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A note on the Universal bicompactification of strictly completely regular bitopological spaces

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Abstract. In this paper we show that the bitopological analogue of the Stone–Čech compactification can be obtained from the Nachbin ordered compactification when restricted to strictly completely regular bitopological spaces.

1. Introduction

The analogue for ordered topological spaces of the Stone–Čech compactification was introduced by NACHBIN in [3]. We will call this compactification the Nachbin ordered compactification. For an ordered space $(X, \tau, \leq)$ we will denote the Nachbin ordered compactification by $\beta_{\leq}(X, \tau, \leq)$. In [5] the analogue of the Stone–Čech compactification was proven to exist for bitopological spaces. We will call this compactification the Universal bicompactification. Following [5], for a bitopological space (X, τ_1, τ_2) we will denote the Universal bicompactification by $\beta_2(X, \tau_1, \tau_2)$.

For bitopological spaces we refer the reader to [5] and [6] and for ordered spaces we refer the reader to [3]. Let 2Top be the category of bitopological spaces and bicontinuous functions, CR2Top the category of completely regular bitopological spaces and bicontinuous functions, CRTopOrd the category of completely regular ordered spaces and continuous order-preserving functions. For an ordered space $(X, \tau, \leq)$, there is a naturally

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associated bitopological space $(X, \tau^\sharp, \tau^\flat)$ where $\tau^\sharp$ consists of the open upper (increasing) sets and $\tau^\flat$ consists of the open lower (decreasing) sets.

We shall use $\mathbb{R}_0$ to denote the set of real numbers with the usual topology and the usual order. For a subset A of the set of real numbers, we use A_0 to denote A as an ordered subspace of $\mathbb{R}_0$, e.g. I_0 denotes the closed interval with the usual topology and the usual order. For an ordered topological space $(X, \tau, \leq)$, let $\mathcal{A}_0$ denote the collection of the decreasing zero sets of continuous real valued order preserving real valued functions and $\mathcal{B}_0$ denote the collection of the increasing zero sets of continuous real valued order reversing real valued functions. In [6] the functor I : CRTopOrd $\to$ CR2Top given by $I(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$, where $\tau_{\mathcal{A}_0}$ is the topology having $\mathcal{A}_0$ as a base for closed sets, and $\tau_{\mathcal{B}_0}$ is the topology having $\mathcal{B}_0$ as a base for closed sets, was discussed.

We now consider the functors M : CR2Top $\to$ CRTopOrd, and S : CRTopOrd $\to$ 2Top, given by $M(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2, \leq_{\tau_1})$, where $x \leq_{\tau_1} y \iff x \in \overline{\{y\}}^{\tau_1}$ and $S(X, \tau, \leq) = (X, \tau^\sharp, \tau^\flat)$, respectively. Example 6 in [1] of a completely regular ordered space which is not a strictly completely regular ordered space shows that the image of S is not necessarily the category CR2Top. In [1] an example of a completely regular ordered space which is not a strictly completely regular ordered space was given. By Proposition 1 in [1], this example shows that the functors I : CRTopOrd $\to$ CR2Top, and S : CRTopOrd $\to$ 2Top are not the same, answering a question raised in [6]. It was observed in [6] that when I and M are restricted to the categories of compact ordered spaces and compact bitopological spaces, they form an equivalence of these categories.

It was shown in [6] that the functor M maps into CRTopOrd. In [6] it was proved that $MI = 1$ and $IM \geq 1$.

The following strengthening of complete regularity in ordered spaces was introduced in [2].

Let $(X, \tau, \leq)$ be a topological ordered space. Then X is said to be a *strictly completely regular ordered space* if

(i) the order on X is semiclosed, i.e. $d(a)$ and $i(a)$ are closed, where $d(a)$ ($i(a)$) is the smallest lower (upper respectively) set containing a.

(ii) X is strongly order convex, i.e. the open upper sets and open lower sets form a subbasis for the topology.

(iii) given a closed lower (upper respectively) set A and a point $x \notin A$, there exists a continuous order-preserving function $f : (X, \tau, \leq) \to I_0$ such that $f(A) = 0$ and $f(x) = 1$, $f(A) = 1$ and $f(x) = 0$ respectively).

We shall use the definition of a compact bitopological space, as defined in [5], i.e. a bispace (X, τ_1, τ_2) is a compact bitopological space if $(X, \tau_1 \vee \tau_2)$ is a compact space. Let $\mathbb{R}_b$ be the real numbers with the upper topology and the lower topology, where the upper topology is the topology which has sets of the form $\{(-\infty, a) : a \in \mathbb{R}\}$ as a base and the lower topology has sets of the form $\{(a, \infty) : a \in \mathbb{R}\}$ as a base. We shall denote the upper topology by u and the lower topology by l. The bitopological space (I, u, l) denotes the closed unit interval $[0, 1]$ with the upper topology and the lower topology. The bitopological space $\beta_2(X, \tau_1, \tau_2)$ is characterised by the universal extension property that every bicontinuous function $f : (X, \tau_1, \tau_2) \to (I, u, l)$ has a bicontinuous extension to $\beta_2(X, \tau_1, \tau_2)$ or equivalently every bicontinuous function $f : (X, \tau_1, \tau_2) \to (K, \upsilon_1, \upsilon_2)$ has a bicontinuous extension to $\beta_2(X, \tau_1, \tau_2)$, for all compact bitopological spaces $(K, \upsilon_1, \upsilon_2)$ [5].

The following strengthening of complete regularity in bitopological spaces was given in [4].

Definition 1. A bitopological space (X, τ_1, τ_2) is a *strictly completely regular bitopological space* if the following conditions are satisfied:

(a) Every $\tau_1 \vee \tau_2$-closed set which is $\leq_{\tau_1}$-decreasing ($\leq_{\tau_1}$-increasing) is τ_1-closed (τ_2-closed).

(b) given a $\tau_1 \vee \tau_2 - closed$, $\leq_{\tau_1}$-decreasing ($\leq_{\tau_1}$-increasing) set A and a point $x \notin A$, there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \to (I, u, l)$ such that $x \in f^{\leftarrow}(\{1\})$ and $A \subseteq f^{\leftarrow}(\{0\})(x \in f^{\leftarrow}(\{0\})$ and $A \subseteq f^{\leftarrow}(\{1\}))$.

2. The Universal bicompactification of a strictly completely regular bitopological space

In [4] it was shown that the functors I and S coincide on strictly completely regular ordered spaces.

The following results were proved in [6].

(ii) $\beta_2 \circ I = I \circ \beta_{\leq}$ and $M \circ \beta_2 \circ I = \beta_{\leq}$.

(iii) $M \circ \beta_2 \neq \beta_{\leq} \circ M$ and $I \circ \beta_{\leq} \circ M \neq \beta_2$.

The above results show that the Nachbin ordered compactification can be obtained from the Universal bicompactification but the Universal bicompactification cannot in general be obtained from the Nachbin ordered compactification in a canonical way. We will show that when restricted to strictly completely regular bitopological spaces the Universal bicompactification can also be obtained from the Nachbin ordered compactification.

Theorem 1. *For a bitopological space (X, τ_1, τ_2) the following are equivalent:*

(i) (X, τ_1, τ_2) *is a strictly completely regular bitopological space*

(ii) $M \circ \beta_2(X, \tau_1, \tau_2) = \beta_\leq \circ M(X, \tau_1, \tau_2)$

(iii) $I \circ \beta_\leq \circ M(X, \tau_1, \tau_2) = \beta_2(X, \tau_1, \tau_2)$.

Proof. (i) $\implies$ (ii) Let $\beta_2(X, \tau_1, \tau_2) = (\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$. Then $M \circ \beta_2(X, \tau_1, \tau_2) = (\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1})$ and $(\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1})$ is a compact ordered space. To show that $(\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1})$ is the Nachbin ordered compactification of $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ we need to show that every continuous order-preserving function $f : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to I_0$ has a continuous order-preserving extension $\overline{f} : (\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1}) \to I_0$. Let $f : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to I_0$ be continuous order-preserving function. By Lemma 1 in [1], $f : (X, (\tau_1 \vee \tau_2)^\sharp, (\tau_1 \vee \tau_2)^\flat) \to (I, u, l)$ is bicontinuous. Since (X, τ_1, τ_2) is strictly completely regular, $(X, \tau_1, \tau_2) = (X, (\tau_1 \vee \tau_2)^\sharp, (\tau_1 \vee \tau_2)^\flat)$ by [4, Proposition 2]. Then $f : (X, \tau_1, \tau_2) \to (I, u, l)$ is bicontinuous. Since $(\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$ is the Universal bicompactification, f has a bicontinuous extension $\overline{f} : (\overline{X}, \overline{\tau}_1, \overline{\tau}_2) \to (I, u, l)$. Then $\overline{f} : (\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1}) \to I_0$ is continuous order-preserving function. Thus $(\overline{X}, \overline{\tau}_1 \vee \overline{\tau}_2, \leq_{\overline{\tau}_1})$ is the Nachbin ordered compactification of $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$. Therefore $M \circ \beta_2(X, \tau_1, \tau_2) = \beta_\leq \circ M(X, \tau_1, \tau_2)$

(ii) $\implies$ (iii) Suppose $M \circ \beta_2(X, \tau_1, \tau_2) = \beta_\leq \circ M(X, \tau_1, \tau_2)$. Then $I \circ \beta_\leq \circ M(X, \tau_1, \tau_2) = I \circ M \circ \beta_2(X, \tau_1, \tau_2) = \beta_2(X, \tau_1, \tau_2)$, since $I \circ M = 1$ on compact bitopological spaces.

(iii) $\implies$ (i) Suppose $I \circ \beta_\leq \circ M(X, \tau_1, \tau_2)$. Put $\beta_2(X, \tau_1, \tau_2) = (\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$. Then

$$\begin{aligned} I \circ \beta_\leq \circ M(X, \tau_1, \tau_2) &= I \circ \beta_\leq(X, \tau_1 \vee \tau_2, \leq_{\tau_1}) = I(\overline{X}, \overline{\tau_1 \vee \tau_2}, \overline{\leq}_{\tau_1}) \\ &= S(\overline{X}, \overline{\tau_1 \vee \tau_2}, \overline{\leq}_{\tau_1}) = (\overline{X}, \overline{\tau_1 \vee \tau_2}^\sharp, \overline{\tau_1 \vee \tau_2}^\flat). \end{aligned}$$

Therefore $(\overline{X}, \overline{\tau_1 \vee \tau_2}^\sharp, \overline{\tau_1 \vee \tau_2}^\flat) = (\overline{X}, \overline{\tau}_1, \overline{\tau}_2)$. Thus $(\overline{X}, \overline{\tau_1 \vee \tau_2}^\sharp, \overline{\tau_1 \vee \tau_2}^\flat)$ is the Universal bicompactification of (X, τ_1, τ_2). To show that (X, τ_1, τ_2) is strictly completely regular, we need only show that $(X, \tau_1, \tau_2) = (X, (\tau_1 \vee \tau_2)^\sharp, (\tau_1 \vee \tau_2)^\flat)$. The result will then follow from [4, Proposition 2]. We know that $\tau_1 \subseteq (\tau_1 \vee \tau_2)^\sharp$. Now let $U \in (\tau_1 \vee \tau_2)^\sharp$. Then $U = F \cap X$ where $F \in \overline{\tau_1 \vee \tau_2}^\sharp$. Since $\overline{\tau_1 \vee \tau_2}^\sharp = \overline{\tau}_1$ we have that $F \in \overline{\tau}_1$. Thus $U = F \cap X \in \tau_1$. Hence $\tau_1 = (\tau_1 \vee \tau_2)^\sharp$. Similarly $\tau_2 = \tau_1 \vee \tau_2^\flat$. Therefore (X, τ_1, τ_2) is a strictly completely regular bitopological space. $\square$

Example 1. Let (Q, us, ls) be the set of rational numbers with the upper Sorgenfrey topology and the lower Sorgenfrey topology (i.e. us has basic open sets of the form $(a, b]$ where $a, b \in Q$, $a < b$, and ls has basic open sets of the form $[a, b)$ where $a, b \in Q$, $a < b$). We observe that $M(Q, us, sl) = (Q$, discrete topology, discrete order$)$. This example shows that $I \circ \beta_{\leq} \circ M \neq \beta_2$. Note that the above theorem does not apply because (Q, us, ls) is not strictly completely regular.

Example 2. Consider the bitopological space $\mathbb{R}_b = (\mathbb{R}, u, l)$. We know that $\mathbb{R}_b$ is a strictly completely regular bitopological space [4]. By Theorem 1, we have that $I \circ \beta_{\leq} \circ M(\mathbb{R}_b) = \beta_2(\mathbb{R}_b)$.

Theorem 2. *Let $(X, \tau, \leq)$ be a completely regular ordered space. The following are equivalent:*

(i) $(X, \tau, \leq)$ *is a strictly completely regular ordered space.*

(ii) $S\beta_{\leq}(X, \tau, \leq) = \beta_2(X, \tau^\sharp, \tau^\flat)$.

(iii) $S(X, \tau, \leq) = I(X, \tau, \leq)$.

Proof. (i) $\implies$ (ii) Suppose $(X, \tau, \leq)$ is strictly completely regular ordered space. By [1], $(X, \tau^\sharp, \tau^\flat)$ is a completely regular bitopological space. By [2], the mapping $j : (X, \tau^\sharp, \tau^\flat) \longrightarrow \beta_2(X, \tau^\sharp, \tau^\flat)$ is a bitopological embedding where $j : (X, \tau, \leq) \longrightarrow \beta_{\leq}(X, \tau, \leq)$ is the Nachbin ordered compactification. It suffices to show that any bicontinuous map $f : (X, \tau^\sharp, \tau^\flat) \longrightarrow (I, u, l)$ can be extended to $\beta_2(X, \tau^\sharp, \tau^\flat)$. If f is bicontinuous then it is continuous and order-preserving as a map from $(X, \tau, \leq)$ to I_0 [1]. Since $j : (X, \tau, \leq) \longrightarrow \beta_{\leq}(X, \tau, \leq)$ is the Nachbin ordered compactification, there is an extension of f, $f' : \beta_{\leq}(X, \tau, \leq) \longrightarrow I_0$. Then f' is bicontinuous as a map from $\beta_2(X, \tau^\sharp, \tau^\flat)$ to (I, u, l). Therefore $\beta_2(X, \tau^\sharp, \tau^\flat)$ is the Universal bicompactification, i.e. $S\beta_{\leq}(X, \tau, \leq) = \beta_2(X, \tau^\sharp, \tau^\flat)$.

(ii) $\implies$ (i) Suppose $S\beta_{\leq}(X, \tau, \leq) = \beta_2(X, \tau^\sharp, \tau^\flat)$.
Let $j : (X, \tau, \leq) \longrightarrow \beta_{\leq}(X, \tau, \leq)$ be the Nachbin ordered compactification.

By assumption $j : (X, \tau^\sharp, \tau^\flat) \longrightarrow \beta_2(X, \tau^\sharp, \tau^\flat)$ is the Universal bicompactification and therefore $(X, \tau^\sharp, \tau^\flat)$ is a completely regular bitopological space. Therefore $(X, \tau, \leq)$ is a strictly completely regular ordered space [1].

(i) $\Longleftrightarrow$ (iii) This follows from [4, Theorem 6]. $\square$

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