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On zeros of reciprocal polynomials

By PIROSKA LAKATOS (Debrecen)

Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that the "coefficient condition"

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|$$

is satisfied.

Moreover, if the "coefficient condition" holds, then all zeros e^{iu_j}, $(j = 1, 2, \dots, m)$ can be arranged such that

$$\left| e^{i\frac{2\pi j}{m+1}} - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero, and all zeros of P_{2n+1} are single.
If $m = 2n$ is even, if the "coefficient condition" holds with equality and if

$$\operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1}\frac{A_n - A_{2n}}{2} \quad (k = 1, 2, \dots, n-1),$$

then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_{2n}. Otherwise all zeros of P_{2n} are single.

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1. Introduction

The Coxeter transformation was introduced in the representation theory of finite dimensional algebras (see [2]). The characteristic polynomial of the Coxeter transformation of an oriented graph whose underlying graph is a wild star is a Salem polynomial (see [3], [4]).

Allowing circles in the underlying graph, the spectral properties of the Coxeter transformations get much more complicated. These properties are related to polynomials of the form

$$l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \geq 2$, $1 \leq k \leq \left[\frac{m}{2}\right]$ and l is a fixed real number.

The zeros of the first expression $l(z^m + z^{m-1} + \cdots + z + 1)$ are

$$\epsilon_j = e^{i \frac{j}{m+1} 2\pi} \quad (j = 1, 2, \ldots, m)$$

the $(m+1)$st roots of unity except 1, *they are on the unit circle*. It is surprising that adding $z^k + z^{m-k}$ to the first expression the polynomial obtained inherits this property. Moreover, *not just all zeros remain on the unit circle but they move away from* ϵ_j *just a little* even if we add a linear combination $\sum_{k=1}^{[\frac{m}{2}]} a_k(z^k + z^{m-k})$ to the expression $l(z^m + z^{m-1} + \cdots + z + 1)$, *provided that* $|l|$ *is large enough*. This leads to the main result of the paper: giving a sufficient condition for reciprocal polynomials to have all of their zeros on the unit circle and also giving the location of the zeros.

Our basic tool is a transformation of semi-reciprocal polynomials called the Chebyshev transformation. Although this transformation seems to be well known we could not find a suitable reference. In Section 2, based on [1], we summarize the properties of the Chebyshev transformation. In Section 3 we formulate our results and prove them. In Section 4 we discuss the necessity of our sufficient condition.

2. The Chebyshev transformation

A polynomial p of the form $p(z) = \sum_{j=0}^{m} a_j z^j$ $(z \in \mathbb{C})$ where $a_j \in \mathbb{C}$ are given numbers with $a_m \neq 0$, $a_j = a_{m-j}$ $\left(j = 0, \ldots, \left[\frac{m}{2}\right]\right)$ is called a *reciprocal polynomial of degree m*.

We need a more general class of reciprocal polynomials (of even degree).

Definition 1. A polynomial p of the form

$$p(z) = \sum_{j=0}^{2n} a_j z^j \quad (z \in \mathbb{C}) \tag{1}$$

where $n \in \mathbb{N}, a_0, \dots, a_{2n} \in \mathbb{R}$ and

$$a_j = a_{2n-j} \ (j = 0, \dots, n-1) \tag{2}$$

is called a *real semi-reciprocal polynomial* of degree at most $2n$. If $a_{2n} \neq 0$ we call p a *real reciprocal polynomial* of degree $2n$.

Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n}$, $p \neq o$ ($o =$ the zero polynomial), then there is an integer k, $0 \leq k \leq n$, such that

$$\begin{gathered} a_{2n} = a_{2n-1} = \cdots = a_{n+k+1} = 0 = a_{n-k-1} = \cdots = a_0 \\ \text{but} \quad a_{n+k} = a_{n-k} \neq 0. \end{gathered} \tag{3}$$

Hence

$$p(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[a_{n+k} \left(z^k + \frac{1}{z^k} \right) + \cdots + a_{n+1} \left(z + \frac{1}{z} \right) + a_n \right]. \tag{4}$$

Let T_j be the jth Chebyshev polynomial of the first kind, defined by

$$T_j(\cos x) = \cos jx \quad (j = 0, 1, \dots).$$

With $z + \frac{1}{z} = x$ we have $z^j + \frac{1}{z^j} = C_j(x)$ $(j = 1, 2, \dots)$ (see e.g. [6], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x \in \mathbb{C},\ j = 1, 2, \dots)$$

are the normalized Chebyshev polynomials of the first kind. For us it will be now more convenient to define C_0 by

$$C_0(x) := T_0(x) \quad (x \in \mathbb{C}).$$

Hence, from (4)

$$p(z) = z^n \sum_{j=0}^{k} a_{n+j} C_j(x) = a_{n+k}\, z^n \prod_{j=1}^{k} (x - \alpha_j) \tag{5}$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \dots, k)$ are the zeros of the polynomial $\sum_{j=0}^{k} a_{n+j} C_j(x)$. Equation (5) remains true in the case when $k = 0$, i.e. $p(z) = a_n z^n$ if we agree that

$$\prod_{j=1}^{0} b_j := 1. \tag{6}$$

Going back to the variable z we get that

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} z\left(z + \frac{1}{z} - \alpha_j\right) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1).$$

With this we have justified

Proposition 1. *Every non-zero polynomial $p \in \mathcal{R}_{2n}$ has the decomposition*

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1) \tag{7}$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $a_{n+k} \neq 0$ for some k with $0 \leq k \leq n$ and the convention (6) *is adopted. If $p \in \mathcal{R}_{2n}$ is a reciprocal polynomial of degree $2n$, then* (7) *holds with $k = n$.*

Definition 2. The Chebyshev transform of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the decomposition (7) is defined by

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j) \tag{8}$$

(with (6) adopted) while for the zero polynomial $p = o$ let

$$\mathcal{T}o(x) = 0. \tag{9}$$

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\leq n$ with real coefficients.

Proposition 2. *The Chebyshev transform $\mathcal{T}$ is an isomorphism of the (real) vector space $\mathcal{R}_{2n}$ onto $\mathcal{P}_n$.*

Proof. (i) *$\mathcal{T}$ preserves the addition and the multiplication by a real constant.* Using (5) and (3) (to include also the zero coefficients into the sum) we can write $\mathcal{T}p$ into the form

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j) = \sum_{j=0}^{k} a_{n+j} C_j(x) = \sum_{j=0}^{n} a_{n+j} C_j(x)$$

and the last form of $\mathcal{T}p$ is valid also for the zero polynomial. Taking now another $q \in \mathcal{R}_{2n}$ with $q(z) = \sum_{j=0}^{2n} b_j z^j$ ($b_j = b_{2n-j}$ for $j = 0, \ldots, n-1$) and constants $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha p + \beta q)(z) = \sum_{j=0}^{2n} (\alpha a_j + \beta b_j) z^j$$

thus

$$\begin{aligned}\mathcal{T}(\alpha p + \beta q)(x) &= \sum_{j=0}^{n} (\alpha a_{n+j} + \beta b_{n+j}) C_j(x) \\ &= \alpha \sum_{j=0}^{n} a_{n+j} C_j(x) + \beta \sum_{j=0}^{n} b_{n+j} C_j(x) = \alpha \left(\mathcal{T}p(x)\right) + \beta \left(\mathcal{T}q(x)\right).\end{aligned}$$

(ii) *$\mathcal{T}$ maps onto $\mathcal{P}_n$.* Every polynomial $\tilde{r} \in \mathcal{P}_n$ can uniquely be written as a (real) linear combination of $C_0, C_1, \ldots, C_n$ in the form $\tilde{r}(x) = \sum_{j=0}^{n} A_{n+j} C_j(x)$ ($A_{n+j} \in \mathbb{R}$). With $r(z) := \sum_{j=0}^{2n} A_j z^j$ where $A_j := A_{2n-j}$ for $j = 0, \ldots, n-1$ we have $r \in \mathcal{R}_{2n}$ and $\mathcal{T}r = \tilde{r}$ proving our claim.

(iii) *$\mathcal{T}$ is one-to-one.* Namely, if $\mathcal{T}p = \mathcal{T}q$ for $p, q \in \mathcal{R}_{2n}$, then $\mathcal{T}p - \mathcal{T}q = \mathcal{T}(p - q) = o$ hence, by (8), (9) $p - q = o$, $p = q$. □

Lemma 1. (i) *Let p be a real reciprocal polynomial of degree $2n$. Then all zeros of p are on the unit circle if and only if all zeros of its Chebyshev transform $\mathcal{T}p$ are in the closed interval $[-2, 2]$.*

(ii) *Moreover, if all zeros α_j of $\mathcal{T}p$ are in $[-2, 2]$, written as $\alpha_j = 2\cos u_j$ with $u_j \in [0, \pi]$ ($j = 1, 2, \ldots, n$), then all zeros of p are given by*

$$e^{\pm i u_j} \quad (j = 1, 2, \ldots, n).$$

The multiplicity of $\alpha_j \neq \pm 2$ is the same as the multiplicities of e^{iu_j} and e^{-iu_j} $(j = 1, 2, \dots, n)$ while in the case of $\alpha_j = \pm 2$ the multiplicities of the corresponding zeros $e^{iu_j} = \pm 1$ of p are doubled.

Proof. (i) *Necessity.* Suppose that all zeros of p are on the unit circle. They can be arranged in conjugate pairs $(\beta_1, \bar{\beta}_1), (\beta_2, \bar{\beta}_2) \dots (\beta_n, \bar{\beta}_n)$. By assumption $|\beta_j|^2 = \beta_j \bar{\beta}_j = 1$, $\bar{\beta}_j = \frac{1}{\beta_j}$ $(j = 1, \dots n)$, hence

$$p(z) = a_{2n} \prod_{j=1}^{n} (z - \beta_j)(z - \bar{\beta}_j) = a_{2n} \prod_{j=1}^{n} \left(z^2 - (\beta_j + \bar{\beta}_j) z + 1 \right)$$

and

$$\mathcal{T} p(x) = a_{2n} \prod_{j=1}^{n} \left(x - (\beta_j + \bar{\beta}_j) \right).$$

It is clear that $|\beta_j + \bar{\beta}_j| = |2\mathrm{Re}\,(\beta_j)| \leq 2|\beta_j| = 2$.

(i) *Sufficiency.* Assume that the Chebyshev transform has the form

$$\mathcal{T} p(x) = a_{2n} \prod_{j=1}^{n} (x - \alpha_j)$$

where $a_{2n} \neq 0$ and $\alpha_j \in [-2, 2]$ $(j = 1, \dots, n)$. Then

$$p(z) = a_{2n} \prod_{j=1}^{n} \left(z^2 - \alpha_j z + 1 \right).$$

Since $\alpha_j \in [-2, 2]$ we have $z^2 - \alpha_j z + 1 = (z - \beta_j)(z - \bar{\beta}_j)$ with $\beta_j \bar{\beta}_j = 1 = |\beta_j|^2$ proving that all zeros $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \dots \beta_n, \bar{\beta}_n$ of p are on the unit circle.

(ii) We have $\alpha_j = 2 \cos u_j = \beta_j + \bar{\beta}_j$. Writing β_j as e^{iq_j} (here we may suppose that $0 \leq q_j \leq \pi$) we obtain that $2 \cos u_j = e^{iq_j} + e^{-iq_j} = 2 \cos q_j$ hence $u_j = q_j$ $(j = 1, 2, \dots, n)$. The statement concerning the multiplicities is obvious. $\square$

3. Results and proofs

Theorem 1. *All zeros of the (real reciprocal) polynomial*

$$(10)\quad h_m(z) = l(z^m + z^{m-1} + \dots + z + 1) + \sum_{k=1}^{[\frac{m}{2}]} a_k \left(z^{m-k} + z^k \right) \quad (z \in \mathbb{C})$$

of degree m where $l, a_1, \dots, a_{[\frac{m}{2}]} \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \geq 2$, are on the unit circle if

$$|l| \geq 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k|. \tag{11}$$

Moreover, if (11) is satisfied, then for even $m = 2n$ all zeros of h_m can be given as

$$e^{iu_j}, \quad e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n-1)$$
$$\frac{n - \frac{1}{2}}{m+1} 2\pi < u_n \leq \pi.$$

In the last inequality $u_n \leq \pi$, we have equality if and only if

$$|l| = 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k| \quad \text{and} \quad \operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k \quad \text{for all } k = 1, 2, \dots, n. \tag{12}$$

If (12) holds, then $-1 = e^{i\pi} = e^{-i\pi}$ is a double zero of h_m and all other zeros are single.

For odd $m = 2n + 1$ all zeros of h_m are single, they can be given as

$$-1, \ e^{iu_j}, \ e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n).$$

Remark 1. The statement concerning the location of the zeros of h_m can also be formulated as follows.

If (11) is satisfied, then all the zeros e^{iu_j} $(j = 1, 2, \dots, m)$ of h_m can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m)$$

where, as in the introduction, ϵ_j are the $(m+1)$st roots of unity, except 1.

Namely, for even $m = 2n$, let u_j $(j = 1, 2, \dots, n)$ be the same as in Theorem 1 and $u_{n+j} := 2\pi - u_{n+1-j}$ $(j = 1, 2, \dots, n)$. If (12) does not hold, then all zeros of h_m are single. If (12) holds, then $u_n = u_{n+1} = \pi$ and $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero and all other zeros are single.

For odd $m = 2n + 1$ let u_j $(j = 1, 2, \dots, n)$ be the same as in Theorem 1, $u_{n+1} := \pi$ and $u_{n+1+j} := 2\pi - u_{n+1-j}$ $(j = 1, 2, \dots, n)$. The number $-1 = e^{iu_{n+1}}$ is always a zero and all zeros are single.

Proof. The basic idea of our proof is the following. Assume that (11) holds and let

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}2\pi \quad (j = 0, \dots, \left[\tfrac{m}{2}\right]).$$

If $m = 2n$ is an even number, we show that $\operatorname{sgn}\mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, 1, \dots, n-1)$ and $\mathcal{T}h_{2n}(x_n) = 0$ if (12) holds, otherwise $\operatorname{sgn}\mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, \dots, n)$.

If $m = 2n + 1$ is odd, then $h_{2n+1}(z) = (z + 1)\bar{h}_{2n}(z)$ with a suitable reciprocal polynomial $\bar{h}_{2n}$ from $\mathcal{R}_{2n}$. We show that $\operatorname{sgn}\mathcal{T}\bar{h}_{2n}(x_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^j$ $(j = 0, 1, \dots, n)$.

Applying Lemma 1 completes the proof.

Case 1: $m = 2n$. With the notation $v_j(z) = z^j + z^{j-1} + \cdots + 1 = \frac{z^{j+1}-1}{z-1}$, $e_j(z) = z^j$, $w_j(z) = z^j + 1$ $(j = 0, 1, \dots)$ we have

$$h_{2n}(z) = lv_{2n}(z) + \sum_{k=1}^{n} a_k e_k(z) \cdot w_{2n-2k}(z),$$

$$\mathcal{T}h_{2n}(x) = l\mathcal{T}v_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot w_{2n-2k})(x).$$

The zeros of v_{2n} are the $(2n + 1)$st roots of unity, except 1: $e^{\frac{2j\pi i}{2n+1}}$ $(j = 1, 2, \dots, 2n)$. They can be arranged into conjugate pairs: $\left(e^{\frac{2j\pi i}{2n+1}}, e^{\frac{2(2n+1-j)\pi i}{2n+1}}\right) = \left(e^{\frac{2j\pi i}{2n+1}}, e^{-\frac{2j\pi i}{2n+1}}\right)$ $(j = 1, \dots, n)$, thus

$$v_{2n}(z) = \prod_{j=1}^{2n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right) = \prod_{j=1}^{n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right)\left(z - e^{-\frac{2j\pi i}{2n+1}}\right)$$

$$= \prod_{j=1}^{n} \left(z^2 - 2\cos\frac{2j\pi}{2n+1} z + 1 \right),$$

$$\mathcal{T} v_{2n}(x) = \prod_{j=1}^{n} \left(x - 2\cos\frac{2j\pi}{2n+1} \right).$$

Similarly, for each $0 \le k \le n$ the zeros of w_{2n-2k} are the $(2n-2k)$ st roots of $-1 : e^{\frac{(2j-1)\pi i}{2n-2k}}$ $(j = 1, \dots, 2n-2k)$. They can be arranged into conjugate pairs

$$\left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{\frac{(2(2n-2k+1-j)-1)\pi i}{2n-2k}} \right) = \left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{-\frac{(2j-1)\pi i}{2n-2k}} \right) \quad (j = 1, \dots, n-k).$$

Therefore

$$w_{2n-2k}(z) = \prod_{j=1}^{2n-2k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k}} \right) = \prod_{j=1}^{n-k} \left(z^2 - 2\cos\frac{(2j-1)\pi}{2n-2k} z + 1 \right),$$

$$\mathcal{T}(e_k w_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2\cos\frac{(2j-1)\pi}{2n-2k} \right).$$

Denote by U_n the nth Chebyshev polynomial of the second kind (see for example in [6]), defined by

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \quad (n = 0, 1, \dots).$$

We claim that

$$\mathcal{T} v_{2n}(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right), \tag{13}$$

$$\mathcal{T}(e_k \cdot w_{2n-2k})(x) = 2T_{n-k}\left(\frac{x}{2}\right). \tag{14}$$

To justify the first identity we note that

$$\begin{aligned} U_n(\cos y) + U_{n-1}(\cos y) &= \frac{\sin(n+1)y + \sin ny}{\sin y} \\ &= 2\frac{\sin\frac{(2n+1)y}{2}\cos\frac{y}{2}}{\sin y} = \frac{\sin\frac{(2n+1)y}{2}}{\sin\frac{y}{2}}. \end{aligned} \tag{15}$$

The right hand side is zero if and only if $y = \frac{2j\pi}{2n+1}$ ($j \in \mathbb{Z} \setminus \{0\}$) hence all zeros of $U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{2j\pi}{2n+1}$ ($j = 1, \dots, n$). Since both sides of (13) are monics which have the same zeros, they are identical.

The zeros of T_p can be calculated easily from their definition, for $p \in \mathbb{N}$ they are

$$\cos\frac{(2j-1)\pi}{2p} \quad (j = 1, \dots, p).$$

Thus for $k < n$ the zeros of the monic $2T_{n-k}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k}$ ($j = 1, \dots, n-k$). They are the same as the zeros of $\mathcal{T}(e_k \cdot w_{2n-2k})$, hence (14) holds. It also holds for $k = n$ since then both sides of (14) are equal to 2.

Next we evaluate $\mathcal{T}h_{2n}$ at the points

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}\,2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $x_j = 2\cos y_j$ with $y_j = \frac{j+\frac{1}{2}}{2n+1}\,2\pi$ we have by (13), (14)

$$\begin{aligned}
\mathcal{T}h_{2n}(x_j) &= l\left(U_n\left(\frac{x_j}{2}\right) + U_{n-1}\left(\frac{x_j}{2}\right)\right) + \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x_j}{2}\right) \\
&= 2\left[\frac{\frac{l}{2}\,\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right] \\
&= 2\left[\frac{\frac{l}{2}(-1)^j}{\sin\frac{y_j}{2}} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right].
\end{aligned}$$

If $j = 0, 1, \dots, n-1$, then $0 < \sin\frac{y_j}{2} < 1$, $\sum_{k=1}^{n} |a_k\cos(n-k)y_j| \le \sum_{k=1}^{n} |a_k|$ and by (11) the sign of the expression in the bracket is $(-1)^j \operatorname{sgn} l$.

If $j = n$, then $y_n = \pi$ and the expression in the bracket is

$$\frac{l}{2}(-1)^n + \sum_{k=1}^{n} a_k(-1)^{n-k} = (-1)^n\left(\frac{l}{2} + \sum_{k=1}^{n} a_k(-1)^k\right).$$

Its sign is $(-1)^n \operatorname{sgn} l$ if in (11) strict inequality holds or if in (11) we have equality and at least for one k ($1 \le k \le n$) we have $\operatorname{sgn} l = \operatorname{sgn}(-1)^k \operatorname{sgn} a_k$. If we have equality in (11) and $\operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k$ for all $k = 1, \dots, n$, then the expression in the bracket is zero.

Thus either $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, \dots, n)$ or $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, 1, \dots, n-1)$ and $\mathcal{T}h_{2n}(x_n) = 0$. In both cases $\mathcal{T}h_{2n}$ has n distinct zeros in the interval $[-2, 2]$. Writing these in the form $2\cos u_j$ with $0 \le u_1 \le u_2 \le \cdots \le u_n \le \pi$ and applying Lemma 1 we can complete the proof in the first case.

Case 2: $m = 2n+1$. We have $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with

$$\bar{h}_{2n}(z) = l\bar{v}_{2n}(z) + \sum_{k=1}^{n} a_k z^k \bar{w}_{2n-2k}(z)$$

where

$$\bar{v}_{2n}(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1 = v_n(z^2),$$

$$\bar{w}_{2n-2k}(z) = \frac{w_{2n+1-2k}(z)}{z+1} = \frac{z^{2n+1-2k}+1}{z+1}.$$

Using the factorization of v_n we get

$$\bar{v}_{2n}(z) = v_n(z^2) = \prod_{j=1}^{n}\left(z^2 - e^{\frac{2j\pi i}{n+1}}\right) = \prod_{j=1}^{n}\left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{\frac{j\pi i}{2n+1}-\pi i}\right).$$

Arranging the zeros of $\bar{v}_{2n}$ into conjugate pairs $\left(e^{\frac{j\pi i}{n+1}}, e^{-\frac{j\pi i}{n+1}}\right)$ $(j=1,\dots,n)$ we have

$$\bar{v}_{2n}(z) = \prod_{j=1}^{n}\left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{-\frac{j\pi i}{n+1}}\right) = \prod_{j=1}^{n}\left(z^2 - 2\cos\frac{2j\pi}{2n+1}z + 1\right)$$

therefore

$$\mathcal{T}\bar{v}_{2n}(x) = \prod_{j=1}^{n}\left(x - 2\cos\frac{j\pi}{n+1}\right).$$

We can easily calculate the zeros of $\bar{w}_{2n-2k}$ (we omit this elementary calculation) and obtain the factorization

$$\begin{aligned}\bar{w}_{2n-2k}(z) &= \prod_{j=1}^{n-k}\left(z - e^{\frac{(2j-1)\pi i}{2n-2k+1}}\right)\left(z - e^{-\frac{(2j-1)\pi i}{2n-2k+1}}\right)\\ &= \prod_{j=1}^{n-k}\left(z^2 - 2\cos\frac{(2j-1)\pi}{2n-2k+1}z + 1\right)\end{aligned}$$

therefore

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2\cos\frac{(2j-1)\pi}{2n-2k+1} \right).$$

Next we show that

$$\mathcal{T}\bar{v}_{2n}(x) = U_n\left(\frac{x}{2}\right), \tag{16}$$

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right). \tag{17}$$

where we have to adopt the convention

$$U_{-1}(x) = 0 \quad (x \in \mathbb{C}). \tag{18}$$

The first identity follows from the fact that the zeros of both sides are the same.

To justify the second we note that

$$\begin{aligned} U_{n-k}(\cos y) - U_{n-k-1}(\cos y) &= \frac{\sin(n-k+1)y - \sin(n-k)y}{\sin y} \\ &= \frac{2\cos\frac{(2n-2k+1)y}{2}\sin\frac{y}{2}}{\sin y} = \frac{\cos\frac{(2n-2k+1)y}{2}}{\cos\frac{y}{2}} \end{aligned}$$

for all $k = 0, \dots, n$ provided that the convention (18) is adopted.

If $k = n$, then both sides of (17) are equal to 1 thus (17) holds. For $k < n$ the right hand side of (17) is zero if and only if $y = \frac{(2j-1)\pi}{2n-2k+1}$ $(j \in \mathbb{Z})$ hence all zeros of $U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k+1}$ $(j = 1, \dots, n-k)$, they are the same as the zeros of $\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})$ proving (17).

By the linearity of the Chebyshev transform and by (16), (17) we have

$$\begin{aligned} \mathcal{T}\bar{h}_{2n}(x) &= l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) \\ &= lU_n\left(\frac{x}{2}\right) + \sum_{k=1}^{n} a_k \left[U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right) \right]. \end{aligned}$$

Next we evaluate $\mathcal{T}\bar{h}_{2n}$ at the points

$$\bar{x}_j = x_j = 2\cos\frac{j+\frac{1}{2}}{2n+2}\,2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $\bar{x}_j = 2\cos\bar{y}_j$ with $\bar{y}_j = \frac{j+\frac{1}{2}}{2n+2}\,2\pi$ we have

$$\begin{aligned}
\mathcal{T}\bar{h}_{2n}(\bar{x}_j) &= 2\left[\frac{l}{2}\frac{\sin(n+1)\bar{y}_j}{\sin\bar{y}_j} + \frac{\sum_{k=1}^{n} a_k \cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right] \\
&= 2\left[\frac{l}{2}\frac{(-1)^j}{\sin\bar{y}_j} + \sum_{k=1}^{n} a_k \frac{\cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right] \\
&= 2\,\frac{\frac{l}{2}(-1)^j + \sum_{k=1}^{n} a_k \sin\frac{\bar{y}_j}{2}\cos\frac{2n-2k+1}{2}\bar{y}_j}{\sin\bar{y}_j}.
\end{aligned}$$

Since $\bar{y}_j \in\,]0,\pi[$ we have $\sin\bar{y}_j > 0$, $0 < \sin\frac{\bar{y}_j}{2} < 1$, $|\cos\frac{2n-2k+1}{2}\bar{y}_j| \le 1$ for all $k = 1, \dots, n$ therefore the sign of the expression in the bracket is $\operatorname{sgn} l\ \operatorname{sgn}(-1)^j$. Thus $\operatorname{sgn}\left(\mathcal{T}\bar{h}_{2n}(x_j)\right) = \operatorname{sgn} l\ \operatorname{sgn}(-1)^j$ $(j = 0, 1, \dots, n)$ proving that $\mathcal{T}\bar{h}_{2n}$ has n different zeros in $[-2, 2]$. Writing these zeros in the form $2\cos u_j$ with $0 \le u_1 \le u_2 \le \dots \le u_n \le \pi$ and applying Lemma 1 the proof is completed in the second case as well. □

We can formulate Theorem 1 in a more symmetric way. This formulation explains, in a certain way, the appearance of the factor 2 in (11).

Theorem 2. *All zeros of the reciprocal polynomial*

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C}) \tag{19}$$

of degree $m \ge 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \ne 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that

$$|A_m| \ge \sum_{k=1}^{m-1} |A_k - A_m|. \tag{20}$$

If (20) holds, then all zeros e^{iu_j} $(j = 1, 2, \dots, m)$ of P_m can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If $m = 2n+1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero and all zeros of P_m are single.

If $m = 2n$ is even

$$
\text{(21)}\qquad \begin{cases} |A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}| \quad \textit{and} \\ \operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1}\dfrac{A_n - A_{2n}}{2} \\ \qquad (k = 1, 2, \dots, n-1) \end{cases}
$$

holds, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_m and all other zeros are single. Otherwise (i.e. if $m = 2n$, (21) does not hold) all zeros of P_m are single.

Proof. Comparing the coefficients of z^j in h_m and P_m we see that for even $m = 2n$

$$
A_{2n} = A_0 = l,\ A_{2n-1} = A_1 = l + a_1, \dots,\ A_{n+1} = A_{n-1} = l + a_{n-1},
$$
$$
A_n = l + 2a_n
$$

thus $l = A_{2n}$, $a_k = A_{2n-k} - A_{2n} = A_k - A_{2n}$ for $k = 1, 2, \dots, n-1$ and $2a_n = A_n - A_{2n}$. Therefore the condition (11)

$$
|l| \geq 2 \sum_{k=1}^{n} |a_k|
$$

can be written as

$$
|A_{2n}| \geq 2 \sum_{k=1}^{n-1} |A_k - A_{2n}| + |A_n - A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}|
$$

which is the same as (20).

For odd $m = 2n+1$ the comparison of the coefficients gives that

$$
A_{2n+1} = A_0 = l,\ A_{2n} = A_1 = l + a_1, \dots,\ A_{n+1} = A_n = l + a_n
$$

thus $l = A_{2n+1}$, $a_k = A_{2n+1-k} - A_{2n+1} = A_k - A_{2n+1}$ for $k = 1, 2, \dots, n$

and (11) can be written as

$$\begin{aligned}|A_{2n+1}| \geq 2\sum_{k=1}^{n}|A_k - A_{2n+1}| &= \sum_{k=1}^{n}\left(|A_k - A_{2n+1}| + |A_{2n+1-k} - A_{2n+1}|\right)\\ &= \sum_{k=1}^{2n}|A_k - A_{2n+1}|\end{aligned}$$

proving (20). The statement concerning the location of the zeros follows from Remark 1. □

4. Necessary and sufficient conditions

If the degree m of P_m is small we can easily obtain necessary and sufficient conditions for all zeros of P_m to be on the unit circle.

If $m = 2$, then $P_2(z) = A_2z^2 + A_1z + A_2 = z\left(A_2(z + \frac{1}{z}) + A_1\right)$ hence $\mathcal{T}P_2(x) = A_2x + A_1$. The only zero of $\mathcal{T}P_2$ is in $[-2, 2]$ if and only if

$$|A_2| \geq \frac{1}{2}|A_1|. \tag{22}$$

This is the criteria for P_2 to have all zeros on the unit circle.

If $m = 3$, then $P_3(z) = A_3z^3 + A_2z^2 + A_2z + A_3 = (z + 1)(A_3z^2 + (A_2 - A_3)z + A_3)$. By (22) the zeros of P_3 are on the unit circle if and only if

$$|A_3| \geq \frac{1}{2}|A_2 - A_3|. \tag{23}$$

If $m=4$, then $P_4(z)=A_4z^4+A_3z^3+A_2z^2+A_3z+A_4=z^2(A_4(z^2+\frac{1}{z^2}) + A_3(z + \frac{1}{z}) + A_2)$ hence with $x = z + \frac{1}{z}$ we get that $\mathcal{T}P_4(x) = A_4(x^2 - 2) + A_3x + A_2$. By Lemma 1 all zeros of P_4 are on the unit circle if and only if the discriminant of $\mathcal{T}P_4$ is non-negative:

$$A_3^2 - 4A_4(A_2 - 2A_4) \geq 0 \tag{24}$$

and

$$-2 \leq x_1, \quad x_2 \leq 2 \tag{25}$$

hold where $x_1 \le x_2$ are the real zeros of $\mathcal{T}P_4$. A simple calculation shows that (24) and (25) are equivalent to

$$2\sqrt{\max\{A_2A_4 - 2A_4^2, 0\}} \le |A_3| \le \min\left\{4|A_4|, |A_4| + \frac{1}{2}A_2 \operatorname{sgn} A_4\right\}. \tag{26}$$

This is the criterion for P_4 to have all of its zeros on the unit circle.

For $m = 2$ (22) holds if and only if

$$A_1 \in [-2|A_2|, 2|A_2|]$$

while (20) gives only the smaller interval

$$A_1 \in [A_2 - |A_2|, A_2 + |A_2|]\,.$$

This shows that (20) for $m = 2$ is not necessary. The situation is similar for $m = 3$.

For $m = 4$ the necessary and sufficient condition (26) is non-linear in the coefficients, while our sufficient condition (20) is linear for all $m \ge 2$. In some special cases we get necessary and sufficient conditions.

Corollary 1. *All zeros of the polynomial*

$$l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \ge 2$, $1 \le k \le \left[\frac{m}{2}\right]$ and l is a fixed positive number, are on the unit circle for all $m \ge 2$ and for all $k = 1, 2, \dots, \left[\frac{m}{2}\right]$ if and only if $l \ge 2$.

Namely, taking $m = 2$, $k = 1$ by (22) all zeros of the resulting polynomial $lz^2 + (l+2)z + l$ are on the unit circle if and only if $l \notin \left(-\frac{2}{3}, 2\right)$ therefore $l \ge 2$. On the other hand if $l \ge 2$, then by Theorem 1 all zeros of the polynomial $l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right)$ are on the unit circle.

Remark 2. A preliminary version of some parts of this paper was reported in [5].

A. Schinzel [7] generalized Theorem 2 to the case of self-inversive polynomials over $\mathbb{C}$, i.e. polynomials $P_m(z) = \sum_{k=0}^{n} A_k z^k$ for which $A_k \in \mathbb{C}$,

$A_m \neq 0$, $\epsilon \bar{A}_k = A_{m-k}$ for all $k = 0, \dots, m$ with a fixed $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$. He proved that all zeros of P_m are on the unit circle, provided that

$$|A_m| \geq \inf \sum_{k=0}^{m} |cA_k - d^{m-j} A_m|,$$

where the infimum is taken over all $c, d \in \mathbb{C}$ and $|d| = 1$.

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PIROSKA LAKATOS
INSTITUTE OF MATHEMATICS AND INFORMATICS
DEBRECEN UNIVERSITY
4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: lapi@math.klte.hu

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