in \mathbb{R}_+.$$

This is of the form (9), with $\theta : \mathbb{R}_+ \to (0,1)$ defined by

$$\theta(x) = \frac{x}{1+x}, \quad x \in \mathbb{R}_+,$$

hence the general solution is given by

$$f(x) = L\left(\frac{x}{1+x}\right) + c, \quad x \in \mathbb{R}_+,$$

for some logarithmic function L and real constant c.

Example 9. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x) + f(y) = f\left(\ln\left[\frac{e^x + e^y}{2}\right]\right) + f\left(x + y - \ln\left[\frac{e^x + e^y}{2}\right]\right), \quad x, y \in \mathbb{R}.$$

This is of the form (9), with $\theta : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\theta(x) = e^x, \quad x \in \mathbb{R},$$

hence the general solution is given by $f(x) = L(e^x) + c$, $x \in \mathbb{R}$, for some logarithmic function L and real constant c. Defining $A : \mathbb{R} \to \mathbb{R}$ by $A(x) := L(e^x)$, we have

$$f(x) = A(x) + c, \quad x \in \mathbb{R},$$

where A is an arbitrary additive function.

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