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On the second variation formula for biharmonic maps to a sphere

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Abstract. We compute the nullity for the following weakly stable biharmonic maps: the identity map $\mathbf{1} : \mathbb{S}^n \to \mathbb{S}^n$ and the canonical inclusion $\boldsymbol{i} : \mathbb{S}^m \to \mathbb{S}^n$.

1. Introduction

A map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is *harmonic* if it is a critical point of the *energy* $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$. The map ϕ is harmonic if and only if its tension field $\tau(\phi) = \operatorname{trace} \nabla d\phi$ vanishes. In the same way, as suggested by J. Eells and J. H. Sampson in [6], a map ϕ is *biharmonic* if it is a critical point of the *bienergy* $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$. G. Y. Jiang has obtained in [7], [8] the first and second variation formula. He has proved that the map ϕ is biharmonic if and only if

$$\tau_2(\phi) = J(\tau(\phi)) = 0,$$

where J is the Jacobi operator of ϕ. Of course, any harmonic map is biharmonic.

B. Y. Chen and S. Ishikawa have shown in [3] that there are no nonharmonic biharmonic submanifolds of $\mathbb{R}^3$. In the same way, in [2], the authors have proved that there are no such submanifolds in $N^3(-1)$, where $N^3(-1)$ is a 3-dimensional manifold with negative constant sectional curvature -1.

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In [1] the authors have given the classification of nonharmonic biharmonic submanifolds of $\mathbb{S}^3$. They are: circles, spherical helices and parallel spheres. Then, in [2], the authors have given some methods to construct examples of nonharmonic biharmonic submanifolds of the unit n-dimensional sphere $\mathbb{S}^n$, for $n > 3$. In this case the family of such submanifolds is much larger.

A harmonic map is an absolute minimum of the bienergy and hence stable. The goal of this paper is to find the second variation formula for biharmonic maps $\phi : (M, g) \to \mathbb{S}^n$ and then to compute the nullity for the simplest two biharmonic maps: the identity map $\mathbf{1} : \mathbb{S}^n \to \mathbb{S}^n$ and the canonical inclusion $\boldsymbol{i} : \mathbb{S}^m \to \mathbb{S}^n$ (Theorem 2.4 and Theorem 2.5).

Notation. We shall work in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall indicate a connected manifold of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi–Civita connection of (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. The indices i, j, k, l take the values $1, 2, \ldots, m$.

2. The second variation formula of the bienergy

Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Assume that M is compact and orientable. The tension field of ϕ is given by $\tau(\phi) = \operatorname{trace} \nabla d\phi$ and the *bienergy* is defined by

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g.$$

The map ϕ is called *biharmonic* if it is a critical point of the bienergy. As we said in the introduction, the first variation formula is given by

$$\frac{d}{dt}\Big|_{t=0} E_2(\phi_t) = \int_M \langle \tau_2(\phi), V \rangle v_g,$$

where v_g is the volume element, V is the variational vector field corresponding to the variation $\{\phi_t\}_{t \in \mathbb{R}}$ of ϕ, and

$$\tau_2(\phi) = -\Delta \tau(\phi) - \operatorname{trace} R^N(d\phi \cdot, \tau(\phi)) d\phi \cdot . \tag{2.1}$$

Now, let $\phi : (M, g) \to \mathbb{S}^n$ be a biharmonic map. We consider a smooth variation $\{\phi_{s,t}\}_{s,t\in\mathbb{R}}$ of ϕ with two parameters s and t, i.e. we consider the smooth map Φ given by

$$\Phi : \mathbb{R} \times \mathbb{R} \times M \to \mathbb{S}^n, \quad \Phi(s, t, p) = \phi_{s,t}(p),$$

where $\Phi(0, 0, p) = \phi_{0,0}(p) = \phi(p)$, $\forall p \in M$.

The corresponding variational vector fields V and W are given by

$$V(p) = \frac{d}{ds}\Big|_{s=0} \phi_{s,0}(p) = d\Phi_{(0,0,p)}\left(\frac{\partial}{\partial s}\right) \in T_{\phi(p)}\mathbb{S}^n,$$

and

$$W(p) = \frac{d}{dt}\Big|_{t=0} \phi_{0,t}(p) = d\Phi_{(0,0,p)}\left(\frac{\partial}{\partial t}\right) \in T_{\phi(p)}\mathbb{S}^n.$$

V and W are sections of $\phi^{-1}T\mathbb{S}^n$, i.e. $V, W \in C(\phi^{-1}T\mathbb{S}^n)$.

The Hessian of E_2 at its critical point ϕ is defined by

$$H(E_2)_\phi(V, W) = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E_2(\phi_{s,t}).$$

Theorem 2.1. *Let $\phi : (M, g) \to \mathbb{S}^n$ be a biharmonic map. Then the Hessian of the bienergy E_2 at ϕ is given by*

$$H(E_2)_\phi(V, W) = \int_M \langle I(V), W\rangle v_g,$$

where

$$\begin{aligned}
(2.2) \qquad I(V) = {} & \Delta(\Delta V) + \Delta\{\operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot - |d\phi|^2 V\} \\
& + 2\langle d\tau(\phi), d\phi\rangle V + |\tau(\phi)|^2 V \\
& - 2\operatorname{trace}\langle V, d\tau(\phi)\cdot\rangle d\phi\cdot - 2\operatorname{trace}\langle \tau(\phi), dV\cdot\rangle d\phi\cdot \\
& - \langle \tau(\phi), V\rangle \tau(\phi) + \operatorname{trace}\langle d\phi\cdot, \Delta V\rangle d\phi\cdot \\
& + \operatorname{trace}\langle d\phi\cdot, \operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot\rangle d\phi\cdot \\
& - 2|d\phi|^2 \operatorname{trace}\langle d\phi\cdot, V\rangle d\phi\cdot \\
& + 2\langle dV, d\phi\rangle \tau(\phi) - |d\phi|^2 \Delta V + |d\phi|^4 V.
\end{aligned}$$

PROOF. We start by computing $\frac{\partial}{\partial t}\big|_{t=0} E_2(\phi_{s,t})$. We have

$$\frac{\partial}{\partial t}\Big|_{t=0} E_2(\phi_{s,t}) = \frac{\partial}{\partial t}\Big|_{t=0} \frac{1}{2}\int_M |\tau(\phi_{s,t})|^2 v_g$$
$$= \int_M \langle \nabla_{\frac{\partial}{\partial t}} \tau(\phi_{s,t}), \tau(\phi_{s,t})\rangle\big|_{t=0} v_g.$$

In order to obtain $\nabla_{\frac{\partial}{\partial t}}\tau(\phi_{s,t})$, let $\{X_i\}_{i=1}^m$ be a geodesic frame field around an arbitrary point $p \in M$. We obtain

$$\nabla_{\frac{\partial}{\partial t}}\tau(\phi_{s,t}) = \nabla_{\frac{\partial}{\partial t}}\Big\{\sum_{i=1}^m (\nabla_{X_i} d\phi_{s,t}(X_i) - d\phi_{s,t}(\nabla_{X_i}X_i))\Big\}$$
$$= \nabla_{\frac{\partial}{\partial t}}\Big\{\sum_{i=1}^m (\nabla_{X_i} d\Phi_s(X_i) - d\Phi_s(\nabla_{X_i}X_i))\Big\},$$

where $\Phi_s(t,p) = \Phi(s,t,p)$. Using the formula

$$\nabla_{\widetilde{X}} d\Phi_s(\widetilde{Y}) - \nabla_{\widetilde{Y}} d\Phi_s(\widetilde{X}) = d\Phi_s([\widetilde{X},\widetilde{Y}]),\ \forall \widetilde{X},\widetilde{Y} \in C(\Phi_s^{-1}T\mathbb{S}^n),$$

we obtain, at p and for $t=0$, the following

$$\nabla_{\frac{\partial}{\partial t}}\tau(\phi_{s,t}) = \sum_{i=1}^m \big\{\nabla_{\frac{\partial}{\partial t}}\nabla_{X_i} d\Phi_s(X_i) - \nabla_{\frac{\partial}{\partial t}} d\Phi_s(\nabla_{X_i}X_i)\big\}$$
$$= \sum_{i=1}^m \Big\{\nabla_{\frac{\partial}{\partial t}}\nabla_{X_i} d\Phi_s(X_i) - \nabla_{\nabla_{X_i}X_i} d\Phi_s\Big(\frac{\partial}{\partial t}\Big) - d\Phi_s\Big(\Big[\frac{\partial}{\partial t}, \nabla_{X_i}X_i\Big]\Big)\Big\}$$
$$= \sum_{i=1}^m \nabla_{\frac{\partial}{\partial t}}\nabla_{X_i} d\Phi_s(X_i) = \sum_{i=1}^m \Big\{R^{\mathbb{S}^n}\Big(d\Phi_s\Big(\frac{\partial}{\partial t}\Big), d\Phi_s(X_i)\Big)d\Phi_s(X_i)$$
$$+ \nabla_{X_i}\nabla_{\frac{\partial}{\partial t}} d\Phi_s(X_i) + \nabla_{[\frac{\partial}{\partial t}, X_i]} d\Phi_s(X_i)\Big\}$$
$$= \sum_{i=1}^m \Big\{R^{\mathbb{S}^n}(W_s, d\Phi_s(X_i))d\Phi_s(X_i)$$
$$+ \nabla_{X_i}\Big(\nabla_{X_i} d\Phi_s\Big(\frac{\partial}{\partial t}\Big) + d\Phi_s\Big(\Big[\frac{\partial}{\partial t}, X_i\Big]\Big)\Big)\Big\}$$
$$= \sum_{i=1}^m R^{\mathbb{S}^n}(W_s, d\Phi_s(X_i))d\Phi_s(X_i) + \sum_{i=1}^m \nabla_{X_i}\nabla_{X_i} W_s$$

$$= -\Delta W_s - \sum_{i=1}^{m} R^{\mathbb{S}^n}(d\Phi_s(X_i), W_s)d\Phi_s(X_i),$$

where

$$W_s(p) = \frac{d}{dt}\Big|_{t=0}\phi_{s,t}(p) = d\Phi_{s,(0,p)}\left(\frac{\partial}{\partial t}\right), \quad W_s \in C(\phi_{s,0}^{-1}T\mathbb{S}^n),\ W_0 = W.$$

Thus $\frac{\partial}{\partial t}\big|_{t=0}E_2(\phi_{s,t})$ is given by

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0}E_2(\phi_{s,t}) &= \int_M \langle -\Delta W_s - \operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, W_s)d\phi_{s,0}\cdot, \tau(\phi_{s,0})\rangle v_g \\ &= \int_M \langle -\Delta\tau(\phi_{s,0}) - \operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0}))d\phi_{s,0}\cdot, W_s\rangle v_g.\end{aligned}$$

Since ϕ is biharmonic, from (2.1) we obtain

$$\begin{aligned}&H(E_2)_\phi(V,W)\\ &\quad= \frac{\partial}{\partial s}\Big|_{s=0}\int_M \langle -\Delta\tau(\phi_{s,0}) - \operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0}))d\phi_{s,0}\cdot, W_s\rangle v_g\\ &\quad= \int_M \langle \nabla_{\frac{\partial}{\partial s}}\left\{-\Delta\tau(\phi_{s,0}) - \operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0}))d\phi_{s,0}\cdot\right\}\big|_{s=0}, W\rangle v_g\\ &\quad= \int_M \langle I(V), W\rangle v_g,\end{aligned}$$

where

$$I(V) = \nabla_{\frac{\partial}{\partial s}}\left\{-\Delta\tau(\phi_{s,0}) - \operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot, \tau(\phi_{s,0}))d\phi_{s,0}\cdot\right\}\big|_{s=0}. \tag{2.3}$$

Next, since

$$\nabla_{\frac{\partial}{\partial s}}\tau(\phi_{s,0})\big|_{s=0} = -\Delta V - \operatorname{trace} R^{\mathbb{S}^n}(d\phi\cdot, V)d\phi\cdot$$

and

$$\operatorname{trace} R^{\mathbb{S}^n}(d\phi\cdot, V)d\phi\cdot = \operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot - |d\phi|^2 V,$$

we get

$$\begin{aligned}\nabla_{\frac{\partial}{\partial s}}\left\{-\Delta\tau(\phi_{s,0})\right\}\big|_{s=0} &= \Delta(\Delta V) + \Delta\{\operatorname{trace}\langle V, d\phi\cdot\rangle d\phi\cdot - |d\phi|^2 V\}\\ &\quad + 2\langle d\tau(\phi), d\phi\rangle V + |\tau(\phi)|^2 V + \operatorname{trace}\langle \tau(\phi), d\phi\cdot\rangle dV\cdot\\ &\quad - 2\operatorname{trace}\langle V, d\tau(\phi)\cdot\rangle d\phi\cdot - \operatorname{trace}\langle \tau(\phi), dV\cdot\rangle d\phi\cdot - \langle \tau(\phi), V\rangle\tau(\phi),\end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial s}}\left\{-\operatorname{trace} R^{\mathbb{S}^n}(d\phi_{s,0}\cdot,\tau(\phi_{s,0}))d\phi_{s,0}\cdot\right\}\big|_{s=0}\\
&\quad=-\operatorname{trace}\langle\tau(\phi),dV\cdot\rangle d\phi\cdot+\operatorname{trace}\langle d\phi\cdot,\Delta V\rangle d\phi\cdot\\
&\qquad+\operatorname{trace}\langle d\phi\cdot,\operatorname{trace}\langle V,d\phi\cdot\rangle d\phi\cdot\rangle d\phi\cdot\\
&\qquad-|d\phi|^2\operatorname{trace}\langle d\phi\cdot,V\rangle d\phi\cdot-\operatorname{trace}\langle\tau(\phi),d\phi\cdot\rangle dV\cdot\\
&\qquad+2\langle dV,d\phi\rangle\tau(\phi)-|d\phi|^2\Delta V-|d\phi|^2\operatorname{trace}\langle V,d\phi\cdot\rangle d\phi\cdot+|d\phi|^4V.
\end{aligned}
\tag{2.5}
$$

Now, replacing (2.4) and (2.5) in (2.3), we obtain (2.2). □

Remark 2.2. We note that formula (2.2) can be also deduced from formula (5.8) in [8].

Corollary 2.3. *Let $\phi:(M,g)\to\mathbb{S}^n$ be a harmonic Riemannian immersion. Then the operator I of ϕ is symmetric, positive semi-definite and*

$$
\ker I=\{V\in C(\phi^{-1}T\mathbb{S}^n)\mid \Delta V=mV-V^T\},
\tag{2.6}
$$

where $V=V^T+V^N$, $V^T\in C(TM)$ and $V^N\in C(NM)$.

Proof. From (2.2) it follows

$$
I(V)=\Delta(\Delta V)-2m\Delta V+m^2V+\Delta V^T+(\Delta V)^T+(1-2m)V^T.
$$

First we shall prove that I is symmetric, i.e. $(I(V),W)=(V,I(W))$, $\forall V,W\in C(\phi^{-1}T\mathbb{S}^n)$, where $(V,W)=\int_M\langle V,W\rangle v_g$ is the usual inner product on the real vector space $C(\phi^{-1}T\mathbb{S}^n)$. Since Δ is a symmetric operator and $\langle V^T,W\rangle=\langle W^T,V\rangle$, in order to prove that I is symmetric we must show that

$$
\int_M\langle\Delta V^T+(\Delta V)^T,W\rangle v_g=\int_M\langle\Delta W^T+(\Delta W)^T,V\rangle v_g.
$$

But

$$
\begin{aligned}
\int_M\langle\Delta V^T,W\rangle v_g&=\int_M\langle V^T,\Delta W\rangle v_g=\int_M\langle V^T,(\Delta W)^T\rangle v_g\\
&=\int_M\langle V,(\Delta W)^T\rangle v_g,
\end{aligned}
$$

and

$$\int_M \langle (\Delta V)^T, W\rangle v_g = \int_M \langle (\Delta V)^T, W^T\rangle v_g = \int_M \langle \Delta V, W^T\rangle v_g$$
$$= \int_M \langle V, \Delta W^T\rangle v_g.$$

So I is a symmetric operator.

In order to prove that J is positive semi-definite, i.e. $(I(V), V) \geq 0$, we start with the following remarks

$$\int_M \langle \Delta V^T, V\rangle v_g = \int_M \langle (\Delta V)^T, V\rangle v_g,$$

and

$$\begin{aligned} I(V) = {}& \Delta\Delta V^T + \Delta\Delta V^N - 2m\Delta V^T - 2m\Delta V^N + m^2 V^T + m^2 V^N \\ & + \Delta V^T + (\Delta V)^T + (1-2m)V^T. \end{aligned}$$

Thus we have

$$\begin{aligned} (I(V), V) = {}& \int_M \{\langle \Delta(\Delta V^T), V\rangle + 2(1-m)\langle \Delta V^T, V\rangle + (m-1)^2\langle V^T, V\rangle \\ & + \langle \Delta(\Delta V^N), V\rangle - 2m\langle \Delta V^N, V\rangle + m^2\langle V^N, V\rangle\} v_g \\ = {}& \int_M \{\langle \Delta(\Delta V^T), V^T\rangle + 2(1-m)\langle \Delta V^T, V^T\rangle + (m-1)^2|V^T|^2 \\ & + \langle \Delta(\Delta V^N), V^N\rangle - 2m\langle \Delta V^N, V^N\rangle + m^2|V^N|^2 \\ & + \langle \Delta(\Delta V^T), V^N\rangle + 2(1-m)\langle \Delta V^T, V^N\rangle \\ & + \langle \Delta(\Delta V^N), V^T\rangle - 2m\langle \Delta V^N, V^T\rangle\} v_g \\ = {}& \int_M \{|\Delta V^T + (1-m)V^T|^2 + |\Delta V^N - mV^N|^2 \\ & + 2(\langle \Delta V^T, \Delta V^N\rangle + (1-2m)\langle \Delta V^T, V^N\rangle)\} v_g \\ = {}& \int_M |\Delta V^T + (1-m)V^T + \Delta V^N - mV^N|^2 v_g \\ = {}& \int_M |\Delta V - mV + V^T|^2 v_g. \end{aligned}$$

From the above relation it follows that I is positive semi-definite and $\ker I$ is given by (2.6). $\square$

In the following we shall consider the simplest two cases of biharmonic maps $\phi : (M, g) \to \mathbb{S}^n$. These maps are harmonic Riemannian immersions, so they are weakly-stable, i.e. the operator I is positive semi-definite.

Theorem 2.4. *The identity map* $\mathbf{1} : \mathbb{S}^n \to \mathbb{S}^n$ *is weakly-stable and*

a) *if* $n = 2$ *then* $\operatorname{nullity}(\mathbf{1}) = 6$,

b) *if* $n > 2$ *then* $\operatorname{nullity}(\mathbf{1}) = \frac{n(n+1)}{2}$.

Proof. In this case $C(\mathbf{1}^{-1}T\mathbb{S}^n) = C(T\mathbb{S}^n)$ and $\Delta V = -\operatorname{trace} \nabla^2 V$. We shall use X to denote a tangent vector field on $\mathbb{S}^n$. By Corollary 2.3, the operator I is given by

$$I(X) = \Delta(\Delta X) - 2(n-1)\Delta X + (n-1)^2 X,$$

and

$$I(X) = 0 \iff \Delta X = (n-1)X.$$

The Hodge decomposition theorem for $C(T\mathbb{S}^n)$ states that

$$C(T\mathbb{S}^n) = \{X \in C(T\mathbb{S}^n) \mid \operatorname{div} X = 0\} \oplus \{\operatorname{grad} f \mid f \in C^\infty(\mathbb{S}^n)\}.$$

This decomposition of $C(T\mathbb{S}^n)$ is orthogonal with respect to the scalar product on the real vector space $C(T\mathbb{S}^n)$, and Δ_H preserves invariantly these subspaces, where, using the musical isomorphisms,

$$\Delta_H(X) = (\overline{\Delta} X^\flat)^\sharp,$$

$\overline{\Delta}$ being the Laplacian which acts on $\Lambda^1(\mathbb{S}^n)$.

It is known that

$$\Delta X = \Delta_H(X) - (n-1)X$$

(see [5], [11]), so

$$I(X) = 0 \iff \Delta_H(X) = 2(n-1)X.$$

From the Hodge decomposition theorem we write $X = Y + \operatorname{grad} f$, $\operatorname{div} Y = 0$ and we obtain

$$\Delta_H(X) = 2(n-1)X \iff \begin{cases} \Delta_H(Y) = 2(n-1)Y \\ \Delta_H \operatorname{grad} f = 2(n-1) \operatorname{grad} f. \end{cases}$$

Consequently

$$I(X) = 0 \iff \begin{cases} Y \text{ is a Killing vector field} \\ \Delta f = 2(n-1)f. \end{cases}$$

It is known that the first eigenvalues of Δ which acts on $C^\infty(\mathbb{S}^n)$ are 0, n, $2(n+1)$, and the eigenvalue n has the multiplicity $n+1$. So $2(n-1)$ is an eigenvalue if and only if $n = 2$, and in this case its multiplicity is 3.

It is well known too that

$$\dim\{Y \in C(T\mathbb{S}^n) \mid Y \text{is a Killing vector field}\} = \frac{n(n+1)}{2}.$$

Now, the theorem follows. □

Theorem 2.5. *The canonical inclusion* $\boldsymbol{i} : \mathbb{S}^m \to \mathbb{S}^n$ *is weakly-stable and*

a) *if* $m = 2$ *then* $\operatorname{nullity}(\boldsymbol{i}) = 3n$,

b) *if* $m > 2$ *then* $\operatorname{nullity}(\boldsymbol{i}) = (n-m)(m+1) + \frac{m(m+1)}{2}$.

Proof. Let $V \in C(N\mathbb{S}^m)$ and $X, Y \in C(T\mathbb{S}^m)$. As $\boldsymbol{i}$ is a totally geodesic map, it results that

$$\nabla_X V = \nabla^\perp_X V,\ \Delta V = \Delta^\perp V,\ \nabla_X Y =^{\mathbb{S}^m} \nabla_X Y,\ \Delta X = -\operatorname{trace}^{\mathbb{S}^m} \nabla^2 X.$$

Again, by Corollary 2.3, the operator I is given by

$$\begin{cases} I(V) = \Delta^\perp(\Delta^\perp V) - 2m\Delta^\perp V + m^2 V \in C(N\mathbb{S}^m) \\ I(X) = \Delta(\Delta X) - 2(m-1)\Delta X + (m-1)^2 X \in C(T\mathbb{S}^m), \end{cases}$$

and

$$\begin{cases} I(V) = 0 \iff \Delta^\perp V = mV \\ I(X) = 0 \iff \Delta X = (m-1)X. \end{cases}$$

Now, let $\{E_{m+1}, \ldots, E_n\}$ be the vector fields which give the trivialisation of $N\mathbb{S}^m$. We have

$$\nabla_X E_{m+1} = \ldots = \nabla_X E_n = 0, \quad \forall X \in C(T\mathbb{S}^m) \tag{2.7}$$

(see [10]). Since any $V \in C(N\mathbb{S}^m)$ can be written as

$$V = f^1 E_{m+1} + \cdots + f^{n-m} E_n,$$

where $f^1, \dots, f^{n-m} \in C^\infty(\mathbb{S}^m)$, from (2.7) we obtain

$$\Delta^\perp V = mV \iff \Delta f^1 = mf^1,\ \dots,\ \Delta f^{n-m} = mf^{n-m}.$$

So we have

$$\dim\{V \in C(N\mathbb{S}^m) \mid I(V) = 0\} = (n-m)(m+1).$$

Now, using Theorem 2.4 and the fact that the kernel of I splits in the direct sum of the kernel of I restricted to $C(N\mathbb{S}^m)$ and the kernel of I restricted to $C(T\mathbb{S}^m)$, we conclude. □

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