Publ. Math. Debrecen
61 / 3-4 (2002), 589–594

Fixed point of a Ljubomir Ćirić's quasi-contraction mapping in a generalized metric space

By B. K. LAHIRI (Kalyani) and PRATULANANDA DAS (Kolkata)

Abstract. We prove a fixed point theorem for a quasi-contraction mapping in a generalized metric space, a concept recently introduced in [1].

1. Introduction

If (X,d) is a complete metric space and $T : X \to X$ is a contraction, i.e. $d(Tx,Ty) \leq \alpha \cdot d(x,y)$, $0 < \alpha < 1$ for all $x, y \in X$ then the widely known Banach's fixed point theorem tells that T has a unique fixed point in X. During the last four decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.

Recently [1] a very interesting generalization was obtained by changing the structure of the space itself. Branciari [1] introduced the concept of a generalized metric space by replacing the triangle inequality by a more general inequality. As such, any metric space is a generalized metric space but the converse is not true [1]. He obtained Banach's fixed point theorem in such a space.

Under the situation, it is now reasonable to consider if some of the remarkable generalizations of Banach's theorem may be obtained in a generalized metric space. In this paper we shall prove Ćirić's fixed point theorem [3] on quasi-contraction mapping, which generalizes Banach's theorem with respect to both the mapping and the space.

Mathematics Subject Classification: Primary 54H25; Secondary 47H10.
Key words and phrases: generalized metric space, quasi-contraction mapping, fixed point.

2. Definitions and known theorem

Let $\mathbb{R}^+$ denote the set of all non-negative real numbers and $\mathbb{N}$ denote the set of all positive integers.

Definition 1 [1]. Let X be a set and $d : X^2 \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from x and y, one has

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$

then we shall say that (X,d) is a generalized metic space (or shortly g.m.s.).

Any metric space is a g.m.s. but the converse is not true [1].

As in the metric setting, any g.m.s. X is a toplogical space with neighbourhood basis given by

$$B = \{B(x,r) \mid x \in X,\ r \in \mathbb{R}^+ - \{0\}\}$$

where $B(x,r) = \{y \in X;\ d(x,y) < r\}$ is the open ball with centre x and radius r.

Definition 2 [1]. Let (X,d) be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $n \geq n_\epsilon$, one has $d(x_{m+n}, x_n) < \epsilon$. (X,d) is called complete if every Cauchy sequence in X is convergent in X.

Let $T : X \to X$ be a mapping where X is a g.m.s. For $A \subset X$ let $\delta(A) = \sup\{d(a,b) \mid a, b \in A\}$ and for each $x \in X$ let

$$O(x,n) = \{x, Tx, T^2x, \ldots, T^nx\}, \quad n = 1, 2, \ldots;$$

$$O(x,\infty) = \{x, Tx, T^2x, \ldots\}.$$

We now consider the following definitions.

Definition 3 (cf. [2], [3]). A g.m.s. X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x,\infty)$ for some $x \in X$ converges in X.

Note 1. Considering $X = [0,1)$ with the usual metric and $Tx = \frac{x}{10}$, $x \in X$ we see that a T-orbitally complete g.m.s. need not be complete.

Definition 4 (cf. [3]). A mapping $T : X \to X$ where X is a g.m.s. is said to be a quasi-contraction if and only if there exists a number q, $0 \le q < 1$ such that

$$d(Tx,Ty) \le q \max\{d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx), d(x,y)\} \tag{1}$$

hold for all $x, y \in X$.

Ćirić's fixed point theorem. *Let T be a quasi-contraction on a metric space M and let M be T-orbitally complete. Then*

(a) T has a unique fixed point u in M,

(b) $\lim_n T^n x = u$, and

(c) $d(T^n x, u) \le \frac{q^n}{1-q} d(x,Tx)$ for every $x \in M$.

Clearly a contraction mapping is a quasi-contraction but the converse is not true (see [3]).

Throughout this paper we shall assume that X is a generalized metric space (g.m.s.).

3. Lemmas

The following lemmas on quasi-contractions will be needed. The proof of Lemma 1 is omitted as it is similar to the proof of Lemma 1 in [3].

Lemma 1. *Let T be a quasi-contraction on X and let n be any positive integer. Then for each $x \in X$ and all positive integers i and j, $i, j \in \{1, 2, \dots, n\}$ we have $d(T^i x, T^j x) \le q \cdot \delta[O(x,n)]$.*

Remark 1. From Lemma 1 it follows that if $x \in X$, then for every positive integer n, there is a positive integer $k \le n$ such that $\delta[O(x,n)] = d(x, T^k x)$.

Lemma 2. *If T is a quasi-contraction on X then*

$$\delta[O(x,\infty)] \le \frac{1}{1-q} \max\{d(x,Tx), d(x,T^2x)\}$$

holds for all $x \in X$.

Proof. Let $x \in X$. Since

$$\delta[O(x,1)] \le \delta[O(x,2)] \le \dots \le \delta[O(x,n)] \le \dots ,$$

we can write

$$\delta[O(x,\infty)] = \sup\{\delta[O(x,n)];\ n \in \mathbb{N}\}.$$

The proof will follow if we show that

$$\delta[O(x,n)] \le \frac{1}{1-q} \max\{d(x,Tx), d(x,T^2x)\} \tag{2}$$

for all $n \in \mathbb{N}$.

Clearly we can assume that $n \ge 3$. By Remark 1 there is a k, $1 \le k \le n$ such that $\delta[O(x,n)] = d(x,T^kx)$. If $k = 1$ or 2, then the proof follows. So let $k \ge 3$. We note that if $x = Tx$ or $x = T^2x$ or $Tx = T^2x$ then $\delta[O(x,n)] = d(x,Tx)$ and (2) is obtained. So we can assume that x, Tx, T^2x are all distinct while Tx, T^2x are also different from T^kx. Then from (1) and Lemma 1

$$\begin{aligned}
d(x,T^kx) &\le d(x,Tx) + d(Tx,T^2x) + d(T^2x,T^kx)\\
&\le d(x,Tx) + q\max\{d(x,Tx), d(x,T^2x)\} + d(TTx,T^{k-1}Tx)\\
&\le (1+q)\max\{d(x,Tx), d(x,T^2x)\} + q\delta[O(Tx,k-1)]\\
&= (1+q)\max\{d(x,Tx), d(x,T^2x)\} + qd(Tx,T^mTx),\ (m \le k-1)\\
&\le (1+q)\max\{d(x,Tx), d(x,T^2x)\} + qq\delta[O(x,m+1)]\\
&\le (1+q)\max\{d(x,Tx), d(x,T^2x)\} + q^2\delta[O(x,n)]\\
&= (1+q)\max\{d(x,Tx), d(x,T^2x)\} + q^2 d(x,T^kx),
\end{aligned}$$

i.e.

$$(1-q^2)d(x,T^kx) \le (1+q)\max\{d(x,Tx), d(x,T^2x)\}.$$

So, $\delta[O(x,n)] = d(x,T^kx) \le \frac{1}{1-q}\max\{d(x,Tx), d(x,T^2x)\}$ and this completes the proof. □

4. Theorem

In this section we prove the main result of the paper.

Theorem 1. *Let T be a quasi-contraction on X and let X be T-orbitally complete. Then*

a) *T has a unique fixed point u in X,*

b) $\lim_{n\to\infty} T^n x = u$ *for every* $x \in X$,

c) $d(T^n x, u) \le \frac{q^n}{1-q} \max\{d(x,Tx), d(x,T^2x)\}$ *for all* $n \in \mathbb{N}$.

Proof. We first note that if T has a fixed point, it must be unique because T is a quasi-contraction. Let $x \in X$ and m and n ($m > n$) be arbitrary positive integers. From Lemma 1 we obtain

$$d(T^n x, T^m x) = d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \le q \cdot \delta[O(T^{n-1}x, m-n+1)]. \tag{3}$$

By Remark 1, there is a k_1, $1 \le k_1 \le m-n+1$ such that

$$\delta[O(T^{n-1}x, m-n+1)] = d(T^{n-1}x, T^{k_1}T^{n-1}x). \tag{4}$$

Again by Lemma 1 we have

$$d(T^{n-1}x, T^{k_1}T^{n-1}x) = d(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \le q \cdot \delta[O(T^{n-2}x, k_1+1)] \le q \cdot \delta[O(T^{n-2}x, m-n+2)].$$

Hence from (3) and (4) we obtain

$$d(T^n x, T^m x) \le q \cdot \delta[O(T^{n-1}x, m-n+1)] \le q^2 \cdot \delta[O(T^{n-2}x, m-n+2)].$$

Proceeding in this way we get

$$d(T^n x, T^m x) \le q \cdot \delta[O(T^{n-1}x, m-n+1)] \le \cdots \le q^n \cdot \delta[O(x,m)].$$

From Lemma 2 it now follows that

$$d(T^n x, T^m x) \le \frac{q^n}{1-q} \max\{d(x,Tx), d(x,T^2x)\}. \tag{5}$$

Since $\lim_{n\to\infty} q^n = 0$, $\{T^n x\}$ is a Cauchy sequence. Since X is T-orbitally complete,

$$\lim_{n\to\infty} T^n x = u \text{ (say)}. \tag{6}$$

Making $m \to \infty$ in (5) and noting that $\{T^n x\}$ is a Cauchy sequence, we can easily show that

$$d(T^n x, u) \le \frac{q^n}{1-q} \max\{d(x,Tx), d(x,T^2x)\}. \tag{7}$$

We shall now show that $Tu = u$. From (1) we have

$$d(Tu, TT^n x) \leq q \max\{d(u, Tu), d(T^n x, T^{n+1} x), d(u, T^{n+1} x), d(T^n x, Tu), d(u, T^n x)\}.$$

Taking the limit as n tends to infinity, by (6) and the fact that $\{T^n x\}$ is a Cauchy sequence, we get $d(Tu, u) \leq q \cdot d(Tu, u)$.

Hence, as $q < 1$, $d(Tu, u) = 0$; hence $Tu = u$. This in conjunction with (6) and (7) completes the proof. □

The authors are thankful to the referees, for their suggestions towards improvements in the presentation of the paper.

References

[1] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen* **57** 1–2 (2000), 31–37.

[2] Lj. B. Ćirić, On contraction type mappings, *Math. Balkanica* **1** (1971), 52–57.

[3] Lj. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* (45) **2** (1974), 267–273.

B. K. LAHIRI
B–1/146 KALYANI
WEST BENGAL – 741235
INDIA

E-mail: indrajit@cal2.vsnl.net.in

PRATULANANDA DAS
DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
KOLKATA – 700 032
INDIA

E-mail: pratulananda@yahoo.co.in

(Received October 19, 2001; revised February 7, 2002)