Publ. Math. Debrecen
61 / 3-4 (2002), 663–669

Jensen's equation and bisymmetry

By GYULA MAKSA (Debrecen)

Abstract. In this note we give short proofs for some known results on the Jensen equation and characterizations of quasi-linear means by the bisymmetry, and of quasi-arithmetic means as given by KOLMOGOROV [6], NAGUMO [9] and DE FINETTI [5].

1. Introduction

Let $I \subset \mathbb{R}$ (the reals) be an interval of positive length, n be a fixed positive integer and $\alpha \in]0,1[$ be fixed. The Jensen equation is

$$f(\alpha u + (1-\alpha)v) = \alpha f(u) + (1-\alpha)f(v) \tag{1}$$

where $f : I^n \to \mathbb{R}$ and (1) holds for all $u, v \in I^n$. A function $B : I^n \to I$ is n-bisymmetric if $n \geq 2$ (fixed) and

$$B\big(B(x_{11},\dots,x_{1n}),\dots,B(x_{n1},\dots,x_{nn})\big) = B\big(B(x_{11},\dots,x_{n1}),\dots,B(x_{1n},\dots,x_{nn})\big) \tag{2}$$

Mathematics Subject Classification: 39B22.
Key words and phrases: Jensen's equation, bisymmetry, quasi-linear means, quasi-arithmetic means.
This research has been supported by OTKA, Grant T-030082 and by FKFP, Grant 0215/2001.

holds for all $x_{ij} \in I$ $(i,j = 1,\dots,n)$. In what follows we consider equation (2) as application of the function $B : I^n \to I$ to the matrix

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}.$$

An element $(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ is called an n-weight if $\lambda_k \in \,]0,1[$ $(k = 1,\dots,n)$ and $\sum_{k=1}^n \lambda_k = 1$. Throughout the paper, for $n \geq 2$, $\mathrm{CSIR}(I^n)$ denotes the class of all continuous functions $g : I^n \to \mathbb{R}$ that are strictly increasing in each variable and n-reflexive, that is,

$$g(x,\dots,x) = x \quad \text{for all } x \in I.$$

In this note we are interested in solutions of (1) and (2) belonging to the class $\mathrm{CSIR}(I^n)$. These solutions are known (see Münnich–Maksa–Mokken [8] and its references). Our aim is to simplify the proofs by pointing out the important role played by the Jensen equation (1) in characterizing quasi-linear means by bisymmetry and quasi-arithmetic means by the axioms of Kolmogorov [6], Nagumo [9] and de Finetti [5].

2. Solutions of the Jensen equation belonging to $\mathrm{CSIR}(I^n)$

These solutions can be obtained from the case $n = 2$ (see Aczél [1]) by induction on n as in [8] or from more general results (see e.g. Kuczma [7] for $\alpha = \frac{1}{2}$ and Daróczy–Maksa [4] for $\alpha \neq \frac{1}{2}$). Here we give an elementary proof, without induction, of the following.

Lemma. *Suppose that $f \in \mathrm{CSIR}(I^n)$ is a solution of equation* (1) *for an $\alpha \in \,]0,1[$. Then*

$$f(u) = \sum_{k=1}^n \lambda_k u_k \tag{3}$$

for all $u = (u_1,\dots,u_n) \in I^n$ and for an n-weight $(\lambda_1,\dots,\lambda_n)$.

Proof. Let $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n) \in I^n$, $a, b \in I$, $a < b$ and $k \in \{1,\dots,n\}$ be fixed. Integrate both sides of (1) on $[a,b]$ with respect to v_k.

Then we have that

$$\frac{1}{1-\alpha}\int_{\alpha u_k+(1-\alpha)a}^{\alpha u_k+(1-\alpha)b} f(\alpha u_1+(1-\alpha)v_1,\dots,\underbrace{t}_{k},\dots,\alpha u_n+(1-\alpha)v_n)\,dt$$

$$=\alpha(b-a)f(u_1,\dots,u_k,\dots,u_n)+(1-\alpha)\int_a^b f(v_1,\dots,v_k,\dots,v_n)\,dv_k,$$

whence we get the existence and continuity of the partial derivative function $\partial_k f$. Therefore f is continuously differentiable. Differentiating both sides of (1) with respect to u_k and with respect to v_k we obtain that $\partial_k f(\alpha u+(1-\alpha)v)=\partial_k f(u)$ and $\partial_k f(\alpha u+(1-\alpha)v)=\partial_k f(v)$, respectively. Thus $f'(u)=f'(v)$, that is, f' is constant. Therefore there exists $(\lambda_0,\lambda_1,\dots,\lambda_n)\in\mathbb{R}^{n+1}$ such that

$$f(u)=\sum_{k=1}^{n}\lambda_k u_k+\lambda_0,\quad (u=(u_1,\dots,u_n)\in I^n).$$

Since f is reflexive and strictly monotone increasing in each variable we have that $\lambda_0=0$ and $(\lambda_1,\dots,\lambda_n)$ is an n-weight. □

3. n-bisymmetric elements of CSIR(I^n)

The basic tool for finding all n-bisymmetric functions belonging to CSIR(I^n) is the classical theorem of J. Aczél for the case $n=2$ (see [1], [2], [3]). The extension of this result to $n>2$ in the not necessarily symmetric case was first proven in [8]. Here we present a much simpler proof than in [8].

Theorem. *A function $B\in$ CSIR(I^n) is a solution of (2), if and only if, there exist an n-weight $(\lambda_1,\dots,\lambda_n)$ and a continuous and strictly increasing function $\varphi: I\to\mathbb{R}$ such that*

$$B(x_1,\dots,x_n)=\varphi^{-1}\Bigg(\sum_{k=1}^{n}\lambda_k\varphi(x_k)\Bigg)\quad (x_1,\dots,x_n\in I).\tag{4}$$

PROOF. The "if" part is obvious. The "only if" part is true for $n=2$ (see e.g. [1]). Let $x, y, u, v \in I$ and apply B to the $n \times n$ matrix

$$\begin{pmatrix} x & y & \dots & y \\ u & v & \dots & v \\ \vdots & \vdots & & \vdots \\ u & v & \dots & v \end{pmatrix}.$$

Then, with the definition

$$M(x,y) = B(x,y,\dots,y), \quad (x,y \in I) \tag{5}$$

(2) implies that

$$M\big(M(x,y), M(u,v)\big) = M\big(M(x,u), M(y,v)\big).$$

Since $M \in \mathrm{CSIR}(I^2)$ we can apply ACZÉL's theorem (see [1], [2], [3]) to get

$$M(x,y) = \varphi^{-1}\big(\alpha\varphi(x) + (1-\alpha)\varphi(y)\big) \qquad (x,y \in I) \tag{6}$$

for a $\varphi : I \to \mathbb{R}$ strictly increasing and continuous function and for an $\alpha \in \,]0,1[$. Let now $(x_1,\dots,x_n), (y_1,\dots,y_n) \in I^n$ and apply B to the $n \times n$ matrix

$$\begin{pmatrix} x_1, & y_1 & \dots & y_1 \\ x_2 & y_2 & \dots & y_2 \\ \vdots & \vdots & & \\ x_n & y_n & \dots & y_n \end{pmatrix}.$$

Then, by (5) and (6), we obtain that

$$\begin{aligned} &B\big(\varphi^{-1}(\alpha\varphi(x_1) + (1-\alpha)\varphi(y_1)\big), \dots, \varphi^{-1}\big(\alpha\varphi(x_n) + (1-\alpha)\varphi(y_n))\big) \\ &\qquad = \varphi^{-1}\big(\alpha\varphi(B(x_1,\dots,x_n)) + (1-\alpha)\varphi(B(y_1,\dots,y_n))\big). \end{aligned} \tag{7}$$

Let $J = \varphi(I)$. Then $J \subset \mathbb{R}$ is an interval of positive length, and for all $u_1,\dots,u_n \in J$, with the substitutions $x_k = \varphi^{-1}(u_k)$ $(k = 1,\dots,n)$ and with the definition

$$f(u_1,\dots,u_n) = \varphi\big(B(\varphi^{-1}(u_1),\dots,\varphi^{-1}(u_n))\big) \tag{8}$$

equation (7) goes over into Jensen equation (1). Applying our lemma we have (3) with some n-weight $(\lambda_1,\dots,\lambda_n)$. Thus (8) and (3) imply (4). □

4. Remarks on the characterizations of quasi-linear and quasi-arithmetic means

In the sense of Kolmogorov [6], Nagumo [9], and de Finetti [5] a quasi-arithmetic mean is a sequence (B_n) of functions $B_n : I^n \to I$ $(n \geq 2)$ with the property that there exists a strictly increasing and continuous function $\varphi : I \to \mathbb{R}$ such that

$$B_n(x_1, \dots, x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k)\right) \tag{9}$$

for all positive integers $n \geq 2$ and for all $(x_1, \dots, x_n) \in I^n$. For a characterization of this sequence (B_n) they used the following system of axioms:

(a) $B_n \in \mathrm{CSIR}(I^n)$ for all $n \geq 2$;

(b) B_n is symmetric for all $n \geq 2$;

(c) $B_n(B_k(x_1, \dots, x_k), \dots, B_k(x_1, \dots, x_k), x_{k+1}, \dots, x_n) = B_n(x_1, \dots, x_n)$ for all $n \geq 2$, for all $2 \leq k \leq n$ and for all $x_1, \dots, x_n \in I$.

Aczél [3] has observed that (b) and (c) imply that B_n is n-bisymmetric for all fixed $n \geq 2$ while the converse is not true. Indeed, let $x_{ij} \in I$ and $y_i = B_n(x_{i1}, \dots, x_{in})$ $(i, j = 1, \dots, n)$. Then (b), (c) and the reflexivity imply that

$$\begin{aligned}
&B_{n^2}(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}) \\
&\quad = B_{n^2}(y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_n, \dots, y_n) \\
&\quad = B_{n^2}(y_1, y_2, \dots, y_n, y_1, y_2, \dots, y_n, \dots, y_1, y_2, \dots, y_n) \\
&\quad = B_{n^2}(B_n(y_1, \dots, y_n), B_n(y_1, \dots, y_n), \dots, B_n(y_1, \dots, y_n)) \\
&\quad = B_n(y_1, \dots, y_n) \\
&\quad = B_n(B_n(x_{11}, \dots, x_{1n}), B_n(x_{21}, \dots, x_{2n}), \dots, B_n(x_{n1}, \dots, x_{nn})).
\end{aligned}$$

This, again by the symmetry of B_{n^2}, implies (2) for B_n instead of B. On the other hand, an easy calculation shows that, (2) is satisfied by any function $B : I^n \to I$ of the form

$$B(x_1, \dots, x_n) = \varphi^{-1}\left(\sum_{k=1}^{n}\lambda_k\varphi(x_k)\right), \quad ((x_1, \dots, x_n) \in I^n) \tag{10}$$

with strictly increasing and continuous $\varphi : I \to \mathbb{R}$ and n-weight $(\lambda_1, \dots, \lambda_n)$. A function B defined by (10) is called quasi-linear mean of n variables with n-weight $(\lambda_1, \dots, \lambda_n)$ (see [3]) and it is symmetric, if $\lambda_1 = \dots = \lambda_n = \frac{1}{n}$. Therefore our theorem characterizes the quasi-linear means of n variables for fixed $n \geq 2$. Furthermore this theorem and our lemma provide the following short proof of the characterization based on the properties (a)–(c) of a quasi-arithmetic mean (B_n): It is obvious that any quasi-arithmetic mean (B_n) has the properties (a)–(c). To prove the converse suppose that (B_n) satisfies (a)–(c). Then $B_n \in \mathrm{CSIR}(I^n)$ is a symmetric solution of (2) for all fixed $n \geq 2$. Therefore, by our theorem, there exists a strictly monotone increasing and continuous function $\varphi_n : I \to \mathbb{R}$ such that

$$(11) \qquad B_n(x_1, \dots, x_n) = \varphi_n^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} \varphi_n(x_k)\right), \quad ((x_1, \dots, x_n) \in I^n).$$

Write $\varphi = \varphi_2$ and use equation (c) for $k = 2$. Then (11) implies that

$$(12) \qquad \varphi_n^{-1}\left(\frac{\varphi_n(x_1) + \varphi_n(x_2)}{2}\right) = \varphi^{-1}\left(\frac{\varphi(x_1) + \varphi(x_2)}{2}\right)$$

if $x_1, x_2 \in I$.

It is well-known (see e.g. [1]) and easy to see that (12) can be reduced to the one-dimensional Jensen equation for $f = \varphi_n \circ \varphi^{-1}$ (with $\alpha = \frac{1}{2}$) and we have that $\varphi_n(x) = a_n \varphi(x) + b_n$ for all $x \in I$ and for some $0 < a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$. Finally, this and (11) imply (9), that is, (B_n) is a quasi-arithmetic mean.

Acknowledgement. The author is grateful to Zsolt Páles for the idea by which the induction argument in the first version of the proof of the Theorem became unnecessary.

References

[1] J. Aczél, Lectures on functional equations and their applications, *Academic Press, New York – London*, 1966.

[2] J. Aczél, The notion of mean values, *Norske Vid. Selsk. Forh. Trondheim* **19** (1946), 83–86.

[3] J. Aczél, On mean values, *Bull. Amer. Math. Soc.* **54** (1948), 392–400.

[4] Z. Daróczy and Gy. Maksa, Functional equations on convex sets, *Acta Math. Hung.* **68**(3) (1995), 187–195.

[5] B. De Finetti, Sul. Concetto di media, *Giornale dell' Istituto Italiano degli Attuari* **2** (1931), 369–396.

[6] A. Kolmogorov, Sur la notion de la moyenne, *Rend. Accad. dei Lincei* (6) **12** (1930), 388–391.

[7] M. Kuczma, An introduction to the theory of functional equations and inequalities, *Państwowe Wydawnictwo Naukowe*, *Warszawa – Kraków – Katowice*, 1985.

[8] Á. Münnich, Gy. Maksa and R. J. Mokken, n-variable bisection, *J. Math. Psychol.* **44** (2000), 569–581.

[9] M. Nagumo, Über eine Klasse der Mittelwerte, *Japan J. Math.* **7** (1930), 71–79.

GYULA MAKSA
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSY OF DEBRECEN
DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: maksa@math.klte.hu

(Received April 18, 2002; revised September 12, 2002)