Publ. Math. Debrecen
61 / 3-4 (2002), 595–602

On the frequency of k-deficient numbers

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Abstract. A number n is said to be *k-deficient* if $\sigma(n) < kn$. We prove that, given $k > 1$ and a function $H(x)$ satisfying $H(x)/(\log\log\log x \cdot \log\log\log\log x) \to +\infty$ then, if n is sufficiently large, there is always a k-deficient number between n and $n + H(n)$.

§1. Introduction

Let $\sigma(n)$ stand for the sum of the divisors of the positive integer n. A number n is called *deficient* if $\sigma(n) < 2n$. It is well known that roughly $\frac{3}{4}$ of the positive integers are deficient. Using a method developed by Galambos [2] and Kátai [3], Sándor [4] proved that if n is sufficiently large, then there is always a deficient number between n and $n + \log^2 n$.

Given a real number $k > 1$, we shall say that a number n is *k-deficient* if $\sigma(n) < kn$. The density of the set of k-deficient numbers exists and steadily decreases to 0 as $k \to 1^+$. We shall prove that, given $k > 1$ and a function $H(x)$ satisfying

$$\lim_{x\to\infty} \frac{H(x)}{\log_3 x \cdot \log_4 x} = +\infty \tag{1}$$

(where $\log_\ell x$ stands for the function $\log x$ iterated ℓ times), then, if $n > n_0 = n_0(k, H)$, there is always a k-deficient number between n and $n + H(n)$.

Mathematics Subject Classification: 11A25, 11N37, 11N60.
Key words and phrases: sum of divisors, deficient numbers.
Research of the first author supported in part by a grant from CRSNG.

Also, letting

$$f(n) = \sum_{p|n} \frac{1}{p} \quad \text{for each } n \geq 2 \tag{2}$$

and given two real numbers $\eta > 0$ and $0 < \xi < 1$, we shall prove that there exists a sequence $\{x_\nu\}$ tending to $+\infty$ such that

$$\min_{x_\nu \leq n \leq x_\nu + \frac{1-\xi}{\eta} \log_3 x_\nu} f(n) > \eta \quad (\nu = 1, 2, \dots).$$

§2. Main results

Theorem 1. *Let f be as in (2) and $H = H(x)$ as in (1), then*

$$\lim_{x \to \infty} \min_{x \leq n \leq x+H} f(n) = 0. \tag{3}$$

Theorem 2. *If $H = H(x)$ satisfies (1), then*

$$\lim_{x \to \infty} \min_{x \leq n \leq x+H} \frac{\sigma(n)}{n} = 1. \tag{4}$$

Thus, given any real number $k > 1$, there exist $n_0 = n_0(k)$ such that, for all integers $n \geq n_0$, the interval $[n, n + H(n)]$ contains at least one k-deficient number.

Theorem 3. *Given two real numbers $\eta > 0$ and $0 < \xi < 1$, there exists a sequence $\{x_\nu\}$ tending to $+\infty$ such that*

$$\min_{x_\nu \leq n \leq x_\nu + \frac{1-\xi}{\eta} \log_3 x_\nu} f(n) > \eta \qquad (\nu = 1, 2, \dots). \tag{5}$$

§3. Preliminary results

In this paper, we use the following notations. For each $x \geq e^{e^{e^e}}$, we let $H = H(x)$ be a function satisfying (1). Let $\varepsilon > 0$ be arbitrarily small but fixed throughout the text. For each $x \geq e$, we set $Y = Y(x, \varepsilon) = 2(\log x)^{1/(1+\varepsilon)}$.

Letting f be as in (2), we define $f_i(n)$, $1 \le i \le 4$, for each integer $n \ge 2$, by

$$f_1(n) = \sum_{\substack{p|n \\ p<Y}} \frac{1}{p}, \qquad f_2(n) = \sum_{\substack{p|n \\ p\ge Y}} \frac{1}{p}, \tag{6}$$
$$f_3(n) = \sum_{\substack{p|n \\ H\le p<Y}} \frac{1}{p}, \qquad f_4(n) = \sum_{\substack{p|n \\ p<H}} \frac{1}{p},$$

so that $f = f_1 + f_2 = f_2 + f_3 + f_4$.

Given $H = H(x)$ and a real number δ, $0 < \delta < \frac{1}{2}$, let

$$\mathcal{M} = \mathcal{M}(\delta, H) = \{n \in \,]x, x+H] : p(n) > H^{\delta}\}, \tag{7}$$

where $p(n)$ stands for the smallest prime factor of n. We write $\#\mathcal{M}$ to denote its cardinality. Finally c stands for a positive constant, not necessarily the same at each occurence.

We shall be using the following known estimates, which are all consequences of the Prime Number Theorem:

$$\prod_{p\le x} p = e^{(1+o(1))x}, \tag{8}$$

$$\sum_{p\le x} \frac{1}{p} = \log\log x + c + o(1), \tag{9}$$

$$\prod_{p\le x} \left(1 - \frac{1}{p}\right) = (1+o(1))\frac{e^{-\gamma}}{\log x}, \quad \text{(here } \gamma \text{ is Euler's constant)} \tag{10}$$

$$\sum_{p>x} \frac{1}{p^2} \ll \frac{1}{x\log x}. \tag{11}$$

Lemma 1. *There exists a real number x_0 such that if $x \ge x_0$, then $f_2(n) < 2\varepsilon$ for all $n \le 2x$.*

Proof. Write $2 = p_1 < p_2 < \dots$ for the sequence of all primes. Given $n \le 2x$, let $q_1, \dots, q_r$ be the prime divisors of n which are larger than Y. Then, writing $s = \pi(Y)$, we have, using (8),

$$p_1p_2\dots p_s q_1q_2\dots q_r \le p_1p_2\dots p_s n \le 2xe^{\frac{3}{2}Y} \le xe^{2Y},$$

provided x is large enough. Moreover since $p_{s+j} \le q_j$ for each positive integer j, we have

$$p_1 p_2 \dots p_{s+r} \le p_1 p_2 \dots p_s q_1 q_2 \dots q_r \le x e^{2Y} \tag{12}$$

and

$$f_2(n) \le \sum_{j=1}^{r} \frac{1}{p_{s+j}}. \tag{13}$$

Clearly inequality (12) implies that

$$\sum_{j=1}^{s+r} \log p_j \le \log x + 2Y,$$

while it follows from (8) that

$$\sum_{j=1}^{r+s} \log p_j = (1+o(1)) p_{s+r}.$$

Whence, combining these last two relations, we have

$$p_{s+r} \le (1+o(1))(\log x + 2Y). \tag{14}$$

Furthermore if follows from (13), (9) and (14) and that

$$\begin{aligned} f_2(n) = \sum_{i=1}^{r} \frac{1}{q_i} \le \sum_{i=1}^{r} \frac{1}{p_{s+i}} &\le \log \frac{\log p_{s+r}}{\log Y} \\ &\le \log \frac{\log(\log x + 2Y)}{\log(2(\log x)^{1/(1+\varepsilon)})} < 2\varepsilon, \end{aligned}$$

if x is sufficiently large.

Lemma 2. *Given $0 < \delta < \frac{1}{2}$ and letting $\mathcal{M}$ be as in (7), there exist two positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that $\lim_{\delta\to 0} c_1(\delta) = \lim_{\delta\to 0} c_2(\delta) = 1$ and*

$$c_1 \le \frac{\#\mathcal{M}}{H \prod_{p \le H^\delta} \left(1 - \frac{1}{p}\right)} \le c_2. \tag{15}$$

Proof. This result follows easily from classical sieve theory, for instance by using Lemma 2.1 of Elliott [1]. □

Lemma 3. *Letting* f_3 *be as in* (6), *we have*

$$\lim_{x\to\infty}\frac{1}{\#\mathcal{M}}\sum_{n\in\mathcal{M}} f_3(n)=0. \tag{16}$$

Proof. If x is sufficiently large, then, using (9),

$$\sum_{x<n\le x+H} f_3(n)\le \sum_{H\le p<Y}\left(\left[\frac{x+H}{p}\right]-\left[\frac{x}{p}\right]\right)$$
$$\le \sum_{H\le p<Y}\frac{1}{p}\ll \log\frac{\log Y}{\log H}.$$

Then, using the left inequality of (15) followed by (10), we get that, due to the choice of $H(x)$ given by (1) and denoting by $\rho(x)$ the quotient $\frac{H(x)}{\log_3 x\log_4 x}$,

$$\frac{1}{\#}\mathcal{M}\sum_{n\in\mathcal{M}} f_3(n)\le c\delta\frac{\log H}{H}\cdot\log\left(\frac{\log Y}{\log H}\right)\ll\frac{\log H}{H}\log\log Y$$
$$\ll\frac{\log_4 x}{\rho(x)\cdot\log_3 x\log_4 x}\cdot\log_3 x=\frac{1}{\rho(x)}=o(1),$$

as $x\to\infty$, which proves (16). □

Lemma 4. *Letting* f_4 *be as in* (6), *we have*

$$\lim_{x\to\infty}\frac{1}{\#\mathcal{M}}\sum_{n\in\mathcal{M}} f_4(n)=0. \tag{17}$$

Proof. We write

$$\sum_{n\in\mathcal{M}} f_4(n)\le\sum_{H^\delta<p<\sqrt{H}}\frac{1}{p}\sum_{\substack{\frac{x}{p}<m\le\frac{x}{p}+\frac{H}{p}\\ p(m)>H^\delta}}1+\sum_{\sqrt{H}\le p\le H}\frac{H}{p^2}=\Sigma_1+\Sigma_2,$$

say. Applying Lemma 2 to estimate the inner sum of Σ_1, we get, using (11),

$$\Sigma_1 \le \frac{2c_2}{\delta} \sum_{p>H^\delta} \frac{H}{p^2 \log H} \le \frac{H}{\delta \log H} \frac{1}{H^\delta \log H^\delta} = \frac{H^{1-\delta}}{\delta^2 \log^2 H}.$$

Since it is clear that $\Sigma_2 < c\sqrt{H}$, it follows from the left inequality of (15), that

$$\frac{1}{\#\mathcal{M}} \sum_{n\in\mathcal{M}} f_4(n) \le \frac{c\delta \log H}{H} \left(\frac{H^{1-\delta}}{\delta^2 \log^2 H} + \sqrt{H} \right)$$

$$\ll \frac{1}{\delta H^\delta \log H} + \frac{\log H}{\sqrt{H}} = o(1) \quad (x \to \infty),$$

which proves (17). □

§4. Proof of the main results

Proof of Theorem 1. Recalling that $f = f_2 + f_3 + f_4$ and using Lemmas 1, 3 and 4, estimate (3) follows. □

Proof of Theorem 2. First observe that, for all $n \ge 2$,

$$\begin{aligned}\frac{\sigma(n)}{n} &= \prod_{p^\alpha \| n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\alpha}\right)\\ &= \exp\left\{ \sum_{p^\alpha \| n} \log \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\alpha}\right)\right\}\\ &< \exp\left\{ 2 \sum_{p|n} \frac{1}{p} \right\} = \exp\left\{ 2f(n) \right\},\end{aligned}$$

where we used the fact that $\log \frac{1}{1-y} < 2y$ for all positive real numbers $y \le \frac{1}{2}$. The result then follows from Theorem 1. □

Proof of Theorem 3. It is enough to prove that given an arbitrary large number X, there exists a number $x > X$ and a particular integer n satisfying

$$\frac{x}{2} < n < x \quad \text{and} \quad \min_{n \le m \le n+r} f(m) > \eta,$$

where $r := [\frac{1-\xi}{\eta} \log_3 x] + 1$.

So we start with a large number $x > X$ with r defined as above. Then, using (9), we have

$$S := \sum_{p<(1-\xi)\log x} \frac{1}{p} = \log_3 x + O(1).$$

We now split the sum S into r sums S_i in such a way that each subsum S_i is larger than η. For each $1 \leq i \leq r$, let $\wp_i$ be the set of primes appearing in the sum S_i and set $P_i = \prod_{p\in\wp_i} p$. Using (8), we have that

$$Q := \prod_{i=1}^{r} P_i = \prod_{p<(1-\xi)\log x} p < e^{(1-\xi/2)\log x} = x^{1-\xi/2}, \tag{18}$$

provided x has been chosen large enough. Then consider the system of congruences

$$\begin{cases} n \equiv 0 \pmod{P_1}, \\ n \equiv -1 \pmod{P_2}, \\ \quad\vdots \\ n \equiv -r+1 \pmod{P_r}. \end{cases}$$

By the Chinese Remainer Theorem, this system of congruences has a solution $n_0 < Q < x^{1-\xi/2}$, because of (18).

Since $n_0 + sQ$, with $s = 0, 1, 2, \ldots$, are all solutions of this system, let us choose s such that

$$\frac{x}{2} < n := n_0 + sQ < x,$$

such a choice being possible because of (18). For such an integer n, we then have that for each integer $m \in [n, n+r-1]$, we have $f(m) \geq \sum_{\substack{p|m \\ p\in\wp_i}} \frac{1}{p} > \eta$ for the appropriate integer i, thus completing the proof of Theorem 3. $\square$

§5. Final remark

It is clear that one can obtain similar results when $f(n)$ is replaced by $f_\alpha(n) := \sum_{p|n} \frac{1}{p^\alpha}$ (for a fixed $\alpha > 0$) and $\sigma(n)$ by $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$.

References

[1] P. D. T. A. Elliott, Probabilistic Number Theory I, *Springer–Verlag*, 1979.

[2] J. Galambos, On a conjecture of Kátai concerning weakly composite numbers, *Proc. Amer. Math. Soc.* **96** (1986), 315–316.

[3] I. Kátai, A minimax theorem for additive functions, *Publ. Math. Debrecen* **30** (1983), 249–252.

[4] J. Sándor, On a method of Galambos and Kátai concerning the frequency of deficient numbers, *Publicationes Mathematicae* **39** (1991), 155–157.

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(Received November 20, 2001; revised January 8, 2002)