Near-ring extensions with a common ideal

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Abstract. This paper considers the inheritance of certain properties by subnear-rings and supernear-rings, with the focus being on prime ideals, 3-prime ideals, and various radicals. Playing a key role in this is a common nonzero ideal shared by the near-ring and its subnear-ring. Sharper results are obtained when this common ideal is essential.

1. Introduction

This paper considers the inheritance, upward and downward, of certain properties of a near-ring and its subnear-rings. The main objects of interest are prime and 3-prime ideals, semiprime ideals, and certain hereditary radicals. The hypothesis of a nonzero ideal shared by the near-ring and subnear-ring of interest plays a crucial role. Some of the results obtained have the flavor of "lying over" theorems for rings; however, these near-ring results are different in detail and require considerably different proof techniques from those in the ring situation. The $\mathcal{A}$-ideals introduced in [3] are useful in obtaining the "lying over" results in Section 3. As might be expected, the distributively generated (d.g.) case is more amenable to analysis.

In Section 2, we give large classes of natural examples and construction techniques to motivate the theory that is subsequently developed. Examples are also given in Section 4 to supplement the theory of centralizing extensions developed there.

Throughout this paper, S will be a (left) near-ring and R a nonzero subnear-ring of S. If X is a nonempty subset of S, then $\underline{\mathbf{r}}_S(X) = \{a \in S \mid Xa = 0\}$, the right annihilator of X in S. The sets $\mathbf{gp}(X)$ and $\mathbf{ngp}(X)$ are the subgroup and normal subgroup of $(S, +)$ generated by X, respectively. Also, $\langle X \rangle_S$ and $\mathbf{nr}_S(X)$ are the ideal and subnear-ring generated by X in S, respectively. For any group G, G' denotes the commutator subgroup of G. In particular, S' is the commutator subgroup of $(S, +)$. A nonzero ideal I of S is called *ideal essential* in S if whenever Y is a nonzero ideal of S, $Y \cap I \neq 0$.

The concepts of prime ideal and prime near-ring are covered extensively in the literature [10]. Here, we use $\mathcal{P}(N)$ for the prime radical of a near-ring N. An ideal X of N is called *3-prime* if whenever $a, b \in N$ are such that $aNb \subseteq X$, then $a \in X$ or $b \in X$. Also, N is a 3-prime near-ring if 0 is a 3-prime ideal of N. The 3-prime radical is denoted by $\mathcal{P}_3(N)$. It is well-known that every 3-prime ideal is prime, but the converse fails to hold even for finite d.g. near-rings [2, Example 1.17]. For more on 3-prime ideals and the 3-prime radical, see [8].

Let S be a near-ring with unity, 1, and suppose that R is a subnear-ring of S such that $1 \in R$. Let X be a nonempty subset of $C_S(R) = \{s \in S \mid sr = rs$ for all $r \in R\}$ with $1 \in X$. Observe that $\mathbf{gp}(RX) = \mathbf{gp}(XR)$ is a left R-subgroup containing R which need not be a right R-subgroup nor a subnear-ring. We say that S is a *(finite) centralizing extension* of R if there exists some (finite) subset X of $C_S(R)$ with $1 \in X$ and $S = \mathbf{gp}(RX)$. Such extensions will be considered in Section 4.

2. Examples and constructions

In this section we give some natural examples and some constructions which illustrate the phenomenon of extensions with a common ideal. In the first example, both the subnear-ring and its extension are d.g. We first recall some terminology and background information.

Let $(G, +)$ be a group, not necessarily abelian, and let K be a nonzero subgroup of G. Then $\mathcal{H}(G, K) = \mathbf{gp}(\mathrm{Hom}(G, K))$ is a d.g. near-ring and is a left $\mathcal{E}(G)$-subgroup of the near-ring $\mathcal{E}(G) = \mathbf{gp}(\mathrm{End}\, G)$. (The first in-depth study of $\mathcal{H}(G, K)$ is found in [5]; also, see [6].) If K is normal in G, then $\mathcal{H}(G, K)$ is a left ideal of $\mathcal{E}(G)$, and if K is fully invariant in G, then $\mathcal{H}(G, K)$ is an ideal of $\mathcal{E}(G)$.

Example 2.1. Take $K \neq G$ with K fully invariant in G. Then the identity mapping, 1_G, is not in $\mathcal{H}(G, K)$. However, $\mathcal{H}(G, K)$ is a common ideal in the near-ring $\mathcal{E}(G)$ and the subnear-ring $\mathbf{gp}(\mathcal{H}(G, K) \cup \{1_G\})$, each of which is a d.g. near-ring with unity. More generally, one can take $R = \mathbf{gp}(\mathcal{H}(G, K) \cup X)$, where X is a subsemigroup of $(\operatorname{End} G, \circ)$ containing 1_G but X is small enough so that $R \neq \mathcal{E}(G)$.

Example 2.2. Let Ω be a subset of $\operatorname{End} G$ which contains $\operatorname{Inn} G$ and is closed under composition. Then $\mathbf{gp}(\Omega)$ is a tame endomorphism near-ring [10, p. 176]. Let $\mathcal{E}_\Omega(G, K) = \{\alpha \in \mathbf{gp}(\Omega) \mid G\alpha \subseteq K\}$. Observe that $\mathcal{E}_\Omega(G, K)$ is a subnear-ring of the d.g. near-ring $(\mathbf{gp}(\Omega), +, \circ)$. Let $X \subseteq \Omega$ be such that $1_G \in X$ and X is closed under composition. Then $R = \mathbf{gp}(\mathcal{E}_\Omega(G, K) \cup X)$ is a near-ring with unity. If K is an Ω-invariant subgroup, then $\mathcal{E}_\Omega(G, K)$ is a common ideal of R and $\mathbf{gp}(\Omega)$. This construction is of particular interest when Ω is $\operatorname{End} G$, $\operatorname{Aut} G$, or $\operatorname{Inn} G$, where we require K to be fully invariant, characteristic, or normal in G, respectively. The special case in which $K = G' \neq G$, $X = \{1_G\}$, and $\Omega = \operatorname{End} G$ presents interesting group theoretic considerations.

Construction 2.3. Let X be a nonempty subset of S. If I is any ideal of S, then I is also an ideal of the near-ring $\mathbf{nr}_S(I \cup X)$. If S has unity, 1, and if $1 \in X$, then $\mathbf{nr}_S(I \cup X)$ has unity as well. In particular, one might take $X = \{1\}$.

Construction 2.4. Let I be a nonzero ideal of R. A standard Zorn's lemma argument yields the existence of a subnear-ring W which is maximal with respect to containing R and having I as an ideal. In principle, there may be many such subnear-rings. (Note that, in the category of rings, there is a unique such subring, namely $\{s \in S \mid sI \subseteq I \text{ and } Is \subseteq I\}$, the idealizer of I in S. However, for S a near-ring, this set need not be a subnear-ring.) When S has unity, a similar Zorn's lemma argument yields a subnear-ring with the same unity element and which has the desired maximality properties.

Further examples illustrating extensions with the pair of near-rings having a common ideal are available in the many papers that give detailed information on ideals, radicals, and subnear-rings of $\mathcal{E}(G)$, $\mathcal{A}(G) = \mathbf{gp}(\operatorname{Aut} G)$, and $\mathcal{I}(G) = \mathbf{gp}(\operatorname{Inn} G)$ for various classes of groups. For more information, including further references to the primary literature, see [10, Chapter 17].

3. Extensions linked by a common ideal

Throughout this section, we make use of the assumption of a common nonzero ideal B shared by the two near-rings $R \subseteq S$ to study the transference of ideal properties. First we need some background information.

Definition 3.1. A nonempty subset Y of a near-ring N is said to satisfy an n-reverting permutation identity if there exists a natural number n and a permutation π on $\{1, 2, \ldots, n\}$ such that $\pi n \neq n$ and

$$y_1 \cdot y_2 \cdots y_n = y_{\pi 1} \cdot y_{\pi 2} \cdots y_{\pi n}$$

for each $y_1, y_2, \ldots, y_n \in Y$.

Examples of n-reverting permutation identities can be found in [1].

Recall [3] that an ideal I of a near-ring N is called an $\mathcal{A}$-*ideal* (or *Andrunakievich ideal*) if for each ideal K of the near-ring I there exists $m \geq 1$ such that $(\langle K \rangle_N)^m \subseteq K$. If each ideal of N is an $\mathcal{A}$-ideal, then N is called an $\mathcal{A}$-*near-ring*. It is known [3] that the class of $\mathcal{A}$-near-rings includes all d.g. near-rings, all strongly regular near-rings, and near-rings which satisfy various permutation or word identities (e.g., n-reverting permutation identities). For a d.g. near-ring N, it is known that for any ideal I of N, then $(\langle K \rangle_N)^4 \subseteq K$ for each ideal K of I [3].

The first result follows immediately from [3, Corollary 3.3] and will be useful in the sequel.

Lemma 3.2. *Let X be a nonempty subset of a d.g. near-ring T. Then $\langle X \rangle_T \subseteq \mathbf{ngp}_T(TX \cup X \cup TXT \cup XT)$.*

Theorem 3.3. *Let S be a prime near-ring. Then each of the following conditions implies that R is prime:*

(i) *S is d.g. and B is n-reverting;*

(ii) *S is zero-symmetric and B is an $\mathcal{A}$-ideal of S which is ideal essential in R.*

PROOF. Let X and Y be ideals of R such that $XY = 0$.

(i) Observe that $B^{n+1}\langle X \rangle_S \subseteq B^n \mathbf{ngp}_B(BX \cup BXS)$. Consider a product of the form $a_1 \ldots a_n a x s$, where $a_1, \ldots, a_n, a \in B$, $x \in X$, and $s \in S$. Such a product can be rewritten, using the permutation identity, as either $axsa'$, where $a' \in B$ or $a''axsa'$, where $a', a'' \in B$. In

either case, $xsa' \in X$, so that the product $a_1 \dots a_n axs$ is in X. Thus $B^{n+1}X \cup B^{n+1}XS \subseteq X$, and hence $B^{n+1}\langle X \rangle_S \subseteq X$ by the preceding lemma. Similarly, $B^{n+1}\langle Y \rangle_S \subseteq Y$. So $B^{n+1}\langle X \rangle_S B^{n+1}\langle Y \rangle_S \subseteq XY = 0$. Since S is prime and $B \neq 0$, we have either $X = 0$ or $Y = 0$.

(ii) Since $B \cap X$ and $B \cap Y$ are ideals of B, there exist positive integers m, n such that

$$(\langle B \cap X \rangle_S)^m (\langle B \cap Y \rangle_S)^n \subseteq (B \cap X)(B \cap Y) \subseteq XY = 0.$$

Since S is prime, it follows that $B \cap X = 0$ or $B \cap Y = 0$. Consequently, $X = 0$ or $Y = 0$.   □

Corollary 3.4. *Let P be a prime ideal of S with $B \not\subseteq P$. Then each of the following conditions implies that $R \cap P$ is a prime ideal of R:*

(i) *S is d.g. and B is n-reverting;*

(ii) *S is zero-symmetric and B is an $\mathcal{A}$-ideal of S which is ideal essential in S.*

PROOF. Let ν be the natural homomorphism from S onto the prime near-ring $\bar{S} = S/P$. Then $\bar{B} = \nu(B)$ is a nonzero ideal of $\bar{S}$ and of $\nu(R + P) = \bar{R} = \nu(R)$. Observe that conditions (i) and (ii) are inherited in the homomorphic image $\bar{S}$. Applying Theorem 3.3, we get that $\bar{R}$ is a prime near-ring. However, P is also an ideal of the near-ring $R + P$, and $\bar{R} = (R + P)/P \cong R/(R \cap P)$. So $R \cap P$ is a prime ideal of R.   □

Lemma 3.5. *Each of the following conditions implies that $B\langle QB \rangle_S \subseteq Q$ for any ideal Q of R:*

(i) *$B^2 \subseteq Q$;*

(ii) *S is a centralizing extension of R;*

(iii) *S is d.g.*

PROOF. Let Q be any ideal of R. As B is an ideal of S, $\langle QB \rangle_S \subseteq B$. So, using (i), $B\langle QB \rangle_S \subseteq B^2 \subseteq Q$.

To prove the conclusion using (ii), suppose that $S = \mathbf{gp}(RX)$ for some $X \subseteq C_S(R)$, and let $a, a' \in B$, $s_1, s_2 \in S$, and $q \in Q$ be given. Then $a'qa \in Q$, so that elements in the generating set of $\langle QB \rangle_S$ satisfy the desired containment. We complete the proof by showing that elements resulting from the application of any ideal property to an element satisfying the desired containment also satisfy the desired containment. To this end,

let $c, d \in S$ be such that $yc, yd \in Q$ for any $y \in B$. Then $a'(s_1 + c - s_1) = a's_1 + a'c - a's_1$ and $a's_1c$ are in Q as $a's_1 \in B \subseteq R$. Let $s_2 = \sum_i \epsilon_i r_i x_i$, where $\epsilon_i \in \{+1, -1\}$, $r_i \in R$, and $x_i \in X$. The term $a'[(s_1 + c)s_2 - s_1 s_2]$ can be written as the sum of terms of the form $a's_1 \epsilon_i r_i x_i$ or $a_i c \epsilon_i r_i x_i$, each of which is in Q. Since $a'(c + d) = a'c + a'd$ and $a'cd$ are in Q as well, the desired result follows from (ii).

Assume (iii). We make use of [3, Corollary 3.3]. Note that $B(SQB) \subseteq BQB \subseteq Q$ and $B(QBS) \subseteq BQB \subseteq Q$. Also, if $q \in Q$, $a, a' \in B$, and $s \in S$, then $a'(s + qa - s) \in Q$ as in the proof in the previous paragraph. Using this repeatedly, one obtains $B \cdot \mathbf{ngp}_S(QB) \subseteq Q$, whence [3, Corollary 3.3] gives the result. $\qquad\square$

The next result is a "lying over" theorem.

Theorem 3.6. *Suppose that Q is a prime ideal of R with $B \not\subseteq Q$. Then each of the following conditions implies that there exists some prime ideal P of S such that $Q = P \cap R$:*

(i) $B^2 \subseteq Q$;

(ii) *S is a centralizing extension of R;*

(iii) *S is d.g.*

PROOF. Let P be an ideal of S maximal with respect to $P \cap R \subseteq Q$ (such a P must exist by Zorn's lemma). Then P is a prime ideal of S by the same argument as in [12, Lemma 2.12.41], and $B \not\subseteq P$. By passing to S/P and $R/(P \cap R)$, we may suppose without loss of generality that $P = 0$ and $B \neq 0$. Since $QB \subseteq B$, $\langle QB \rangle_S \cap R = \langle QB \rangle_S$. From the preceding lemma, we have that $B \langle QB \rangle_S \subseteq Q$ in all three cases. Since $B \not\subseteq Q$, it follows that $\langle QB \rangle_S \subseteq Q$, and so $\langle QB \rangle_S = 0$ by the maximality of P. Thus $Q = 0$. So in the general case, $Q = P \cap R$ as desired. $\qquad\square$

Proposition 3.7. *Let S be zero-symmetric and suppose that R is a 3-prime near-ring. Then $\bar{S} = S/\underline{\mathbf{r}}_S(B)$ is a 3-prime near-ring.*

PROOF. Let $\bar{b}, \bar{c} \in \bar{S}$ be the respective images of $b, c \in S$ under the natural homomorphism, and suppose that $\bar{b}\bar{S}\bar{c} = \{\bar{0}\}$. Then $BbSc = \{0\}$, which implies that $BbRc = \{0\}$. If $\bar{c} \neq \bar{0}$, then there is some nonzero $a \in B$ such that $ac \neq 0$. So $BbRac = \{0\}$. As R is 3-prime, this says that $Bb = \{0\}$, or $\bar{b} = \bar{0}$. $\qquad\square$

4. Centralizing extensions

This section is concerned with a near-ring analogue to the concept of centralizing (or liberal) extensions studied in [11].

First we show how centralizing extensions arise naturally for near-rings by giving a general construction.

Construction 4.1. Let N be a near-ring with unity 1 and let Γ be a subsemigroup of $(\mathcal{D}(N), \cdot)$, the multiplicative semigroup of right distributive elements of N, which contains 1. Define the following

$$D_\Gamma = \{d \in \mathcal{D}(N) \mid d\gamma = \gamma d \text{ for each } \gamma \in \Gamma\};$$

$$E_\Gamma = \{k \in \mathbf{gp}(\mathcal{D}(N)) \mid k\gamma = \gamma k \text{ for each } \gamma \in \Gamma\};$$

$$H_\Gamma = \mathbf{gp}(D_\Gamma).$$

Then $\mathbf{gp}(E_\Gamma \cdot \Gamma)$ and $\mathbf{gp}(H_\Gamma \cdot \Gamma)$ are centralizing extensions of the near-rings E_Γ and H_Γ, respectively. Observe that $\mathbf{gp}(H_\Gamma \cdot \Gamma)$ is d.g.

For a concrete realization of this construction, let G be a group and let N be the near-ring of all self-maps on G. In this case, $\mathbf{gp}(E_\Gamma \cdot \Gamma)$ and $\mathbf{gp}(H_\Gamma \cdot \Gamma)$ are subnear-rings of the centralizer near-ring N_Γ. Furthermore, if G is finite, then for any choice of Γ, $\mathbf{gp}(E_\Gamma \cdot \Gamma)$ and $\mathbf{gp}(H_\Gamma \cdot \Gamma)$ are finite centralizing extensions of E_Γ and H_Γ, respectively.

Proposition 4.2. *Let S be a near-ring with unity 1, and let R be a subnear-ring of S with $1 \in R$. Suppose that $\phi : S \to \bar{S}$ is a homomorphism onto a ring $\bar{S}$, with $K = \ker \phi \subseteq R$. If S is a finite centralizing extension of R, then for each prime ideal P of R with $K \subseteq P$, there exists a 3-prime ideal Q of S such that $Q \cap R = P$.*

PROOF. Observe that the quotient $\bar{R} = R/K$ is a unitary subring of the ring $\bar{S}$. Let X be a finite subset of $C_S(R)$ such that $S = \mathbf{gp}(RX)$. Then $\bar{X} = \phi X$ is a finite subset of $C_{\bar{S}}(\bar{R})$, say $\bar{X} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$, such that $\bar{S} = \sum_i \bar{R}\bar{x}_i$. Furthermore, any prime ideal P of R with $K \subseteq P$ is a prime ideal of $\bar{R}$. Using [11, Theorem 4.1], there is a prime ideal $\bar{Q}$ of $\bar{S}$ such that $\bar{Q} \cap \bar{R} = \bar{P}$; then Q, the pre-image of $\bar{Q}$, is a 3-prime ideal with $K \subseteq Q$ and $Q \cap R = P$. $\square$

Note that the kernel K plays the role of the common ideal in the above. The next example illustrates the type of extension under consideration here.

Example 4.3. We illustrate Proposition 4.2 with an example connecting Construction 4.1 with group rings. Let G be a finite group and let Γ be a subsemigroup of $(\operatorname{End} G, \circ)$ with $1_G \in \Gamma$ as in the latter part of 4.1. Let S_1 be the group ring $K[G]$, where K is an arbitrary ring with unity, and let $S_2 = \mathbf{gp}(E_\Gamma \cdot \Gamma)$. (Similarly, one could use $S_2 = \mathbf{gp}(H_\Gamma \cdot \Gamma)$.) Since a direct product of centralizing extensions is again a centralizing extension, $S = S_1 \oplus S_2$ together with the projection mapping $\phi : S \to S_1$ gives the situation of Proposition 4.2.

Corollary 4.4. *Let S be a d.g. near-ring with unity, 1, and let R be a unitary subnear-ring of S such that $S' \subseteq R$. If S is a finite centralizing extension of R, then for each prime ideal P of R with $S' \subseteq P$, there exists a 3-prime ideal Q of S such that $Q \cap R = P$.*

PROOF. Observe that the quotient S/S' is a ring and apply Proposition 4.2. $\qquad\square$

Corollary 4.5. *Let S and R be as in Corollary 4.4. If $(S, +)$ is solvable, then*

(i) *for each prime ideal P of R there exists a 3-prime ideal Q of S such that $Q \cap R = P$; and*

(ii) *$\mathcal{P}(S) \cap R = \mathcal{P}_3(S) \cap R \subseteq \mathcal{P}(R)$, where $\mathcal{P}$ and $\mathcal{P}_3$ are the prime and 3-prime radicals, respectively.*

PROOF. Since $(S, +)$ is solvable and S is d.g., S' is multiplicatively nilpotent [10, Corollary 9.49]. So S' is contained in every prime ideal of R and in every prime ideal of S (S' is an ideal of R as well). Furthermore, every prime ideal of S is a 3-prime ideal. Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be the set of all prime ideals of R. Since $S' \subseteq P_\lambda$ for every $\lambda \in \Lambda$, it follows that for a given $\lambda \in \Lambda$ there exists a 3-prime ideal Q_λ of S such that $Q_\lambda \cap R = P_\lambda$. So

$$\mathcal{P}(R) = \bigcap_{\lambda \in \Lambda} (Q_\lambda \cap R) = \Big(\bigcap_{\lambda \in \Lambda} Q_\lambda \Big) \cap R \supseteq \mathcal{P}_3(S) \cap R. \qquad \square$$

5. Radicals

In this final section we assume that S is zero-symmetric and that R and S have a common nonzero ideal B. We use ideal essentiality of B in R or in S to obtain the transference of semisimplicity between R and S of some radicals in general and some specific examples of radicals in particular. (For details on the radical theory of near-rings, see [9] and [10].)

Lemma 5.1. *Let ρ be a Hoehnke radical satisfying $\rho(B) = B \cap \rho(S) = B \cap \rho(R)$ and $\mathcal{P}(S) \subseteq \rho(S)$.*

(i) *If B is ideal essential in S and $\rho(R) = 0$, then $\rho(S) = 0$.*

(ii) *If B is ideal essential in S and $\rho(S) = 0$, then $\rho(R) = 0$.*

(iii) *If B is ideal essential in R and $\rho(S) = 0$, then $\rho(R) = 0$.*

PROOF. (i) Assume $\rho(S) \neq 0$. Then $0 \neq B \cap \rho(S) \subseteq \rho(B)$. However, $\rho(B) \subseteq B \cap \rho(R) = 0$, a contradiction.

(ii) Observe that $\rho(S) = 0$ implies that $\rho(B) = 0$. Thus $B \cap \rho(R) \subseteq \rho(B) = 0$. Suppose $\rho(R) \neq 0$. Then $I = \langle \rho(R) \rangle_S \cap B$ is a nonzero ideal of S. Hence $I \cdot \rho(R) = 0$, and so $I^2 = 0$. By hypothesis, $I \subseteq \rho(S) = 0$, a contradiction. Thus $\rho(R) = 0$.

(iii) Observe that $\rho(S) = 0$ implies $\rho(B) = 0$. Then $B \cap \rho(R) \subseteq \rho(B) = 0$, and since B is ideal essential in R, we must have $\rho(R) = 0$. □

Note that in Lemma 5.1, the condition $\mathcal{P}(S) \subseteq \rho(S)$ was only used in the proof of part (ii). This lemma applies to many of the most studied near-ring radicals, as the next result shows.

Proposition 5.2. *In either of the following situations, parts* (i) *through* (iii) *of Lemma 5.1 hold.*

(i) *ρ is either of the Jacobson radicals J_2 or J_3, or ρ is either the Brown–McCoy radical or the equiprime radical;*

(ii) *S and R are each $\mathcal{A}$-near-rings and ρ is either $\mathcal{P}, \mathcal{P}_3$, the completely prime radical, or the upper nil radical.*

PROOF. Part (i) follows from Lemma 5.1, [7], and [9]. Part (ii) is a consequence of Lemma 5.1 and [4, Corollary 12]. □

Note that if R and S are both d.g., then the statements in (i)–(iii) of Lemma 5.1 hold for the four radicals mentioned in (ii) of Proposition 5.2. Examples of such d.g. near-rings R and S with an appropriate nonzero common ideal B can be found among the endomorphism near-rings in Section 2 and in [10, Chapter 11].

Another consequence of Proposition 5.2(ii) is the following: if S is d.g. and semiprime and B is essential among nilpotent ideals of R (i.e., $B \cap X \neq 0$ for each nonzero nilpotent ideal X of R), then R is semiprime.

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