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Higher-order generalizations of Hadamard's inequality

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Abstract. In this paper we derive generalizations of Hadamard's classical inequality for higher-order convex functions. In the proof the remainder formula of the Hermite–Fejér interpolation and a smoothing technique is used.

1. Introduction

Hadamard's classical inequality [2] provides the following lower and upper estimates for the integral average of a convex function $f : [a, b] \to \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

An account of various generalizations of Hadamard-type inequalities can be found in a recent book [1] by S. S. Dragomir and C. E. M. Pearce. Interesting historical remarks are due to Mitrinović and Lacković [6].

If $f : [a, b] \to \mathbb{R}$ is supposed to be monotone increasing, an analogous "Hadamard-type" inequality can trivially be derived:

$$f(a) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b).$$

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Our goal is to generalize these inequalities when $f : [a,b] \to \mathbb{R}$ is *n-monotone* or, in other terms, $(n-1)$-*convex*, that is,

$$(-1)^n \begin{vmatrix} f(x_0) & \dots & f(x_n) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \geq 0$$

whenever $a \leq x_0 < \cdots < x_n \leq b$. Obviously, a function is 1-monotone if and only if it is monotone increasing; similarly, a function is 2-monotone if and only if it is convex.

In a series of papers [8]–[18], T. Popoviciu introduced and investigated the notion of higher-order convexity. A summary of these results can be found in the book [19] and also in [5]. In our investigations, we need the following two results of T. Popoviciu. The first characterizes n-monotonicity in terms of the nth derivative of f.

Theorem A ([5, Theorem 1. p. 387]). *Assume that* $f : \,]a,b[\, \to \mathbb{R}$ *is an* n *times differentiable function. Then* f *is* n*-monotone if and only if* $f^{(n)}(x) \geq 0$ *for all* $x \in \,]a,b[$.

The second result states that, for $n \geq 2$, n-monotonicity implies regularity properties of f.

Theorem B ([5, Theorem 1. p. 391]). *Assume that* $f : \,]a,b[\, \to \mathbb{R}$ *is an* n*-monotone function and* $n \geq 2$. *Then* f *is* $(n-2)$ *times differentiable and* $f^{(n-2)}$ *is continuous.*

Applying Theorem A, we will be able to prove Hadamard-type inequalities by using Gauss-type quadrature formulae and their remainder terms for smooth enough functions.

For the general case, when $f : [a,b] \to \mathbb{R}$ is supposed to be continuous only and n-monotone, a smoothing technique is developed to get Hadamard-type inequalities. As an application, we derive Hadamard-type inequalities for 3-, 4-, 5-, 6-, 8-, 10-, and 12-monotone functions.

2. Gauss-type quadrature formulae and remainder terms

Let $f, g : [a,b] \to \mathbb{R}$ and $\rho : [a,b] \to \,]0,+\infty[$ be continuous functions. The functions f and g are said to be *ρ-orthogonal on* $[a,b]$ if

$$\langle f, g\rangle_\rho := \int_a^b fg\rho = 0.$$

We say that a system of polynomials is an *orthogonal polynomial system on* $[a,b]$ *with respect to the weight function* ρ if each member of the system is ρ-orthogonal to the others on $[a,b]$. Define the *moments* of ρ by

$$m_n := \int_a^b x^n \rho(x)dx \qquad (n = 0, 1, 2, \dots).$$

It is easy to check, that

$$P_n(x) := \begin{vmatrix} 1 & m_0 & \dots & m_{n-1} \\ x & m_1 & \dots & m_n \\ \vdots & \vdots & \ddots & \vdots \\ x^n & m_n & \dots & m_{2n-1} \end{vmatrix}$$

is the nth degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function ρ, since it is immediate to see that P_n is ρ-orthogonal to the polynomials $1, x, \dots, x^{n-1}$.

Let us consider the following

$$\int_a^b f(x)\rho(x)dx = \sum_{k=1}^n c_k f(\xi_k) \tag{1}$$

$$\int_a^b f(x)\rho(x)dx = c_0 f(a) + \sum_{k=1}^n c_k f(\xi_k) \tag{2}$$

$$\int_a^b f(x)\rho(x)dx = \sum_{k=1}^n c_k f(\xi_k) + c_{n+1} f(b) \tag{3}$$

$$\int_a^b f(x)\rho(x)dx = c_0 f(a) + \sum_{k=1}^n c_k f(\xi_k) + c_{n+1} f(b) \tag{4}$$

Gauss-type quadrature formulae, where the constants $c_0, c_1, \dots, c_n, c_{n+1}$ and $\xi_1, \dots, \xi_n \in \,]a,b[$ are to be determined so that (1)–(3), and (4) be exact when f is a polynomial of degree at most $2n-1$, $2n$, $2n$, and $2n+1$, respectively. We shall distinguish four cases.

Case A.

Theorem 1. *Let P_n be the nth degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function ρ. Then* (1) *is exact for polynomials f with* $\deg f \le 2n-1$ *if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n and*

$$c_k = \int_a^b \frac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx. \tag{5}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]a,b[$, and $c_k \ge 0$ for all $k = 1, \dots, n$.

This theorem follows easily from well known results in numerical analysis [3], [4], [20]. For the sake of completeness, we provide a proof.

Proof. Assume that $\xi_1, \dots, \xi_n$ are the zeros of P_n. Denote by $L_k : [a,b] \to \mathbb{R}$ $(k = 1, \dots, n)$ the primitive Lagrange interpolation polynomials:

$$L_k(x) := \begin{cases} \dfrac{P_n(x)}{(x-\xi_k)P_n'(\xi_k)} & \text{if } x \ne \xi_k \\ 1 & \text{if } x = \xi_k. \end{cases}$$

If Q is a polynomial with $\deg Q \le 2n-1$, then using Euclidean algorithm Q can be written in the form

$$Q = PP_n + R$$

such that $\deg P, \deg R \le n-1$. The inequality $\deg P \le n-1$ implies that

$$\langle P, P_n \rangle_\varrho = 0,$$

while $\deg R \le n-1$ yields that R is equal to its Lagrange interpolation polynomial:

$$R = \sum_{k=1}^{n} R(\xi_k)L_k.$$

Therefore, by the definition of $c_1, \dots, c_n$ in (5),

$$\int_a^b Q\rho = \int_a^b PP_n\rho + \int_a^b R\rho = \sum_{k=1}^n R(\xi_k) \int_a^b L_k\rho$$
$$= \sum_{k=1}^n c_k R(\xi_k) = \sum_{k=1}^n c_k \big(P(\xi_k)P_n(\xi_k) + R(\xi_k)\big) = \sum_{k=1}^n c_k Q(\xi_k).$$

That is, (1) is exact for polynomials of degree at most $2n-1$.

Conversely, assume that (1) is exact for polynomials of degree at most $2n-1$. Let $Q(x) := (x-\xi_1)\dots(x-\xi_n)$ and let be P a polynomial with $\deg P \le n-1$. Then $\deg PQ \le 2n-1$, thus

$$\int_a^b PQ\rho = c_1 P(\xi_1)Q(\xi_1) + \dots + c_n P(\xi_n)Q(\xi_n) = 0.$$

Therefore, Q is ρ-orthogonal to P. Using the uniqueness of P_n, we get that $P_n = a_n Q$ and $\xi_1, \dots, \xi_n$ are the zeros of P_n. Furthermore, (1) is exact if we substitute $f := L_k$ and $f := L_k^2$, respectively. The first substitution gives (5), while the second one shows the nonnegativity of c_k. □

Case B. Denote by ρ_a the weight function defined by

$$\rho_a(x) := (x-a)\rho(x) \qquad (x \in [a,b]).$$

Theorem 2. *Let P_n be the nth degree member of the orthogonal polynomial-system on $[a,b]$ with respect to the weight function ρ_a. Then (2) is exact for polynomials f with $\deg f \le 2n$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n,*

$$c_0 = \frac{1}{P_n^2(a)} \int_a^b P_n^2(x)\rho(x)dx \tag{6}$$

and

$$c_k = \frac{1}{\xi_k - a} \int_a^b \frac{P_n(x)(x-a)}{(x-\xi_k)P_n'(\xi_k)}\rho(x)dx. \tag{7}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]\,a,b\,[$, and $c_k \ge 0$ for all $k = 0, 1, \dots, n$.

Proof. Assume that (2) is exact for polynomials of degree at most $2n$. If P is a polynomial with $\deg P \le 2n-1$, then

$$\int_a^b P\rho_a = \int_a^b (x-a)P(x)\rho(x)dx = c_1(\xi_1-a)P(\xi_1)+\cdots+c_n(\xi_n-a)P(\xi_n).$$

Applying Theorem 1 to the weight function ρ_a and the constants

$$c_{a;k} := c_k(\xi_k - a)$$

we get, that $\xi_1,\dots,\xi_n$ are the zeros of P_n, and the constants $c_{a;k}$ ($k = 1,\dots,n$) can be computed by the formula (5). Substituting $f := P_n^2$ into (2), we obtain that

$$c_0 = \frac{1}{P_n^2(a)}\int_a^b P_n^2\rho.$$

Thus, we get that (6) and (7) are valid and $c_k \ge 0$ for $k = 1,\dots,n$.

Conversely, assume that $\xi_1,\dots,\xi_n$ are the zeros of P_n, and the constants $c_1,\dots,c_n$ are given by the formula (7) and $c_0 = \int_a^b \rho-(c_1+\ldots+c_n)$. If P is a polynomial with $\deg P \le 2n$, then there exists a polynomial Q with $\deg Q \le 2n-1$ such that

$$P(x) = Q(x)(x-a) + P(a).$$

By Theorem 1,

$$\int_a^b Q\rho_a = c_{a;1}Q(\xi_1) + \cdots + c_{a;n}Q(\xi_n)$$

holds. Thus

$$\begin{aligned}\int_a^b P(x)\rho(x)dx &= \int_a^b \big(Q(x)(x-a)+P(a)\big)\rho(x)dx\\ &= \sum_{k=1}^n c_k(\xi_k-a)Q(\xi_k) + \sum_{k=0}^n P(a)c_k\\ &= c_0P(a) + \sum_{k=1}^n c_k\big((\xi_k-a)Q(\xi_k)+P(a)\big)\\ &= c_0P(a) + \sum_{k=1}^n c_kP(\xi_k),\end{aligned}$$

which yields that (2) is exact for polynomials of degree at most $2n$. Therefore, substituting $f := P_n^2$ into (2), we get (6). $\square$

Case C. Denote by ρ^b the weight function defined by

$$\rho^b(x) := (b-x)\rho(x) \qquad (x \in [a,b]).$$

Theorem 3. *Let P_n be the nth degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function ρ^b. Then (3) is exact for polynomials f with $\deg f \leq 2n$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n,*

$$c_k = \frac{1}{b-\xi_k} \int_a^b \frac{P_n(x)(b-x)}{(x-\xi_k)P_n'(\xi_k)} \rho(x)dx \tag{8}$$

and

$$c_{n+1} = \frac{1}{P_n^2(b)} \int_a^b P_n^2(x)\rho(x)dx. \tag{9}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]\,a,b\,[$, and $c_k \geq 0$ for all $k = 1, \dots, n, n+1$.

Hint. Applying a similar argument as in the previous proof for the weight function ρ^b, one can get the statement of the theorem. $\square$

Case D. Denote by ρ_a^b the weight function defined by

$$\rho_a^b(x) := (b-x)(x-a)\rho(x) \qquad (x \in [a,b]).$$

Theorem 4. *Let P_n be the nth degree member of the orthogonal polynomial-system on $[a,b]$ with respect to the weight function ρ_a^b. Then (4) is exact for polynomials f with $\deg f \leq 2n+1$ if and only if $\xi_1, \dots, \xi_n$ are the zeros of P_n,*

$$c_0 = \frac{1}{(b-a)P_n^2(a)} \int_a^b P_n^2(x)(b-x)\rho(x)dx, \tag{10}$$

$$c_k = \frac{1}{(b-\xi_k)(\xi_k-a)} \int_a^b \frac{P_n(x)(b-x)(x-a)}{(x-\xi_k)P_n'(\xi_k)} \rho(x)dx, \tag{11}$$

and

$$c_{n+1} = \frac{1}{(b-a)P_n^2(b)} \int_a^b P_n^2(x)(x-a)\rho(x)dx. \tag{12}$$

Furthermore, $\xi_1, \dots, \xi_n$ are pairwise distinct elements of $]\,a, b\,[$, and $c_k \geq 0$ for all $k = 0, 1, \dots, n,\ n+1$.

Hint. Using Theorem 2 or Theorem 3 and applying a similar argument as in the previous proof for the weight-function ρ_a^b, one can get the statement of the theorem. A more direct proof can also be done by using Theorem 3. For deriving (10) and (12), substitute $f(x) := (b-x)P_n^2(x)$ and $f(x) := (x-a)P_n^2(x)$ into (4). □

Remainder term for the Hermite interpolation formula. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, $x_1, \dots, x_n$ be pairwise distinct elements of $[a, b]$, and $1 \leq r \leq n$ be a fixed integer. Denote by H the Hermite interpolation polynomial satisfying the following conditions:

$$H(x_k) = f(x_k) \quad (k = 1, \dots, n)$$
$$H'(x_k) = f'(x_k) \quad (k = 1, \dots, r).$$

We recall that $\deg H = n + r - 1$. From a well known result, (c.f. [3, Section 5.3, pp. 230–231]), if f is $(n+r)$-times differentiable then, for all $x \in [a, b]$, there exists η such that

$$f(x) - H(x) = \frac{\omega_n(x)\omega_r(x)}{(n+r)!} f^{(n+r)}(\eta), \tag{13}$$

where

$$\omega_k(x) = (x - x_1) \cdots (x - x_k).$$

3. Smoothing n-monotone functions

It is well known that there exists a function φ which possesses the following properties:

(i) $\varphi : \mathbb{R} \to \mathbb{R}_+$ is $\mathcal{C}^\infty$, i.e., it is infinitely many times differentiable;

(ii) $\operatorname{supp} \varphi \subset [-1, 1]$;

(iii) $\int_{\mathbb{R}} \varphi = 1$.

Using φ, we define for all $\varepsilon > 0$ the function φ_ε by

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\Big(\frac{x}{\varepsilon}\Big) \qquad (x \in \mathbb{R}).$$

Then, one can easily check that φ_ε satisfies the following conditions:

(i') $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is $\mathcal{C}^\infty$;

(ii') $\operatorname{supp} \varphi_\varepsilon \subset [-\varepsilon, \varepsilon]$;

(iii') $\int_{\mathbb{R}} \varphi_\varepsilon = 1$.

Let $I \subset \mathbb{R}$ be a nonempty open interval, $f : I \to \mathbb{R}$ be a continuous function, and $\varepsilon > 0$. We will denote the convolution of f and φ_ε by f_ε, that is,

$$f_\varepsilon(x) := \int_{\mathbb{R}} \bar{f}(y)\varphi_\varepsilon(x-y)dy \qquad (x \in \mathbb{R}),$$

where $\bar{f}(y) = f(y)$ if $y \in I$, otherwise $\bar{f}(y) = 0$. We recall, that $f_\varepsilon \to f$ uniformly as $\varepsilon \to 0$ on each compact subinterval of I, and f_ε is infinitely many times differentiable on $\mathbb{R}$; these important results can be found for example in [21, p. 549].

Theorem 5. *Let $I \subset \mathbb{R}$ be a nonempty open interval, $f : I \to \mathbb{R}$ be an n-monotone continuous function. Then, for all compact subintervals $[a, b]$ of I, there exists a sequence of n-monotone and $\mathcal{C}^\infty$ functions (f_k) which converges uniformly to f on $[a, b]$.*

Proof. Choose $a, b \in I$ and $\varepsilon_0 > 0$ such that the relation $[a - \varepsilon_0, b + \varepsilon_0] \subset I$ hold. We show that the function $\tau_\varepsilon f : [a, b] \to \mathbb{R}$ defined by

$$\tau_\varepsilon f(x) := f(x - \varepsilon) \qquad (x \in [a, b])$$

is n-monotone on $[a, b]$ for $\varepsilon \in]0, \varepsilon_0[$. Let $a \le x_0 < \cdots < x_n \le b$ and $k \le n - 1$ be fixed. Using induction, we are going to verify the equality

$$\begin{vmatrix} \tau_\varepsilon f(x_0) & \dots & \tau_\varepsilon f(x_n) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{k-1} & \dots & x_n^{k-1} \\ x_0^k & \dots & x_n^k \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} \tau_\varepsilon f(x_0) & \dots & \tau_\varepsilon f(x_n) \\ 1 & \dots & 1 \\ x_0 - \varepsilon & \dots & x_n - \varepsilon \\ \vdots & \ddots & \vdots \\ (x_0 - \varepsilon)^{k-1} & \dots & (x_n - \varepsilon)^{k-1} \\ x_0^k & \dots & x_n^k \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix}. \tag{14}$$

If $k = 1$, then this equation obviously holds. Assume, for a fixed positive integer $k \leq n-2$, that the equation remains true. By the binomial theorem,

$$x^k = \binom{k}{0}\varepsilon^k + \binom{k}{1}\varepsilon^{k-1}(x-\varepsilon) + \cdots + \binom{k}{k}(x-\varepsilon)^k,$$

which means, that $(x-\varepsilon)^k$ is the linear combination of the elements $1, x-\varepsilon, \ldots, (x-\varepsilon)^k, x^k$. Therefore, adding the adequate linear combination of the 2nd,…, $(k+1)$st rows to the $(k+2)$nd row, we get that the equation

$$\begin{vmatrix} \tau_\varepsilon f(x_0) & \ldots & \tau_\varepsilon f(x_n) \\ 1 & \ldots & 1 \\ x_0-\varepsilon & \ldots & x_n-\varepsilon \\ \vdots & \ddots & \vdots \\ (x_0-\varepsilon)^{k-1} & \ldots & (x_n-\varepsilon)^{k-1} \\ x_0^k & \ldots & x_n^k \\ x_0^{k+1} & \ldots & x_n^{k+1} \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \ldots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} \tau_\varepsilon f(x_0) & \ldots & \tau_\varepsilon f(x_n) \\ 1 & \ldots & 1 \\ x_0-\varepsilon & \ldots & x_n-\varepsilon \\ \vdots & \ddots & \vdots \\ (x_0-\varepsilon)^{k-1} & \ldots & (x_n-\varepsilon)^{k-1} \\ (x_0-\varepsilon)^k & \ldots & (x_n-\varepsilon)^k \\ x_0^{k+1} & \ldots & x_n^{k+1} \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}$$

holds. That is, (14) holds for all fixed positive k $(1 \leq k \leq n-1)$. Particularly, if $k = n-1$, we get the n-monotonicity of $\tau_\varepsilon f$. Using integral transformation and the previous result,

$$\begin{aligned} &(-1)^n \begin{vmatrix} f_\varepsilon(x_0) & \ldots & f_\varepsilon(x_n) \\ 1 & \ldots & 1 \\ x_0 & \ldots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \ldots & x_n^{n-1} \end{vmatrix} \\ &= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \bar{f}(t)\varphi_\varepsilon(x_0-t) & \ldots & \bar{f}(t)\varphi_\varepsilon(x_n-t) \\ 1 & \ldots & 1 \\ x_0 & \ldots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \ldots & x_n^{n-1} \end{vmatrix} dt \end{aligned}$$

$$= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \bar{f}(x_0 - s) & \dots & \bar{f}(x_n - s) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \varphi_\varepsilon(s) ds$$

$$= \int_{\mathbb{R}} (-1)^n \begin{vmatrix} \tau_s f(x_0) & \dots & \tau_s f(x_n) \\ 1 & \dots & 1 \\ x_0 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_0^{n-1} & \dots & x_n^{n-1} \end{vmatrix} \varphi_\varepsilon(s) ds \geq 0$$

and we get, that f_ε is n-monotone on $[a, b]$ for $\varepsilon \in \,]0, \varepsilon_0[$.

To complete the proof, choose a positive integer n_0 such that the relation $\frac{1}{n_0} < \varepsilon_0$ hold. If $\varepsilon_k := \frac{1}{n_0 + k}$ $(k = 1, 2, \dots)$ and $f_k := f_{\varepsilon_k}$, then $\varepsilon_k \in \,]0, \varepsilon_0[$, thus $(f_k)_{k=1}^\infty$ satisfies the requirements of the theorem. □

4. Generalized Hadamard-inequalities

Our main results concern the cases of odd and even order of convexity separately. First we deal with odd order convex functions.

Theorem 6. *Let, for $n \geq 0$,*

$$p_n(x) := \begin{vmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{n+1} \\ x & \frac{1}{3} & \dots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x^n & \frac{1}{n+2} & \dots & \frac{1}{2n+1} \end{vmatrix},$$

then p_n has n pairwise distinct roots in $]0, 1[$. Denote these roots by $\lambda_1, \dots, \lambda_n$, and

$$\alpha_0 := \frac{1}{p_n^2(0)} \int_0^1 p_n^2(x) dx,$$

$$\alpha_k := \frac{1}{\lambda_k} \int_0^1 \frac{p_n(x) x}{(x - \lambda_k) p_n'(\lambda_k)} dx \quad (k = 1, \dots, n).$$

Then the following inequalities hold for any $m = 2n+1$-monotone function $f : [a, b] \to \mathbb{R}$:

$$\alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \leq \frac{1}{b-a} \int_a^b f(x)dx$$
$$\leq \sum_{k=1}^{n} \alpha_k f\big(\lambda_k a + (1-\lambda_k)b\big) + \alpha_0 f(b).$$

Proof. Observe that p_n is the nth degree orthogonal polynomial on $[0, 1]$ with respect to the weight function $\rho(x) := x$ (c.f. the beginning of Section 2). First we prove the statement for the special case when $a = 0$, $b = 1$ and $f : [0, 1] \to \mathbb{R}$ is supposed to be $m = 2n + 1$ times differentiable. In this case, $f^{(2n+1)} \geq 0$ on $]0, 1[$, according to Theorem A.

Let H be the $2n$th degree Hermite interpolation polynomial which possesses the following properties:

$$H(0) = f(0),$$
$$H(\lambda_k) = f(\lambda_k) \quad (k = 1, \dots, n),$$
$$H'(\lambda_k) = f'(\lambda_k) \quad (k = 1, \dots, n).$$

By (13), for all $x \in [0, 1]$, there exists $\eta \in \,]0, 1[$ such that

$$f(x) - H(x) = \frac{x(x-\lambda_1)^2 \cdots (x-\lambda_n)^2}{(2n+1)!} f^{(2n+1)}(\eta);$$

therefore, for all $x \in [0, 1]$,

$$f(x) \geq H(x).$$

Since H is of degree $2n$, applying Theorem 2, we get that

$$\int_0^1 f(x)dx \geq \int_0^1 H(x)dx = \alpha_0 H(0) + \sum_{k=1}^{n} \alpha_k H(\lambda_k)$$
$$= \alpha_0 f(0) + \sum_{k=1}^{n} \alpha_k f(\lambda_k).$$

Now we suppose that $a, b \in \mathbb{R}$ $(a < b)$, but $f : [a,b] \to \mathbb{R}$ is still $m = 2n+1$ times differentiable. Define the function $F : [0,1] \to \mathbb{R}$ by

$$F(t) := f\big((1-t)a + tb\big).$$

Then, F is m-times differentiable and m-monotone on $[0,1]$. It is easy to check that

$$\int_0^1 F(x)dx = \frac{1}{b-a}\int_a^b f(x)dx.$$

The previous result applied to the function F, yields

$$\alpha_0 f(a) + \sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \le \frac{1}{b-a}\int_a^b f(x)dx.$$

Finally, let $f : [a,b] \to \mathbb{R}$ be an arbitrary m-monotone function. Without the loss of generality we may assume that $m > 1$; by Theorem B, in this case f is continuous. Choose $\varepsilon > 0$. According to Theorem 5, there exists a sequence of $\mathcal{C}^\infty$ functions $(f_i)_{i=1}^\infty$ whose members are defined on $[a,b]$, $f_i \to f$ uniformly on $[a+\varepsilon, b-\varepsilon]$, and f_i is m-monotone on $[a+\varepsilon, b-\varepsilon]$. Then, applying the previous step on the interval $[a+\varepsilon, b-\varepsilon]$, we get

$$\begin{aligned}\alpha_0 f_i(a+\varepsilon) + \sum_{k=1}^{n} \alpha_k f_i\big((1-\lambda_k)(a+\varepsilon) + \lambda_k(b-\varepsilon)\big)\\ \le \frac{1}{b-a-2\varepsilon}\int_{a+\varepsilon}^{b-\varepsilon} f_i(x)dx.\end{aligned}$$

Letting $i \to \infty$ and then $\varepsilon \to 0$, we get the left hand side inequality to be proved.

Now define the function $F : [a,b] \to \mathbb{R}$ by

$$F(x) := -f(a+b-x).$$

Then F is m-monotone on $[a,b]$. Using the left hand side inequality for F, the right hand side inequality for f follows. □

Our second main result offers Hadamard-type inequalities for even-order convex functions.

Theorem 7. *Let, for $n \geq 1$,*

$$p_n(x) := \begin{vmatrix} 1 & 1 & \dots & \frac{1}{n} \\ x & \frac{1}{2} & \dots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^n & \frac{1}{n+1} & \dots & \frac{1}{2n} \end{vmatrix},$$

$$q_n(x) := \begin{vmatrix} 1 & \frac{1}{2\cdot 3} & \dots & \frac{1}{n(n+1)} \\ x & \frac{1}{3\cdot 4} & \dots & \frac{1}{(n+1)(n+2)} \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & \frac{1}{(n+1)(n+2)} & \dots & \frac{1}{(2n-1)2n} \end{vmatrix}$$

then p_n has n, and q_n has $n-1$ pairwise distinct roots in $]0,1[$. Denote these roots by $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_{n-1}$, respectively. Let

$$\alpha_k = \int_0^1 \frac{p_n(x)}{(x-\lambda_k)p_n'(\lambda_k)} dx \quad (k = 1, \dots, n),$$

and

$$\beta_0 := \frac{1}{q_n^2(0)} \int_0^1 q_n^2(x)(1-x)dx,$$

$$\beta_k := \frac{1}{(1-\mu_k)\mu_k} \int_0^1 \frac{q_n(x)x(1-x)}{(x-\mu_k)q_n'(\mu_k)} dx \quad (k = 1, \dots, n-1),$$

$$\beta_n := \frac{1}{q_n^2(1)} \int_0^1 q_n^2(x)x dx,$$

then the following inequalities hold for any $m = 2n$-monotone function $f : [a,b] \to \mathbb{R}$:

$$\sum_{k=1}^{n} \alpha_k f\big((1-\lambda_k)a + \lambda_k b\big) \leq \frac{1}{b-a} \int_a^b f(x)dx$$

$$\leq \beta_0 f(a) + \sum_{k=1}^{n-1} \beta_k f\big((1-\mu_k)a + \mu_k b\big) + \beta_n f(b).$$

An inequality analogous to the left hand side inequality was also established by T. Popoviciu in [12].

PROOF. Observe that p_n is the nth degree orthogonal polynomial on $[0,1]$ with respect to the weight function $\rho(x) := 1$; similarly, q_n is the $(n-1)$st degree orthogonal polynomial on $[0,1]$ with respect to the weight function $\rho(x) := (1-x)x$. First, just as before, we prove the statement for the special case when $a=0$, $b=1$ and $f:[0,1]\to\mathbb{R}$ is supposed to be $m=2n$ times differentiable. In this case, $f^{(2n)}\geq 0$ on $]0,1[$ according to Theorem A.

Let H be the $(2n-1)$st degree Hermite interpolation polynomial which possesses the following properties:

$$\begin{aligned} H(\lambda_k) &= f(\lambda_k), \\ H'(\lambda_k) &= f'(\lambda_k) \quad (k=1,\dots,n). \end{aligned}$$

By (13), for all $x\in[0,1]$, there exists $\eta\in\ ]0,1[$ such that

$$f(x)-H(x)=\frac{(x-\lambda_1)^2\dots(x-\lambda_n)^2}{(2n)!}f^{(2n)}(\eta).$$

Therefore, for all $x\in[0,1]$,

$$f(x)\geq H(x).$$

Since H is of degree $2n-1$, applying Theorem 1, we get that

$$\int_0^1 f(x)dx \geq \int_0^1 H(x)dx=\sum_{k=1}^n \alpha_k H(\lambda_k)=\sum_{k=1}^n \alpha_k f(\lambda_k).$$

Now let H be the $(2n-1)$st degree Hermite interpolation polynomial which possesses the following properties:

$$\begin{aligned} H(0) &= f(0), \\ H(\mu_k) &= f(\mu_k), \\ H'(\mu_k) &= f'(\mu_k) \quad (k=1,\dots,n-1), \\ H(1) &= f(1). \end{aligned}$$

By (13), for all $x\in[0,1]$, there exists $\eta\in\ ]0,1[$ such that

$$f(x)-H(x)=\frac{(x-1)x(x-\mu_1)^2\dots(x-\mu_{n-1})^2}{(2n)!}f^{(2n)}(\eta).$$

Therefore, for $x \in [0,1]$,

$$f(x) \leq H(x).$$

Since H is of degree $2n-1$, applying Theorem 4, we get that

$$\begin{aligned}\int_0^1 f(x)dx \leq \int_0^1 H(x)dx &= \beta_0 H(0) + \sum_{k=1}^{n-1} \beta_k H(\mu_k) + \beta_n H(1) \\ &= \beta_0 f(0) + \sum_{k=1}^{n-1} \beta_k f(\mu_k) + \beta_n f(1).\end{aligned}$$

From this point, an analogous argument as in the previous proof gives the statement of the theorem, for arbitrary interval $[a,b]$ without differentiability assumptions on the function f. □

5. Applications: 2-, 3-, 4-, 5-, 6-, 8-, 10- and 12-monotone functions

In the subsequent corollaries we state Hadamard-type inequalities in those cases when the roots of the polynomials in Theorem 6 and Theorem 7 can explicitly be computed.

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is a 2-monotone (i.e. convex) function, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Corollary 2. *If $f : [a,b] \to \mathbb{R}$ is a 3-monotone function, then the following inequalities hold:*

$$\frac{1}{4}f(a) + \frac{3}{4}f\left(\frac{a+2b}{3}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{3}{4}f\left(\frac{2a+b}{3}\right) + \frac{1}{4}f(b).$$

Corollary 3. *If $f : [a,b] \to \mathbb{R}$ is a 4-monotone function, then the following inequalities hold:*

$$\frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}a+\frac{3-\sqrt{3}}{6}b\right)+\frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}a+\frac{3+\sqrt{3}}{6}b\right)$$
$$\leq \frac{1}{b-a}\int\limits_a^b f(x)dx \leq \frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b).$$

Corollary 4. *If $f : [a,b] \to \mathbb{R}$ is a 5-monotone function, then the following inequalities hold:*

$$\frac{1}{9}f(a)+\frac{16+\sqrt{6}}{36}f\left(\frac{4+\sqrt{6}}{10}a+\frac{6-\sqrt{6}}{10}b\right)$$
$$+\frac{16-\sqrt{6}}{36}f\left(\frac{4-\sqrt{6}}{10}a+\frac{6+\sqrt{6}}{10}b\right) \leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \frac{16-\sqrt{6}}{36}f\left(\frac{6+\sqrt{6}}{10}a+\frac{4-\sqrt{6}}{10}b\right)$$
$$+\frac{16+\sqrt{6}}{36}f\left(\frac{6-\sqrt{6}}{10}a+\frac{4+\sqrt{6}}{10}b\right)+\frac{1}{9}f(b).$$

Corollary 5. *If $f : [a,b] \to \mathbb{R}$ is a 6-monotone function, then the following inequalities hold:*

$$\frac{5}{18}f\left(\frac{5+\sqrt{15}}{10}a+\frac{5-\sqrt{15}}{10}b\right)+\frac{4}{9}f\left(\frac{a+b}{2}\right)$$
$$+\frac{5}{18}f\left(\frac{5-\sqrt{15}}{10}a+\frac{5+\sqrt{15}}{10}b\right) \leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \frac{1}{12}f(a)+\frac{5}{12}f\left(\frac{5+\sqrt{5}}{10}a+\frac{5-\sqrt{5}}{10}b\right)$$
$$+\frac{5}{12}f\left(\frac{5-\sqrt{5}}{10}a+\frac{5+\sqrt{5}}{10}b\right)+\frac{1}{12}f(b).$$

In the other cases analogous statements can be formulated by applying Theorem 7. For simplicity, instead of writing down these corollaries explicitly, we shall present a list which contains the roots of p_n (denoted by λ_k), and the coefficients α_k for the left hand side inequality, furthermore the roots of q_n (denoted by μ_k), and the coefficients β_k for the right hand side inequality, respectively.

Case $m = 8$**.** The roots of p_4:

$$\frac{1}{2} - \frac{\sqrt{525+70\sqrt{30}}}{70}, \quad \frac{1}{2} - \frac{\sqrt{525-70\sqrt{30}}}{70},$$
$$\frac{1}{2} + \frac{\sqrt{525-70\sqrt{30}}}{70}, \quad \frac{1}{2} + \frac{\sqrt{525+70\sqrt{30}}}{70};$$

the corresponding coefficients:

$$\frac{1}{4} - \frac{\sqrt{30}}{72}, \quad \frac{1}{4} + \frac{\sqrt{30}}{72}, \quad \frac{1}{4} + \frac{\sqrt{30}}{72}, \quad \frac{1}{4} - \frac{\sqrt{30}}{72}.$$

The roots of q_4:

$$\frac{1}{2} - \frac{\sqrt{21}}{14}, \quad \frac{1}{2}, \quad \frac{1}{2} + \frac{\sqrt{21}}{14};$$

the corresponding coefficients:

$$\frac{1}{20}, \quad \frac{49}{180}, \quad \frac{16}{45}, \quad \frac{49}{180}, \quad \frac{1}{20}.$$

Case $m = 10$**.** The roots of p_5:

$$\frac{1}{2} - \frac{\sqrt{245+14\sqrt{70}}}{42}, \quad \frac{1}{2} - \frac{\sqrt{245-14\sqrt{70}}}{42},$$
$$\frac{1}{2}, \quad \frac{1}{2} + \frac{\sqrt{245-14\sqrt{70}}}{42}, \quad \frac{1}{2} + \frac{\sqrt{245+14\sqrt{70}}}{42};$$

the corresponding coefficients:

$$\frac{322-13\sqrt{70}}{1800}, \quad \frac{322+13\sqrt{70}}{1800}, \quad \frac{64}{225}, \quad \frac{322+13\sqrt{70}}{1800}, \quad \frac{322-13\sqrt{70}}{1800}.$$

The roots of q_5:

$$\frac{1}{2}-\frac{\sqrt{147+42\sqrt{7}}}{42},\quad \frac{1}{2}-\frac{\sqrt{147-42\sqrt{7}}}{42},$$
$$\frac{1}{2}+\frac{\sqrt{147-42\sqrt{7}}}{42},\quad \frac{1}{2}+\frac{\sqrt{147+42\sqrt{7}}}{42};$$

the corresponding coefficients:

$$\frac{1}{30},\quad \frac{14-\sqrt{7}}{60},\quad \frac{14+\sqrt{7}}{60},\quad \frac{14+\sqrt{7}}{60},\quad \frac{14-\sqrt{7}}{60},\quad \frac{1}{30}.$$

Case $m = 12$ (right hand side inequality). The roots of q_6:

$$\frac{1}{2}-\frac{\sqrt{495+66\sqrt{15}}}{66},\quad \frac{1}{2}-\frac{\sqrt{495-66\sqrt{15}}}{66},$$
$$\frac{1}{2},\quad \frac{1}{2}+\frac{\sqrt{495-66\sqrt{15}}}{66},\quad \frac{1}{2}+\frac{\sqrt{495+66\sqrt{15}}}{66};$$

the corresponding coefficients:

$$\frac{1}{42},\quad \frac{124-7\sqrt{15}}{700},\quad \frac{124+7\sqrt{15}}{700},\quad \frac{128}{525},$$
$$\frac{124+7\sqrt{15}}{700},\quad \frac{124-7\sqrt{15}}{700},\quad \frac{1}{42}.$$

During the investigations of the higher-order cases, we were able to use the symmetry of the roots of the orthogonal polynomials with respect to 1/2, and therefore the calculations lead to solving at most quadratic equations. The first case where "casus irreducibilis" appears, is the 7-monotone case; similarly, this is the reason for presenting only the right hand side inequality when the function was supposed to be 12-monotone.

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