*Publ. Math. Debrecen*
**62/1-2** (2003), 7–42

# The structure of symplectic groups associated with a quadratic extension of fields

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**Abstract.** Given a quadratic extension $L/K$ of fields and a regular alternating space $(V, f)$ of finite dimension over $L$, we determine the isometry group of a $K$-subspace $W$ of $V$ which does not split into the orthogonal sum of two proper $K$-subspaces, $W$ being neither an $L$-space nor a $K$-substructure.

## 1. Introduction

If we are given a field extension $L/K$ and a vector space $V$ of finite dimension over $L$, then $V$ can be viewed as a vector space over $K$ by restriction of scalars. P. Rabau deals in [10] with the classification of all $K$-subspaces of $V$, or with the determination of all $\mathbf{GL}_L(V)$-orbits of $K$-subspaces of $V$. He finds that the number of such orbits is independent of the fields and it is finite just if the degree of the extension is $\leq 3$ (of course, in case of infinite fields).

If $V$ is equipped with an $L$-valued regular alternating form $f$, then $V$ has, as well, a natural structure as a symplectic space $(V, f')$ over $K$. This gives rise to a natural embedding of the symplectic group $Sp_L(V, f)$ as a subgroup of the symplectic group $Sp_K(V, f')$ and $Sp_L(V, f)$-orbits of totally $f'$-isotropic $K$-subspaces of $V$ can be considered. D. S. Kim and P. Rabau investigated this situation in [7] and they found that the number of $Sp_L(V, f)$-orbits of totally $f'$-isotropic $K$-subspaces of $V$ is finite just if $L$

*Mathematics Subject Classification:* 14L30, 14L35.
*Key words and phrases:* symplectic groups, double coset spaces, $L$-automorphic functions.
Research supported by M.U.R.S.T.

is a quadratic extension of $K$ and, moreover, this number is independent of the fields being cosidered. Besides, P. Rabau in [11] analyzed the structure of the orbits in greater detail, in particular working out the structure of their stabilizers in $Sp_L(V, f)$.

The results obtained by Kim and Rabau extend works of Garrett [4] and Piatetski–Shapiro and Rallis [5], who had worked out some special cases for applications to the Rankin–Selberg method for explicit construction of automorphic $L$-functions.

In [2] the authors devoted their attention to classify $Sp_L(V, f)$-orbits of arbitrary $K$-subspaces of $V$ in the case where $L$ is a quadratic extension of the ground field $K$. Of course, matters can be reduced to classify orbits of $K$-subspaces $W$ which do not split into the orthogonal sum of two proper $K$-subspaces. (Krull–Remak–Schmidt Theorem). The most interessant orbits are the ones where $W$ is neither an $L$-subspace nor a $K$-substructure in $V$ (i.e. the natural homomorphism $W \otimes_K L \to V$ is injective). Up to an isometry, there exist precisely $\dim_L V - 1$ such subspaces splitting into three different classes ($K$-subspaces of first, second, or third kind). The classification is independent of the fields.

In this paper we determine the isometry group of an indecomposable $K$-subspace $W$ as above giving a Levi decomposition of it. It turns out that it is not solvable precisely if $W$ is of third kind, a Levi factor being $\mathbf{SL}_2(L)$ in the latter case, a one dimensional torus otherwise.

From the point of view of Aschbacher's Theorem, the paper can be regarded as studying the interaction of two Aschbacher classes of subgroups of the symplectic groups (subfield and subspace stabilizers), and one could envisage a programme of considering other Aschbacher classes (see [1] and [9], or the survey [8]). However, the paper treats a very natural case and produces a complete result in usable form.

## 2. Notation

Throughout this paper the following notation will be used:

- $F_+$ the additive group of a field $F$;
- $F^\times$ the multiplicative group of a field $F$;
- $L$ a quadratic extension $K(\eta)$ of a given field $K$ of characteristic $\neq 2$;

| | |
|---|---|
| $(V, f)$ | a regular alternating vector space over $L$; |
| $f^E$ | the restriction of $f$ at the $K$-subspace $E$ of $V$ (i.e., $f^E : E \times E \to L$ and $f^E(x', x'') = f(x', x'')$ for all $x', x'' \in E$); |
| $(f_1^E, f_2^E)$ | the components of $f^E$ over $K$ (i.e., $f_i^E : E \times E \to K$ and $f^E(x', x'') = f_1^E(x', x'') + \eta f_2^E(x', x'')$ for all $x', x'' \in E$); |
| $\langle v_1, \ldots, v_r \rangle_F$ | the $F$-subspace of $V$ generated by the vectors $v_1, \ldots, v_r$, where $F = K$, or $F = L$; |
| $LE$ | the $L$-subspace of $V$ generated by the $K$-subspace $E$; |
| $E^{\perp_Y}$ | the subset of vectors in $Y \subset V$ orthogonal to every vector in $E$ (i.e., $f(x, y) = 0$ for all $x \in E$, $y \in Y$); |
| $\mathrm{comp}_L E$ | the $L$-component of $E$ (i.e., the largest $L$-subspace of $V$ contained in $E$); |
| $\mathbf{Iso}(E)$ | the group of isometries of $E$ (i.e., the group of invertible $K$-linear transformations $\sigma$ of $E$ preserving $f$, which means $f(\sigma(x'), \sigma(x'')) = f(x', x'')$ for all $x', x'' \in E$); |
| $\sigma_L$ | the extension of $\sigma \in \mathbf{Iso}(E)$ to the alternating $L$-space $(LE, f^{LE})$ (i.e., $\sigma_L((a+\eta b)x) = a\sigma(x)+\eta b\sigma(x)$ for all $a, b \in K$ and $x \in E$); |
| $W$ | an indecomposable $K$-subspace of $V$ with nontrivial $L$-component (i.e., $W$ is not the direct sum of two proper subspaces and $\mathbf{0} \neq \mathrm{comp}_L W \neq W$); |
| $\mathbf{I}_m$ | the identity matrix of dimension $m$; |
| $\mathbf{B}$ | the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. |

If $E$ is a $K$-subspace of $V$, a *basis of $E$ over $L$* consists of vectors $\varepsilon_1, \ldots, \varepsilon_m$, $e_1, \ldots, e_n$, linear independent over $L$, with $\varepsilon_1, \ldots, \varepsilon_m$ generating $\mathrm{comp}_L E$, i.e.

$$E = \langle \varepsilon_1, \ldots, \varepsilon_m \rangle_L \oplus \langle e_1, \ldots, e_n \rangle_K.$$

Of course, a basis of $E$ over $L$ is a basis of $LE$, also.

If $\sigma \in \mathbf{GL}_K(E)$, a *representation of* $\sigma$ *(over* $L$*)* is a nonsingular matrix $M_\sigma$ representing $\sigma_L$ with respect to a basis of $E$ over $L$. As $\mathrm{comp}_L E$ is characteristic, $M_\sigma$ has the shape

$$\begin{pmatrix} M' & \mathbf{0} \\ M'' & M''' \end{pmatrix}$$

with $M' \in \mathbf{GL}_m(L)$, $M'' \in \mathbf{Mat}_{n\times m}(L)$ and $M''' \in \mathbf{GL}_n(K)$.

A *representation of* $f^E$ is a skew-symmetric matrix representing $f^{LE}$ with respect to a basis of $E$ over $L$.

## 3. The structure of $(W, f^W)$

$W$ has an obvious decomposition

$$W = C \oplus X,$$

where $C = \mathrm{comp}_L W$ and $X$ is a $K$-substructure. The fact that $W$ is indecomposable implies that (see [2], Propositions 4.1 and 6.9)

**Proposition 3.1.** *$C$ is a totally isotropic subspace of $L$-dimension $\leq 2$.*

Therefore, the subspace $C^{\perp_W}$, consisting of all vectors in $W$ orthogonal to each vector in $C$, has a decomposition

$$C^{\perp_W} = C \perp U$$

for a suitable $K$-substructure $U$. There are only three cases where $C$ coincides with the own orthogonal space $C^{\perp_W}$. In [2] we denoted them by $\mathbf{H_{11}}$, $\mathbf{H_{12}}$ and $\mathbf{H_{24}}$. We shall deal with these cases in the last section. For now we assume $\dim_K U > 0$. Then, (see [2], Proposition 6.8)

**Proposition 3.2.** *$U$ does not split into the direct sum of two orthogonal subspaces.*

For the pair of integers $(\dim_L C, \dim_K U)$ just three possibilities occur (see [2], Theorems 6.9 and 7.3)

**Proposition 3.3.** *Write the rank of $f^U$ as $2p-2$ for an integer $p \geq 1$. Then, just one of the following occurs*

1. $\dim_L C = 1$ *and* $\dim_K U = 2p - 1$*;*
2. $\dim_L C = 1$ *and* $\dim_K U = 2p$ *with* $p$ *even;*
3. $\dim_L C = 2$ *and* $\dim_K U = 2p$ *with* $p$ *even.*

In [2] we called $W$ of *first, second*, or *third kind* according as whether 1, 2, or 3 occurs.

In [2] we determined the three possible canonical representations for the induced form $f^W$ corresponding to the three different kinds of $W$. Here we need to give such representations in a more suitable way.

**Proposition 3.4.** *The form* $f^W$ *has a representation of the shape*

$$M_{f^W} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_m & \eta\mathbf{I}_m & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\eta\mathbf{I}_m & \mathbf{0} & \mathbf{0} & -B_1 & -B_2 \\ \mathbf{0} & \mathbf{0} & {}^{\mathrm{t}}B_1 & \mathbf{0} & A_p \\ \mathbf{0} & \mathbf{0} & {}^{\mathrm{t}}B_2 & -{}^{\mathrm{t}}A_p & \mathbf{0} \end{pmatrix},$$

*where*

$$m = \begin{cases} 2 & \textit{if } W \textit{ is of } 3^{\mathrm{rd}} \textit{ kind}; \\ 1 & \textit{otherwise}; \end{cases}$$

$$B_1 = \begin{cases} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbf{Mat}_{1\times p}(K) & \textit{if } W \textit{ is of } 1^{\mathrm{st}} \textit{ kind}; \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbf{Mat}_{1\times p}(K) & \textit{if } W \textit{ is of } 2^{\mathrm{nd}} \textit{ kind}; \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbf{Mat}_{2\times p}(K) & \textit{if } W \textit{ is of } 3^{\mathrm{rd}} \textit{ kind}; \end{cases}$$

$$B_2 = \begin{cases} \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbf{Mat}_{1\times (p-1)}(K) & \textit{if } W \textit{ is of } 1^{\mathrm{st}} \textit{ kind}; \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathbf{Mat}_{1\times p}(K) & \textit{if } W \textit{ is of } 2^{\mathrm{nd}} \textit{ kind}; \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbf{Mat}_{2\times p}(K) & \textit{if } W \textit{ is of } 3^{\mathrm{rd}} \textit{ kind}. \end{cases}$$

*More precisely,*

$$\bar{M} = \begin{pmatrix} \mathbf{0} & -B_1 & -B_2 \\ {}^{\mathrm{t}}B_1 & \mathbf{0} & A_p \\ {}^{\mathrm{t}}B_2 & -{}^{\mathrm{t}}A_p & \mathbf{0} \end{pmatrix}$$

*is a matrix of rank*

$$r = \begin{cases} 2(p+1) & \textit{if } W \textit{ is of } 3^{\mathrm{rd}} \textit{ kind}; \\ 2p & \textit{otherwise}; \end{cases}$$

*and*

$$\bar{A}_p = \begin{pmatrix} \mathbf{0} & A_p \\ -{}^{\mathrm{t}}A_p & \mathbf{0} \end{pmatrix}$$

*is a representation of $f^U$ with $A_p$ one of the following:*

a)

$$A_p = \begin{pmatrix} \eta & 0 & \dots & 0 & 0 \\ 1 & \eta & \ddots & \vdots & 0 \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \eta & 0 \\ 0 & \vdots & \ddots & 1 & \eta \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathbf{Mat}_{p\times(p-1)}(L)$$

*if $W$ is of first kind,*

b) $A_p = \mathbf{J}_p + \eta\mathbf{I}_p$, *where*

$$\mathbf{J}_p = \begin{pmatrix} \mathbf{J}_2 & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{I}_2 & \mathbf{J}_2 & \ddots & & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_2 & \mathbf{J}_2 \end{pmatrix} \in \mathbf{GL}_p(K)$$

*and*

$$\mathbf{J}_2 = \begin{pmatrix} 0 & \eta^2 \\ 1 & 0 \end{pmatrix}$$

*if $W$ is of second, or third kind.*

PROOF. The first part of the claim follows from [2] (see Proposition 5.1, Lemma 6.6 and Theorem 6.9). In [2] again (see Section 7) it is proved that $A_p$ has the shape $a)$ in case $W$ is of first kind. So, we have only to prove that $A_p$ has the shape $b)$ if $W$ is not of first kind.

Thanks to SCHARLAU's Theorem [12] and Proposition 6.8 in [2], the alternating form $f^U$ has a representation of the shape

$$\begin{pmatrix} \mathbf{0} & N_1 + \eta N_2 \\ -{}^{\mathrm{t}}N_1 - \eta\, {}^{\mathrm{t}}N_2 & \mathbf{0} \end{pmatrix},$$

with $N_1, N_2 \in \mathbf{Mat}_{p\times p}(K)$, and the $K$-vector space $K^p$ is indecomposable as a $K[N_1, N_2]$-module (i.e., $K^p$ does not split into the direct sum of two proper subspaces stable under both $N_1$ and $N_2$). As $\mathrm{rank}(f^U) = 2p-2$, we have that $\mathrm{rank}(N_1+\eta N_2) = p-1$. Looking at DIEUDONNÉ [3], we see that such a situation occurs precisely if both $N_1$ and $N_2$ are nonsingular and $K^p$ is an indecomposable $K[N_1 N_2^{-1}]$-module. In such a case, we may take $N_2 = \mathbf{I}_p$ and for $N_1$ a matrix having $(x^2-\eta^2)^{\frac{p}{2}}$ as the minimal polinomial (see the proof of Theorem 7.3 in [2]). As the characteristic of $K$ is $\neq 2$, this means that there exists a basis of $U$ with respect to which we obtain for $f^U$ the required representation. □

## 4. The group $\mathbf{Iso}(W)$

A basis $\mathcal{B}$ of $W$ over $L$ giving a representation of $f^W$ as in Proposition 3.4 consists of vectors

$$\varepsilon_1, \varepsilon_2, e'_1, e'_2, e''_1, e''_2, u'_1, \dots, u'_p, u''_1, \dots, u''_p \qquad \text{if } W \text{ is of } 3^{\text{rd}} \text{ kind},$$
$$\varepsilon_1, e'_1, e''_1, u'_1, \dots, u'_p, u''_1, \dots, u''_q \qquad \text{otherwise},$$

where

$$q = \begin{cases} p-1 & \text{if } W \text{ is of } 1^{\text{st}} \text{ kind}, \\ p & \text{otherwise}, \end{cases}$$

the vectors $\varepsilon_i$ generate $C$ and $u'_i$, $u''_j$ generate $U$. In [2] we called "symplectic" a basis such as $\mathcal{B}$.

Any representation in this section will always be referred to $\mathcal{B}$.

Manifestly, the $L$-component $C$ of $W$ is a characteristic subspace of $W$; thus, an isometry $\sigma \in \mathbf{Iso}(W)$ leaves both $C$ and $C^{\perp_W}$ stable. Consequently, $\sigma$ is represented by a matrix of the shape

$$M_\sigma = \begin{pmatrix} L_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L_1 & X_{11} & X_{12} & Y_{11} & Y_{12} \\ L_2 & X_{21} & X_{22} & Y_{21} & Y_{22} \\ L_3 & \mathbf{0} & \mathbf{0} & Z_{11} & Z_{12} \\ L_4 & \mathbf{0} & \mathbf{0} & Z_{21} & Z_{22} \end{pmatrix},$$

with $L_0 \in \mathbf{GL}_m(L)$, $L_1, L_2 \in \mathbf{Mat}_{m\times m}(L)$, $L_3 \in \mathbf{Mat}_{p\times m}(L)$, $L_4 \in \mathbf{Mat}_{q\times m}(L)$, $X_{ij} \in \mathbf{Mat}_{m\times m}(K)$, $Y_{i1} \in \mathbf{Mat}_{m\times p}(K)$, $Y_{i2} \in \mathbf{Mat}_{m\times q}(K)$ $(i, j = 1, 2)$, $Z_{11} \in \mathbf{Mat}_{p\times p}(K)$, $Z_{12}, {}^{\mathrm{t}}Z_{21} \in \mathbf{Mat}_{p\times q}(K)$, $Z_{22} \in \mathbf{Mat}_{q\times q}(K)$ and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbf{GL}_{2m}(K), \qquad Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathbf{GL}_{p+q}(K).$$

The $L$-subspace $C^{\perp_{LW}}$, of vectors in $LW$ orthogonal to each vector in $C$, splits into the direct sum

$$C^{\perp_{LW}} = C \oplus LU \oplus D = LC^{\perp_W} \oplus D,$$

where

$$D = \begin{cases} \langle e''_1 - \eta e'_1,\, e''_2 - \eta e'_2 \rangle_L & \text{if } W \text{ is of } 3^{\text{rd}} \text{ kind;} \\ \langle e''_1 - \eta e'_1 \rangle_L & \text{otherwise.} \end{cases}$$

Of course, $C^{\perp_{LW}}$ is stable under $\sigma_L$, hence $\sigma(e''_i) - \eta\sigma(e'_i) \in C^{\perp_{LW}}$ and this, in turn, says

$$X_{11} = X_{22}, \qquad X_{21} = \eta^2 X_{12}.$$

We shall write $X_1$ and $X_2$ instead of $X_{11}$ and $X_{12}$.

We have a well defined $L$-valued alternating form $\bar{f}$ on the factor space $\bar{C} = C^{\perp_{LW}}/C$ by putting

$$\bar{f}(x+C, y+C) = f(x,y) \qquad (x, y \in C^{\perp_{LW}}).$$

A representation of $\bar{f}$ is the matrix $\bar{M}$ given in Proposition 3.4, which is nonsigular just if $W$ is not of second kind, i.e. the alternating space $(\bar{C}, \bar{f})$ is regular if $W$ is not of second kind.

The matrix representing the isometry

$$\bar{\sigma}_L : x + C \mapsto \sigma(x) + C$$

of $(\bar{C}, \bar{f})$ is

$$\tilde{Z} = \begin{pmatrix} X_1 - \eta X_2 & Y_{21} - \eta Y_{11} & Y_{22} - \eta Y_{12} \\ \mathbf{0} & Z_{11} & Z_{12} \\ \mathbf{0} & Z_{21} & Z_{22} \end{pmatrix}.$$

Consequently, $\det \tilde{Z} = 1$ if $W$ is not of second kind.

As $C^{\perp_W}$ is stable under $\sigma$, the isometry $\bar{\sigma}_L$ fixes $LC^{\perp_W}/C$ and induces there an isometry represented by $Z$. Since the matrix representing the restriction of $\bar{f}$ at $LC^{\perp_W}/C$ is the matrix

$$\bar{A}_p = \begin{pmatrix} \mathbf{0} & A_p \\ -{}^{\mathrm{t}}A_p & \mathbf{0} \end{pmatrix}$$

which represents $f_U$, we conclude that $Z$ represents an isometry in $\mathbf{Iso}(U)$, also. Clearly, there is a homomorphism

$$\psi : \mathbf{Iso}(W) \to \mathbf{Iso}(U), \quad \psi : M_\sigma \mapsto Z.l$$

Now, to ask that $\sigma \in \mathbf{Iso}(W)$ is equivalent to require that the conditions

$$f(\sigma(\varepsilon_i), \sigma(e'_j)) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad f(\sigma(\varepsilon_i), \sigma(e''_j)) = \begin{cases} \eta & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{1}$$

$$f(\sigma(e'_i), \sigma(e'_j)) = 0, \quad f(\sigma(e'_i), \sigma(e''_j)) = 0, \quad f(\sigma(e''_i), \sigma(e''_j)) = 0, \tag{2}$$

$$f(\sigma(e'_i), \sigma(u'_k)) = 0, \qquad f(\sigma(e'_i), \sigma(u''_k)) = 0, \tag{3}$$

$$f(\sigma(e''_i),\sigma(u'_j)) = f(e''_i,u'_j), \qquad f(\sigma(e''_i),\sigma(u''_j)) = f(e''_i,u''_j) \tag{4}$$

hold. In terms of matrices, conditions (1) mean

$$L_0{}^{\mathrm{t}}(X_1+\eta X_2) = \mathbf{I}_m.$$

Also, equations (2) turn respectively into

$$\begin{aligned}(X_1+\eta X_2)\,{}^{\mathrm{t}}L_1 - L_1{}^{\mathrm{t}}(X_1+\eta X_2) &= Y_{11}A_p{}^{\mathrm{t}}Y_{12} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{11} \\ &\quad - X_2(B_1{}^{\mathrm{t}}Y_{11} + B_2{}^{\mathrm{t}}Y_{12}) + (Y_{11}{}^{\mathrm{t}}B_1 + Y_{12}{}^{\mathrm{t}}B_2)\,{}^{\mathrm{t}}X_2;\end{aligned} \tag{5}$$

$$\begin{aligned}(X_1+\eta X_2)\,{}^{\mathrm{t}}L_2 - \eta L_1{}^{\mathrm{t}}(X_1+\eta X_2) &= Y_{11}A_p{}^{\mathrm{t}}Y_{22} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{21} \\ &\quad - X_2(B_1{}^{\mathrm{t}}Y_{21} + B_2{}^{\mathrm{t}}Y_{22}) + (Y_{11}{}^{\mathrm{t}}B_1 + Y_{12}{}^{\mathrm{t}}B_2)^{\mathrm{t}}X_1;\end{aligned} \tag{6}$$

$$\begin{aligned}\eta((X_1+\eta X_2)\,{}^{\mathrm{t}}L_2 - L_2{}^{\mathrm{t}}(X_1+\eta X_2)) &= Y_{21}A_p{}^{\mathrm{t}}Y_{22} - Y_{22}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{21} \\ &\quad + (Y_{21}{}^{\mathrm{t}}B_1 + Y_{22}{}^{\mathrm{t}}B_2)\,{}^{\mathrm{t}}X_1 - X_1(B_1{}^{\mathrm{t}}Y_{21} + B_2{}^{\mathrm{t}}Y_{22});\end{aligned} \tag{7}$$

whereas equations (3) give

$$(X_1+\eta X_2)\,{}^{\mathrm{t}}L_3 = Y_{11}A_p{}^{\mathrm{t}}Z_{12} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Z_{11} - X_2(B_1{}^{\mathrm{t}}Z_{11} + B_2{}^{\mathrm{t}}Z_{12}); \tag{8}$$

$$(X_1+\eta X_2)\,{}^{\mathrm{t}}L_4 = Y_{11}A_p{}^{\mathrm{t}}Z_{22} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Z_{21} - X_2(B_1{}^{\mathrm{t}}Z_{21} + B_2{}^{\mathrm{t}}Z_{22}); \tag{9}$$

and equations (4) (using (8) and (9))

$$\begin{aligned}B_1 &= (X_1-\eta X_2)(B_1{}^{\mathrm{t}}Z_{11} + B_2{}^{\mathrm{t}}Z_{12}) \\ &\quad + (Y_{22}-\eta Y_{12})\,{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Z_{11} - (Y_{21}-\eta Y_{11})\,A_p{}^{\mathrm{t}}Z_{12};\end{aligned} \tag{10}$$

$$\begin{aligned}B_2 &= (X_1-\eta X_2)(B_1{}^{\mathrm{t}}Z_{21} + B_2{}^{\mathrm{t}}Z_{22}) \\ &\quad + (Y_{22}-\eta Y_{12})\,{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Z_{21} - (Y_{21}-\eta Y_{11})A_p{}^{\mathrm{t}}Z_{22}.\end{aligned} \tag{11}$$

The following proposition summarizes the above discussion:

**Proposition 4.1.** *A $K$-linear transformation $\sigma$ of $W$ is an isometry of $W$ if and only if it has a representation of the shape*

$$M_\sigma = \begin{pmatrix} {}^{\mathrm{t}}(X_1+\eta X_2)^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L_1 & X_1 & X_2 & Y_{11} & Y_{12} \\ L_2 & \eta^2 X_2 & X_1 & Y_{21} & Y_{22} \\ L_3 & \mathbf{0} & \mathbf{0} & Z_{11} & Z_{12} \\ L_4 & \mathbf{0} & \mathbf{0} & Z_{21} & Z_{22} \end{pmatrix},$$

*where*

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

*represents an isometry of $U$ and $L_1, L_2 \in \mathbf{Mat}_{m\times m}(L)$, $L_3 \in \mathbf{Mat}_{p\times m}(L)$, $L_4 \in \mathbf{Mat}_{q\times m}(L)$, $Y_{11}, Y_{21} \in \mathbf{Mat}_{m\times p}(K)$, $Y_{12}, Y_{22} \in \mathbf{Mat}_{m\times q}(K)$ satisfy equations* (5)–(11).

*Moreover, we have:*

1. *The mapping $M_\sigma \mapsto Z$ yields a group homomorphism $\psi : \mathbf{Iso}(W) \to \mathbf{Iso}(U)$;*
2. $\det(X_1 - \eta X_2)\det Z = 1$, *provided $W$ is not of second kind.*

Now, we need to deal separately with the cases where $W$ is of first, second, or third kind.

### 4.1. First kind case

Before determining the group $\mathbf{Iso}(W)$, we need to know the group $\mathbf{Iso}(U)$. A matrix

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \in \mathbf{GL}_{2p-1}(K),$$

with $Z_{11} \in \mathbf{Mat}_{p\times p}(K)$, $Z_{12}, {}^{\mathrm{t}}Z_{21} \in \mathbf{Mat}_{p\times(p-1)}(K)$,
$Z_{22} \in \mathbf{Mat}_{(p-1)\times(p-1)}(K)$, represents an isometry $\tau \in \mathbf{Iso}(U)$ (with respect to fixed basis $u'_1, \ldots, u'_p, u''_1, \ldots, u''_{p-1}$ of $U$) just if the matrices $Z_{ij}$ satisfy

$$Z_{11} A_p {}^{\mathrm{t}}Z_{12} - Z_{12} {}^{\mathrm{t}}A_p {}^{\mathrm{t}}Z_{11} = \mathbf{0},$$

$$Z_{11} A_p {}^{\mathrm{t}}Z_{22} - Z_{12} {}^{\mathrm{t}}A_p {}^{\mathrm{t}}Z_{21} = A_p, \tag{12}$$

$$Z_{21} A_p {}^{\mathrm{t}}Z_{22} - Z_{22} {}^{\mathrm{t}}A_p {}^{\mathrm{t}}Z_{21} = \mathbf{0},$$

where

$$A_p = \begin{pmatrix} \eta & 0 & \dots & 0 & 0 \\ 1 & \eta & \ddots & \vdots & 0 \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \eta & 0 \\ 0 & \vdots & \ddots & 1 & \eta \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathbf{Mat}_{p\times(p-1)}(L).$$

As for any $a \in K$, $a \neq 0$, the matrix

$$\begin{pmatrix} a\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & a^{-1}\mathbf{I}_{p-1} \end{pmatrix}$$

represents a transformation in $\mathbf{Iso}(U)$, in order to determine $Z$, we may confine our attention to the case where $\det Z = 1$. We shall prove that equations (12) imply $Z_{11} = \mathbf{I}_p$, $Z_{12} = \mathbf{0}$, $Z_{22} = \mathbf{I}_{p-1}$.

The components $f_1^U$ and $f_2^U$ of $f^U$ over $K$ are respectively represented by

$$\begin{pmatrix} \mathbf{0} & A_p^{(1)} \\ -\,{}^{\mathrm{t}}A_p^{(1)} & \mathbf{0} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{0} & A_p^{(2)} \\ -\,{}^{\mathrm{t}}A_p^{(2)} & \mathbf{0} \end{pmatrix},$$

where

$$A_p^{(1)} = \begin{pmatrix} 0 \ \dots \ 0 \\ \boxed{\mathbf{I}_{p-1}} \end{pmatrix}, \quad A_p^{(2)} = \begin{pmatrix} \boxed{\mathbf{I}_{p-1}} \\ 0 \ \dots \ 0 \end{pmatrix}.$$

As $Z \in \mathbf{GL}_{2p-1}(K)$, $\tau$ preserves both $f_1^U$ and $f_2^U$. It follows that $\tau$ stabilizes both the $K$-subspace $R'$ of vectors in $U$ orthogonal with respect to

$f_2^U$ to each vector in $U$ and the $K$-subspace $R''$ of vectors in $U$ orthogonal with respect to $f_1^U$ to each vector in $R'$. It turns out that

$$R' = \langle u'_p \rangle_K, \qquad R'' = \langle u'_1, \dots, u'_p,\, u''_1, \dots, u''_{p-2} \rangle_K.$$

This implies that the $p^{\mathrm{th}}$ row and the last column of $Z$ are null, apart from the $p^{\mathrm{th}}$ and the last entry, respectively. Hence, we have

$$Z_{11} = \begin{pmatrix} \bar{Z}_{11} & C \\ \mathbf{0} & c \end{pmatrix}, \quad Z_{12} = \begin{pmatrix} \bar{Z}_{12} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, \quad Z_{22} = \begin{pmatrix} \bar{Z}_{22} & \mathbf{0} \\ D & d \end{pmatrix},$$

with $\bar{Z}_{11} \in \mathbf{Mat}_{(p-1)\times(p-1)}(K)$, $\bar{Z}_{12} \in \mathbf{Mat}_{(p-1)\times(p-2)}(K)$,
$\bar{Z}_{22} \in \mathbf{Mat}_{(p-2)\times(p-2)}(K)$, $c, d \in K$, ${}^{\mathrm{t}}C \in K^{p-1}$, $D \in K^{p-2}$. Decompose $Z_{21}$ into blocks

$$Z_{21} = \begin{pmatrix} \bar{Z}_{21} & T_1 \\ T_2 & b \end{pmatrix}$$

with $\bar{Z}_{21} \in \mathbf{Mat}_{(p-2)\times(p-1)}(K)$, ${}^{\mathrm{t}}T_1 \in K^{p-2}$, $T_2 \in K^{p-1}$, $b \in K$. Obviously, $\bar{Z}_{21}$, $\bar{Z}_{12}$, and $\bar{Z}_{22}$ occur only for $p > 2$. Now, equations (12) give

$$\begin{aligned} &\bar{Z}_{11} A_{p-1} {}^{\mathrm{t}}\bar{Z}_{12} - \bar{Z}_{12} {}^{\mathrm{t}}A_{p-1} {}^{\mathrm{t}}\bar{Z}_{11} = \mathbf{0}, \\ &\bar{Z}_{11} A_{p-1} {}^{\mathrm{t}}\bar{Z}_{22} - \bar{Z}_{12} {}^{\mathrm{t}}A_{p-1} {}^{\mathrm{t}}\bar{Z}_{21} = A_{p-1}, \\ &\bar{Z}_{21} A_{p-1} {}^{\mathrm{t}}\bar{Z}_{22} - \bar{Z}_{22} {}^{\mathrm{t}}A_{p-1} {}^{\mathrm{t}}\bar{Z}_{21} = \mathbf{0}. \end{aligned} \tag{13}$$

Let $p = 2$. Then $Z_{12} = \mathbf{0}$ and from (12) we infer $Z_{11} = \mathbf{I}_2$ e $Z_{22} = \mathbf{I}_1$ because we are assuming $\det Z = 1$. Let $p > 2$ and suppose that equations (13) imply $\bar{Z}_{11} = \mathbf{I}_{p-1}$, $\bar{Z}_{12} = \mathbf{0}$, $\bar{Z}_{22} = \mathbf{I}_{p-2}$. Then, the second equation in (12) says that $Z_{11} = \mathbf{I}_p$, $Z_{12} = \mathbf{0}$, $Z_{22} = \mathbf{I}_{p-1}$. The inductive argument shows that

**Proposition 4.2.**

$$\mathbf{Iso}(U) \simeq \left\{ \begin{pmatrix} a\mathbf{I}_p & \mathbf{0} \\ \bar{Z} & a^{-1}\mathbf{I}_{p-1} \end{pmatrix} \in \mathbf{GL}_{(2p-1)}(K) : \bar{Z} A_p = {}^{\mathrm{t}}A_p {}^{\mathrm{t}}\bar{Z},\ a \in K^{\times} \right\}.$$

*Remark 4.3.* Notice that the condition $\bar{Z}A_p = {}^{\mathrm{t}}A_p{}^{\mathrm{t}}\bar{Z}$ forces the entries $z_{ij}$ of $\bar{Z}$ to satisfy the condition $z_{hk} = z_{rs}$ if $h + k = r + s$. Hence, $\mathbf{Iso}(U)$ is the semidirect product of $K_+^{2p-2}$ by $K^\times$. In particular, for $p = 1$ $\mathbf{Iso}(U)$ is a one-dimensional torus over $K$.

Now, go back to the isometry $\sigma$. The representation $M_\sigma$ in Proposition 4.1 reduces to

$$M_\sigma = \begin{pmatrix} (x_1+\eta x_2)^{-1} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \lambda_1 & x_1 & x_2 & Y_{11} & Y_{12} \\ \lambda_2 & \eta^2 x_2 & x_1 & Y_{21} & Y_{22} \\ L_3 & \mathbf{0} & \mathbf{0} & a\mathbf{I}_p & \mathbf{0} \\ L_4 & \mathbf{0} & \mathbf{0} & \bar{Z} & a^{-1}\mathbf{I}_{p-1} \end{pmatrix} \tag{14}$$

with $a, x_1, x_2 \in K$, $\lambda_1, \lambda_2, \in L$, ${}^{\mathrm{t}}L_3 \in L^p$, ${}^{\mathrm{t}}L_4 \in L^{p-1}$, $Y_{11}, Y_{21} \in K^p$, $Y_{12}, Y_{22} \in K^{p-1}$ are subject to the conditions given in Proposition 4.1 and $\bar{Z} \in \mathbf{Mat}_{(p-1)\times p}(K)$ satisfies $\bar{Z}A_p = {}^{\mathrm{t}}A_p{}^{\mathrm{t}}\bar{Z}$. Furthermore, Claim 2 in Proposition 4.1 guarantees that $(x_1 - \eta x_2)a = 1$, which means $x_2 = 0$ and $x_1 = a^{-1}$. This reduces the conditions for the entries in (14) to the following (notice that (5) and (7) vanish if $m = 1$):

$$\begin{gathered} \lambda_2 - \eta\lambda_1 - Y_{11}{}^{\mathrm{t}}B_1 = a(Y_{11}A_p{}^{\mathrm{t}}Y_{22} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{21}), \\ L_3 = -a^2A_p{}^{\mathrm{t}}Y_{12}, \\ L_4 = {}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{11} - a\bar{Z}A_p{}^{\mathrm{t}}Y_{12}, \\ \mathbf{0} = (Y_{22} - \eta Y_{12})\,{}^{\mathrm{t}}A_p, \\ (Y_{21} - \eta Y_{11})A_p - B_1{}^{\mathrm{t}}\bar{Z} = a(Y_{22} - \eta Y_{12})\,{}^{\mathrm{t}}A_p{}^{\mathrm{t}}\bar{Z}, \end{gathered}$$

hence $\lambda_2 - \eta\lambda_1 = Y_{11}{}^{\mathrm{t}}B_1$, $L_3 = \mathbf{0}$, $L_4 = {}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{11}$, $(Y_{21} - \eta Y_{11})A_p = B_1{}^{\mathrm{t}}\bar{Z}$, $Y_{12} = Y_{22} = \mathbf{0}$. These equations say that the representation of $\sigma$ with respect to the basis

$$\bar{\mathcal{B}} = \left\{\varepsilon; u'_1, \ldots, u'_p, e', e'', u''_1, \ldots, u''_{p-1}\right\}$$

of $W$ over $L$ has the shape

$$\begin{pmatrix} a\mathbf{I}_{p+1} & \mathbf{0} \\ X & a^{-1}\mathbf{I}_{p+1} \end{pmatrix},$$

where all the entries of $X \in \mathbf{Mat}_{p+1}(L)$ are elements in $K$, apart from the ones of the first column. As the representation of $f^W$ with respect to $\bar{\mathcal{B}}$ is

$$\begin{pmatrix} \mathbf{0} & M \\ -{}^{\mathrm{t}}M & \mathbf{0} \end{pmatrix}$$

with

$$M = \begin{pmatrix} 1 & \eta & 0 & \dots & 0 & 0 \\ 0 & 1 & \eta & \ddots & \vdots & 0 \\ \vdots & 0 & 1 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & \ddots & \eta & 0 \\ 0 & 0 & \vdots & \ddots & 1 & \eta \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{15}$$

it turns out that

$$XM = {}^{\mathrm{t}}M{}^{\mathrm{t}}X$$

is a necessary and sufficient condition in order that $\sigma \in \mathbf{Iso}(W)$. So, we have

**Theorem 4.4.** *Let $W$ be of first kind and let $\sigma \in \mathbf{GL}_K(W)$. Then, $\sigma \in \mathbf{Iso}(W)$ precisely if $\sigma$ has a representation of the shape*

$$\begin{pmatrix} a\mathbf{I}_{p+1} & \mathbf{0} \\ X & a^{-1}\mathbf{I}_{p+1} \end{pmatrix}, \tag{16}$$

*where $a \in K^\times$ and $X = (x_{ij}) \in \mathbf{Mat}_{(p+1)\times(p+1)}(L)$ is subject to the conditions*

a) *$x_{ij} \in K$ for $j > 2$;*

b) *$XM = {}^{\mathrm{t}}M{}^{\mathrm{t}}X$, where $M$ is the matrix (15).*

*Hence,* $\mathbf{Iso}(W)$ *is a solvable algebraic group of dimension* $2p+3$ *over* $K$.

### 4.2. Second kind case

Manifestly, the structure of the group $\mathbf{Iso}(U)$ is completely different if $W$ is not of first kind, because $(LU, f^{LU})$ is a regular alternating space, the matrix $A_p$ being nonsingular. Before describing the group $\mathbf{Iso}(U)$, we need to introduce the following sets of matrices:

$$\mathcal{T}' = \{X \in \mathbf{Mat}_{2\times 2}(K) : X\mathbf{J}_2 = \mathbf{J}_2 X\} = \left\{ \begin{pmatrix} a & \eta^2 b \\ b & a \end{pmatrix} : a, b \in K \right\};$$

$$\mathcal{S}' = \{X \in \mathbf{Mat}_{2\times 2}(K) : X^{\mathrm{t}}\mathbf{J}_2 = \mathbf{J}_2 X\} = \left\{ \begin{pmatrix} \eta^2 a & b \\ b & a \end{pmatrix} : a, b \in K \right\};$$

$$\mathcal{S}'' = \{X \in \mathbf{Mat}_{2\times 2}(K) : X\mathbf{J}_2 = {}^{\mathrm{t}}\mathbf{J}_2 X\} = \left\{ \begin{pmatrix} a & b \\ b & \eta^2 a \end{pmatrix} : a, b \in K \right\};$$

$$\mathcal{T}'' = \{X \in \mathbf{Mat}_{2\times 2}(K) : X^{\mathrm{t}}\mathbf{J}_2 = {}^{\mathrm{t}}\mathbf{J}_2 X\} = \left\{ \begin{pmatrix} a & b \\ \eta^2 b & a \end{pmatrix} : a, b \in K \right\}.$$

Notice that $\mathcal{T}'$, $\mathcal{S}'$, $\mathcal{S}''$, $\mathcal{T}''$ are closed under addition, $\mathcal{T}'$ and $\mathcal{T}''$ are closed under multiplication and the following occur:

$$\begin{aligned} \mathcal{T}'' &= {}^{\mathrm{t}}\mathcal{T}', \qquad & \mathcal{S}'\mathcal{S}'' &= \mathcal{T}', \\ \mathcal{T}'\mathcal{S}' &= \mathcal{S}', \qquad & \mathcal{T}''\mathcal{S}'' &= \mathcal{S}''. \end{aligned} \tag{17}$$

**Proposition 4.5.** *Let* $W$ *be not of first kind. Then, the group* $\mathbf{Iso}(U)$ *corresponds to the group of matrices* $M_\tau \in \mathbf{GL}_{2p}(K)$ *of the shape*

$$M_\tau = \begin{pmatrix} T' & S' \\ S'' & T'' \end{pmatrix}$$

*with* ${}^{\mathrm{t}}T'T'' - S''S' = \mathbf{I}_p$, *where*

$$T' = \begin{pmatrix} T'_1 & \mathbf{0} & \dots & \mathbf{0} \\ T'_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ T'_{\frac{p}{2}} & \dots & T'_2 & T'_1 \end{pmatrix}, \quad S' = \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & S'_1 \\ \vdots & \iddots & \iddots & S'_2 \\ \mathbf{0} & \iddots & \iddots & \vdots \\ S'_1 & S'_2 & \dots & S'_{\frac{p}{2}} \end{pmatrix},$$

$$S'' = \begin{pmatrix} S''_{\frac{p}{2}} & \dots & S''_2 & S''_1 \\ \vdots & \iddots & \iddots & \mathbf{0} \\ S''_2 & \iddots & \iddots & \vdots \\ S''_1 & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \quad T'' = \begin{pmatrix} T''_1 & T''_2 & \dots & T''_{\frac{p}{2}} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T''_2 \\ \mathbf{0} & \dots & \mathbf{0} & T''_1 \end{pmatrix}$$

*and* $T'_i \in \mathcal{T}'$, $S'_i \in \mathcal{S}'$, $S''_i \in \mathcal{S}''$, $T''_i \in \mathcal{T}''$ *for all* $i = 1, \dots, \frac{p}{2}$.

Proof. An isometry $\tau \in \mathbf{Iso}(U)$ must preserve both the components $f_1^U$ and $f_2^U$ of $f^U$ over $K$. This means that the matrix $M_\tau$ representing $\tau$ with respect to the fixed basis $u'_1, \dots, u'_p$, $u''_1, \dots, u''_p$ of $U$ satisfies the identities

$$M_\tau \bar{A}_p^{(1)}{}^{\mathrm{t}}M_\tau = \bar{A}_p^{(1)} \quad \text{and} \quad M_\tau \bar{A}_p^{(2)}{}^{\mathrm{t}}M_\tau = \bar{A}_p^{(2)},$$

where

$$\bar{A}_p^{(1)} = \begin{pmatrix} \mathbf{0} & \mathbf{J}_p \\ -{}^{\mathrm{t}}\mathbf{J}_p & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \bar{A}_p^{(2)} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_p \\ -\mathbf{I}_p & \mathbf{0} \end{pmatrix}.$$

Therefore, we have $M_\tau$ is a matrix of the symplectic group $\mathbf{Sp}_{2p}(K)$. In particular, $\det M_\tau = 1$. Let $p = 2$. Then,

$$M_\tau = \begin{pmatrix} T' & S' \\ S'' & T'' \end{pmatrix}$$

with ${}^{\mathrm{t}}T'T'' - S''S' = \mathbf{I}_2$ and $T' \in \mathcal{T}'$, $S' \in \mathcal{S}'$, $S'' \in \mathcal{S}''$, $T'' \in \mathcal{T}''$.

Assume now $p \geq 4$. Since the radical of the alternating $L$-space $(LU, f^{LU})$ is generated by the vectors $u'_1 - \eta u'_2$ and $u''_p - \eta u''_{p-1}$, the matrix

$M_\tau$ representing $\tau$ has the shape

$$\begin{pmatrix} T_1' & \mathbf{0} & \dots & \mathbf{0} & S_1' \\ * & * & \dots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ * & * & \dots & * & * \\ S_1'' & \mathbf{0} & \dots & \mathbf{0} & T_1'' \end{pmatrix} \tag{18}$$

with $T_1' \in \mathcal{T}'$, $S_1' \in \mathcal{S}'$, $S_1'' \in \mathcal{S}''$, $T_1'' \in \mathcal{T}''$. For all $S' \in \mathcal{S}'$ the matrix

$$M(S') = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \dots & \dots & \mathbf{0} & S' \\ \mathbf{0} & \ddots & & & \iddots & \mathbf{0} \\ \vdots & \ddots & \mathbf{I}_2 & S' & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_2 & & \vdots \\ \vdots & \iddots & \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_2 \end{pmatrix} \tag{19}$$

represents a transformation in $\mathbf{Iso}(U)$. Also, up to multiply (18) on the left by a matrix (19), we may assume that $T_1' \neq \mathbf{0}$. Then, multiplication on the right by $M(-T_1'^{-1}S_1')$ allows one to put $S_1' = \mathbf{0}$ in (18).

The isometry $\tau_L$ of $(LU, f^{LU})$ fixes both the subspace $U' = \langle u_1', u_2' \rangle_L$ and the subspace $U'^{\perp_{LU}} = \langle u_1', \dots, u_p', u_3'', \dots, u_p'' \rangle_L$ orthogonal to $U'$. Consequently,

$$x + U' \mapsto \tau_L(x) + U'$$

is a well defined linear transformation $\bar{\tau}_L : \bar{U} \to \bar{U}$ of the $(2p-4)$-dimensional $L$-space

$$\bar{U} = U'^{\perp_{LU}}/U' = \langle u_3' + U', \dots, u_p' + U', u_3'' + U', \dots, u_p'' + U' \rangle_L.$$

Furthermore, putting

$$\bar{f}(x + U', y + U') = f(x, y)$$

for all $x, y \in U'^{\perp_{LU}}$, we obtain an alternating form on $\bar{U}$ which is, of course, preserved by $\bar{\tau}_L$. Clearly, the matrix representing $\bar{f}$, with respect to the indicated basis of $\bar{U}$, is the matrix $\bar{A}_{p-2}$. As we want to proceed by induction on $p$, let us assume that the matrix representing $\bar{\tau}_L$ with respect to the above basis of $\bar{U}$ has the shape

$$M_{\bar{\tau}} = \begin{pmatrix} \bar{T}' & \bar{S}' \\ \bar{S}'' & \bar{T}'' \end{pmatrix}$$

with ${}^{\mathrm{t}}\bar{T}'\bar{T}'' - \bar{S}''\bar{S}' = \mathbf{I}_{p-2}$ and

$$\bar{T}' = \begin{pmatrix} \bar{T}'_1 & \mathbf{0} & \dots & \mathbf{0} \\ \bar{T}'_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \bar{T}'_{\frac{p}{2}-1} & \dots & \bar{T}'_2 & \bar{T}'_1 \end{pmatrix}, \quad \bar{S}' = \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & \bar{S}'_1 \\ \vdots & \iddots & \iddots & \bar{S}'_2 \\ \mathbf{0} & \iddots & \iddots & \vdots \\ \bar{S}'_1 & \bar{S}'_2 & \dots & \bar{S}'_{\frac{p}{2}-1} \end{pmatrix},$$

$$\bar{S}'' = \begin{pmatrix} \bar{S}''_{\frac{p}{2}-1} & \dots & \bar{S}''_2 & \bar{S}''_1 \\ \vdots & \iddots & \iddots & \mathbf{0} \\ \bar{S}''_2 & \iddots & \iddots & \vdots \\ \bar{S}''_1 & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \quad \bar{T}'' = \begin{pmatrix} \bar{T}''_1 & \bar{T}''_2 & \dots & \bar{T}''_{\frac{p}{2}-1} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{T}''_2 \\ \mathbf{0} & \dots & \mathbf{0} & \bar{T}''_1 \end{pmatrix},$$

where $\bar{T}'_i \in \mathcal{T}'$, $\bar{S}'_i \in \mathcal{S}'$, $\bar{S}''_i \in \mathcal{S}''$, $\bar{T}''_i \in \mathcal{T}''$ for all $i = 1, \dots, \frac{p}{2} - 1$. Thus, $M_\tau$ is written as

$$M_\tau = \begin{pmatrix} T'_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Q_{21} & \bar{T}' & \mathbf{0} & \bar{S}' \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & \bar{S}'' & \mathbf{0} & \bar{T}'' \end{pmatrix} \tag{20}$$

for suitable $Q_{31}, Q_{33} \in \mathbf{Mat}_{2\times 2}(K)$ and ${}^{\mathrm{t}}Q_{21}, Q_{32}, Q_{34}, {}^{\mathrm{t}}Q_{41} \in \mathbf{Mat}_{2\times(p-2)}(K)$. In particular, looking at (18), we see that $\bar{S}''_1 = \mathbf{0}$ and consequently $\bar{T}''_1 \neq \mathbf{0}$, which means that $\bar{T}''$ is nonsingular. Therefore, the matrix

$$\begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-2} & \mathbf{0} & -\bar{S}'\bar{T}''^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{p-2} \end{pmatrix} \tag{21}$$

represents a transformation in $\mathbf{Iso}(U)$ and this allows one to put $\bar{S}' = \mathbf{0}$ in (20). Now, if we impose the identity $M_\tau \bar{A}_p^{(2)}{}^{\mathrm{t}}M_\tau = \bar{A}_p^{(2)}$, we find for $M_\tau$ the conditions

$$\bar{T}'' = {}^{\mathrm{t}}\bar{T}'^{-1}, \qquad Q_{33} = {}^{\mathrm{t}}T_1'^{-1}, \qquad Q_{34} = -\,{}^{\mathrm{t}}T_1'^{-1}\,{}^{\mathrm{t}}Q_{21}\,{}^{\mathrm{t}}\bar{T}'^{-1}, \tag{22}$$

$$\begin{aligned} Q_{31}T_1'^{-1} + Q_{32}{}^{\mathrm{t}}Q_{34} &= {}^{\mathrm{t}}T_1'^{-1}\,{}^{\mathrm{t}}Q_{31} + Q_{34}{}^{\mathrm{t}}Q_{32}, \\ Q_{32}{}^{\mathrm{t}}\bar{T}'^{-1} &= {}^{\mathrm{t}}T_1'^{-1}\,{}^{\mathrm{t}}Q_{41} + Q_{34}\bar{S}''. \end{aligned} \tag{23}$$

Write $\bar{A}_p^{(1)}$ as

$$\bar{A}_p^{(1)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H & \mathbf{J}_{p-2} \\ -{}^{\mathrm{t}}\mathbf{J}_2 & -\,{}^{\mathrm{t}}H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\,{}^{\mathrm{t}}\mathbf{J}_{p-2} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where ${}^{\mathrm{t}}H \in \mathbf{Mat}_{2\times(p-2)}$ is the matrix

$${}^{\mathrm{t}}H = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}.$$

In view of (22), imposing the identity $M_\tau \bar{A}_p^{(1)}{}^{\mathrm{t}}M_\tau = \bar{A}_p^{(1)}$, we obtain

$$Q_{21}\mathbf{J}_2 + \bar{T}'H = \mathbf{J}_{p-2}Q_{21} + HT_1'. \tag{24}$$

Set ${}^{\mathrm{t}}Q_{21} = ({}^{\mathrm{t}}T_2' \dots {}^{\mathrm{t}}T_{\frac{p}{2}}')$ with $T_i' \in \mathbf{Mat}_{2\times 2}(K)$. Then, equation (24) gives

$$T_2'\mathbf{J}_2 + \bar{T}_1' = \mathbf{J}_2T_2' + T_1' \tag{25}$$

and

$$T'_{i+1}\mathbf{J}_2 + \bar{T}'_i = \mathbf{J}_2 T'_{i+1} + T'_i \tag{26}$$

for all $i = 2, \ldots, \frac{p}{2} - 1$. Equation (25) says that $\bar{T}'_1 = T'_1$ and $T'_2 \in \mathcal{T}'$. Then, equations (26) give $T'_i = \bar{T}'_i$ and $T'_{i+1} \in \mathcal{T}'$ for all $i = 2, \ldots, \frac{p}{2} - 1$. So, $M_\tau$ has the shape

$$M_\tau = \begin{pmatrix} T' & \mathbf{0} \\ * & * \end{pmatrix}$$

with

$$T' = \begin{pmatrix} T'_1 & \mathbf{0} & \ldots & \mathbf{0} \\ T'_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ T'_{\frac{p}{2}} & \ldots & T'_2 & T'_1 \end{pmatrix}$$

and $T'_i \in \mathcal{T}'$ for all $i = 1, \ldots, \frac{p}{2}$. Now, the matrix

$$\begin{pmatrix} T' & \mathbf{0} \\ \mathbf{0} & {}^{\mathrm{t}}T'^{-1} \end{pmatrix} \tag{27}$$

represents a transformation in $\mathbf{Iso}(U)$. Therefore, we may assume $T' = \mathbf{I}_p$ and, thanks to (22) and (23), the matrix $M_\tau$ takes the shape

$$M_\tau = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-2} & \mathbf{0} & \mathbf{0} \\ Q_{31} & Q_{32} & \mathbf{I}_2 & \mathbf{0} \\ {}^{\mathrm{t}}Q_{32} & \bar{S}'' & \mathbf{0} & \mathbf{I}_{p-2} \end{pmatrix}$$

with $Q_{31}$ symmetric. Imposing again the condition $M_\tau \bar{A}_p^{(1)} {}^{\mathrm{t}}M_\tau = \bar{A}_p^{(1)}$, we obtain

$$Q_{31}\mathbf{J}_2 + Q_{32}H = {}^{\mathrm{t}}\mathbf{J}_2 Q_{31} + {}^{\mathrm{t}}H {}^{\mathrm{t}}Q_{32}, \tag{28}$$

$${}^{\mathrm{t}}\mathbf{J}_2 Q_{32} + {}^{\mathrm{t}}H\bar{S}'' = Q_{32}\mathbf{J}_{p-2}. \tag{29}$$

Set $Q_{32} = \left(S''_{\frac{p}{2}-1} \dots S''_1\right)$ with $S''_i \in \mathbf{Mat}_{2\times 2}(K)$. Then, (29) turns into

$$\begin{aligned} {}^{\mathrm{t}}\mathbf{J}_2 S''_1 &= S''_1 \mathbf{J}_2, \\ {}^{\mathrm{t}}\mathbf{J}_2 S''_k + \bar{S}''_k &= S''_k \mathbf{J}_2 + S''_{k-1} \qquad (k = 2, \dots, \frac{p}{2} - 1). \end{aligned}$$

The first equation gives $S''_1 \in \mathcal{S}''$. Thus, in view of the second equation, we infer $\bar{S}''_2 = S''_1$ and $S''_2 \in \mathcal{S}''$. Now, an iterative argument leads to conclude that $\bar{S}''_k = S''_{k-1}$ and $S''_k \in \mathcal{S}''$ for $k = 3, \dots, \frac{p}{2} - 1$. Furthermore, $Q_{31} = S''_{\frac{p}{2}} \in \mathcal{S}''$ follows from condition (28). Summing up, $M_\tau$ has the shape

$$M_\tau = \begin{pmatrix} \mathbf{I}_p & \mathbf{0} \\ S'' & \mathbf{I}_p \end{pmatrix}, \tag{30}$$

with

$$S'' = \begin{pmatrix} S''_{\frac{p}{2}} & \dots & S''_2 & S''_1 \\ \vdots & \iddots & \iddots & \mathbf{0} \\ S''_2 & \iddots & \iddots & \vdots \\ S''_1 & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}$$

and $S''_i \in \mathcal{S}''$. Therefore, that matrices (19), (21), (27) and (30) generate the group $\mathbf{Iso}(U)$ and, in view of (17), the claim is proved. □

*Remark 4.6.* It turns out from Proposition 4.5 that the group $\mathbf{Iso}(U)$ is the semidirect product of its unipotent radical $R_u$, which has dimension $3(p-2)$ over $K$, by the special linear group $\mathbf{SL}_2(L)$. More precisely, $R_u$ is the kernel of the composed group homomorphism $\varphi : \mathbf{Iso}(U) \to \mathbf{SL}_2(L)$

$$\begin{pmatrix} T' & S' \\ S'' & T'' \end{pmatrix} \mapsto \begin{pmatrix} T'_1 & S'_1 \\ S''_1 & T''_1 \end{pmatrix} \mapsto X = \begin{pmatrix} t'_{11} + \eta t'_{12} & -s'_{12} - \eta s'_{11} \\ -s''_{12} - \eta s''_{11} & t''_{11} + \eta t''_{12} \end{pmatrix},$$

where $\det X = 1$ follows from ${}^{\mathrm{t}}T'_1 T''_1 - S'_1 S''_1 = \mathbf{I}_2$. Notice that $R_u$ has descending central series

$$R_u = \mathbf{K}_1 \triangleright \mathbf{K}_2 \triangleright \dots \triangleright \mathbf{K}_{\frac{p}{2}} = \mathbf{1}_{R_u},$$

where, for $i > 1$, $\mathbf{K}_i = [\mathbf{K}_{i-1}, R_u]$ consists of the matrices in $\mathbf{Iso}(U)$ with $T_1' = T_1'' = \mathbf{I}_2$, $S_1' = S_1'' = \mathbf{0}$, and $T_h' = T_h'' = S_h' = S_h'' = \mathbf{0}$ for all $1 < h \leq i$. In particular, the nilpotency class of $R_u$ is $\frac{p}{2} - 1$ and $\mathbf{K}_i$ is an extension of $\mathbf{K}_{i-1}$ by $K_+^6$.

In view of Proposition 4.1 and Proposition 4.5, the isometry $\sigma$ has, with respect to the fixed basis $\mathcal{B}$, the representation

$$\begin{pmatrix} (x_1+\eta x_2)^{-1} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \lambda_1 & x_1 & x_2 & Y_{11} & Y_{12} \\ \lambda_2 & \eta^2 x_2 & x_1 & Y_{21} & Y_{22} \\ L_3 & \mathbf{0} & \mathbf{0} & T' & S' \\ L_4 & \mathbf{0} & \mathbf{0} & S'' & T'' \end{pmatrix} \tag{31}$$

with $x_1, x_2 \in K$, $\lambda_1, \lambda_2 \in L$, ${}^{\mathrm{t}}L_3, {}^{\mathrm{t}}L_4 \in L^p$, $Y_{ij} \in K^p$, $T', S', S'', T'' \in \mathbf{Mat}_{p\times p}(K)$ fulfilling the required conditions.

We claim that $x_2 = 0$, as well as $S' = \mathbf{0}$. We shall prove this using induction on $p$.

The alternating space $(LW, f^{LW})$ is not regular, since it has the line $R$ generated by the vector $u_1' - \eta u_2'$ as the radical. As $R$ is characteristic, we have $S_1' = \mathbf{0}$. Consequently, $\sigma$ stabilizes both the subspace $Q = \langle u_1', u_2' \rangle_K$ and the subspace of vectors in $W$ orthogonal to $Q$, which is the $K$-subspace

$$Q^{\perp_W} = \langle \varepsilon \rangle_L \oplus \langle e', e'', u_1', \ldots, u_p', u_3'', \ldots, u_p'' \rangle_K.$$

Clearly, $(x + Q, y + Q) \mapsto f(x, y)$ yields an $L$-valued alternating form $\bar{f}$ on the factor space $\bar{Q} := Q^{\perp_W}/Q$ and $x + Q \mapsto \sigma(x) + Q$ is a $K$-linear transformation $\bar{\sigma}$ of $\bar{Q}$ preserving $\bar{f}$.

Let $p = 2$. Then, $S' = S_1' = \mathbf{0}$ and $(\bar{Q}, \bar{f})$ is a nonregular alternating space over $L$, the radical $\bar{R}$ of which is generated by the vector $e'' - \eta e'$. Consequently, $\bar{x} + \bar{R} \mapsto \bar{\sigma}(\bar{x}) + \bar{R}$ yields an isometry of a regular alternating plane and this, in turn, says that $(x_1 + \eta x_2)^{-1} x_1 = 1$, i.e. $x_2 = 0$.

Assume $p \geq 4$. Then, the matrix representing $\bar{f}$ is the matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_1 & \eta\mathbf{I}_1 & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\eta\mathbf{I}_1 & \mathbf{0} & \mathbf{0} & -B_1 & -B_2 \\ \mathbf{0} & \mathbf{0} & {}^{\mathrm{t}}B_1 & \mathbf{0} & A_{p-2} \\ \mathbf{0} & \mathbf{0} & {}^{\mathrm{t}}B_2 & -{}^{\mathrm{t}}A_{p-2} & \mathbf{0} \end{pmatrix}$$

and we can use induction in order to conclude that $x_2 = 0$ and $S' = \mathbf{0}$ in any case. We shall write $a$ instead of $x_1$.

Now, ${}^{\mathrm{t}}T'T'' - S''S' = \mathbf{I}_p$ yields $T'' = {}^{\mathrm{t}}T'^{-1}$. So, the conditions for the entries in (31) given by Proposition 4.1 reduce to the following:

$$\begin{aligned} & a(\lambda_2 - \eta\lambda_1 - Y_{12}{}^{\mathrm{t}}B_2) = Y_{11}A_p{}^{\mathrm{t}}Y_{22} - Y_{12}{}^{\mathrm{t}}A_p{}^{\mathrm{t}}Y_{21}, \\ & aL_3 = -T'A_p{}^{\mathrm{t}}Y_{12}, \\ & aL_4 = {}^{\mathrm{t}}T'^{-1\mathrm{t}}A_p{}^{\mathrm{t}}Y_{11} - S''A_p{}^{\mathrm{t}}Y_{12}, \\ & (Y_{22} - \eta Y_{12})\,{}^{\mathrm{t}}A_p = \mathbf{0}, \\ & (Y_{21} - \eta Y_{11})A_p = B_2(a\mathbf{I}_p - T'). \end{aligned} \tag{32}$$

The last two equations turn into

$$Y_{22} = Y_{12}{}^{\mathrm{t}}\mathbf{J}_p, \tag{33}$$

$$Y_{21} = Y_{11}\mathbf{J}_p, \tag{34}$$

$$Y_{12}({}^{\mathrm{t}}\mathbf{J}_p^2 - \eta^2\mathbf{I}_p) = \mathbf{0}, \tag{35}$$

$$Y_{11}(\mathbf{J}_p^2 - \eta^2\mathbf{I}_p) = B_2(a\mathbf{I}_p - T'). \tag{36}$$

In view of (34) and (33), the first equation in (32) gives

$$\lambda_2 = \eta\lambda_1 + Y_{12}{}^{\mathrm{t}}B_2. \tag{37}$$

Furthermore, if we set $Y_{11} = (y'_1 \ldots y'_p)$ and $Y_{12} = (y''_1 \ldots y''_p)$, then, (35) yields $y''_k = 0$ for $k = 1, \ldots, p-2$ and (36) states that the entries of the matrix $T'$ are functions of $y'_3, \dots, y'_p$. In particular, with the symbolism of Proposition 4.5, we have, $T'_1 = a\mathbf{I}_2$ and we can state

**Theorem 4.7.** *Let $W$ be of second kind and let $\sigma \in \mathbf{GL}_K(W)$. Then, $\sigma \in \mathbf{Iso}(W)$ precisely if there is a triangular representation of $\sigma$ of the shape*

$$\left(\begin{array}{ccccc|ccccc}
a^{-1}\mathbf{I}_1 & \mathbf{10} & & & & & & & & \\
\mathbf{0} & a\mathbf{I}_2 & \ddots & & & & & & & \\
\vdots & T'_2 & \ddots & \mathbf{0} & & & & & & \\
\mathbf{0} & \vdots & \ddots & a\mathbf{I}_2 & \mathbf{0} & & & & & \\
L_{\frac{p}{2}} & T'_{\frac{p}{2}} & \dots & T'_2 & a\mathbf{I}_2 & \mathbf{0} & & & & \\
\hline
L_{\frac{p}{2}+1} & T'_{\frac{p}{2}+1} & \mathbf{0} & \dots & \mathbf{0} & a^{-1}\mathbf{I}_2 & \mathbf{0} & & & \\
L_{\frac{p}{2}+2} & T'_{\frac{p}{2}+2} & \ddots & \ddots & \vdots & T'_{p+2} & a^{-1}\mathbf{I}_2 & \mathbf{0} & & \\
\vdots & \vdots & \ddots & T'_{\frac{p}{2}+1} & \mathbf{0} & \vdots & \ddots & \ddots & \ddots & \\
L_p & T'_p & \dots & T'_{\frac{p}{2}+2} & T'_{\frac{p}{2}+1} & T'_{3\frac{p}{2}} & \dots & T'_{p+2} & a^{-1}\mathbf{I}_2 & \mathbf{0} \\
L_{p+1} & Y_1 & \dots & Y_{\frac{p}{2}-1} & Y_{\frac{p}{2}} & Y_{\frac{p}{2}+1} & \mathbf{0} & \dots & \mathbf{0} & a\mathbf{I}_2
\end{array}\right),$$

*where $a \in K^\times$, ${}^{\mathrm{t}}L_i \in L^2$, $Y_j \in \mathbf{Mat}_{2\times 2}(K)$ and*

$$T'_h \in \mathcal{T}' = \left\{ \begin{pmatrix} x & \eta^2 y \\ y & x \end{pmatrix} : x, y \in K \right\},$$

*are subject to the conditions:*

a) $Y_{\frac{p}{2}}, Y_{\frac{p}{2}+1} \in \mathcal{T}'$,

b) $Y_k - {}^{\mathrm{t}}\mathbf{B} Y_{k+1} \in \mathcal{T}'$ $\quad (k = 1, \dots, \frac{p}{2} - 1;\ p > 2)$,

c) $T'_k + Y_{\frac{p}{2}-k+2}\mathbf{J}_2 + \mathbf{J}_2 Y_{\frac{p}{2}-k+2} = \mathbf{0}$ $\quad (k = 2, \dots, \frac{p}{2};\ p > 2)$,

d) $\sum_{\substack{i+j=k \\ i,j>0}} T'_i T'_{p+j} = \mathbf{0}$ $\quad (k = 3, \dots, \frac{p}{2} + 1;\ p > 2)$ $(T'_1 = a\mathbf{I}_2,\ T'_{p+1} = a^{-1}\mathbf{I}_2)$,

e) $L_{\frac{p}{2}} = -(\mathbf{B} + {}^{\mathrm{t}}\mathbf{B})\, {}^{\mathrm{t}}Y_{\frac{p}{2}+1}\binom{1}{\eta}$,

f) $a \sum_{\substack{i+j=k\\ i,j>0}} {}^{\mathrm t}T'_i L_{\frac{p}{2}+j} = (\mathbf{B} + {}^{\mathrm t}\mathbf{B})\, {}^{\mathrm t}Y_{\frac{p}{2}-k+2}\binom{1}{\eta}$ $\qquad (k = 2, \ldots, \frac{p}{2};\ p > 2)$ $(T'_1 = a\mathbf{I}_2)$,

g) $a \sum_{\substack{i+j=\frac{p}{2}+1\\ i,j>0}} {}^{\mathrm t}T'_i L_{\frac{p}{2}+j} = (\mathbf{B} + {}^{\mathrm t}\mathbf{B})\, {}^{\mathrm t}(Y_1 - aY_{\frac{p}{2}+1}T'_{\frac{p}{2}+1})\binom{1}{\eta}$ $\qquad (T'_1 = a\mathbf{I}_2)$,

h) $(-1\ \eta)L_{p+1} + (1\ 0)\, {}^{\mathrm t}Y_{\frac{p}{2}+1}\binom{0}{1} = 0.$

Proof. The claimed triangular representation is the one with respect to the basis

$$\tilde{\mathcal{B}} = \{\varepsilon; u'_1, \ldots, u'_p, u''_p, \ldots, u''_1, e'', e'\}$$

of $W$ over $L$. In fact, with respect to $\tilde{\mathcal{B}}$, conditions (33) and (34) turn into *a*) and *b*), condition (36) becomes c), d) arises from $T'' = {}^{\mathrm t}T'^{-1}$, e)–g) translate the second and third condition in (32), h) is condition (37). □

*Remark 4.8.* Conditions from a) to h) in Theorem 4.7 give $5p + 4$ independent algebraic conditions over $K$. So, $\mathbf{Iso}(W)$ is a solvable algebraic group of dimension $2p + 5$ over $K$. In particular, the unipotent radical $R_u$ is nonabelian for any $p$; for $p > 2$ it is a nilpotent group of class $\frac{p}{2}$ (of class 2 if $p = 2$).

### 4.3. Third kind case

We know (Proposition 3.4) that the structure of the group $\mathbf{Iso}(U)$ is the same both in the second and in the third kind case. So, Theorem 4.1 implies that, with respect to the fixed basis $\mathcal{B}$, the isometry $\sigma$ has the representation

$$M_\sigma = \begin{pmatrix} (X_1 + \eta X_2)^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ L_1 & X_1 & X_2 & Y_{11} & Y_{12} \\ L_2 & \eta^2 X_2 & X_1 & Y_{21} & Y_{22} \\ L_3 & \mathbf{0} & \mathbf{0} & T' & S' \\ L_4 & \mathbf{0} & \mathbf{0} & S'' & T'' \end{pmatrix} \tag{25}$$

where $L_1, L_2 \in \mathbf{Mat}_{2\times 2}(L)$, $L_3, L_4 \in \mathbf{Mat}_{p\times 2}(L)$, $Y_{ij} \in \mathbf{Mat}_{2\times p}(K)$,

$X_i \in \mathbf{Mat}_{2\times 2}(K)$ with

$$\begin{pmatrix} X_1 & X_2 \\ \eta^2 X_2 & X_1 \end{pmatrix} \in GL_4(K), \qquad X = X_1 + \eta X_2 \in GL_2(L),$$

fulfilling (5)–(11), and $T', S', S'', T'' \in \mathbf{Mat}_{p\times p}(K)$ subjected to the conditions given in Proposition 4.5. Comparing the components over $K$ of equations (10) and (11), we find

$$B_1 = X_1 B_1 {}^{\mathrm{t}}T' + X_1 B_2 S' + \eta^2 Y_{11} S' - \eta^2 Y_{12} {}^{\mathrm{t}}T' - Y_{21}\mathbf{J}_p S' + Y_{22} {}^{\mathrm{t}}\mathbf{J}_p {}^{\mathrm{t}}T'; \tag{26}$$

$$\mathbf{0} = X_2 B_1 {}^{\mathrm{t}}T' + X_2 B_2 S' - Y_{11}\mathbf{J}_p S' + Y_{12} {}^{\mathrm{t}}\mathbf{J}_p {}^{\mathrm{t}}T' + Y_{21} S' - Y_{22} {}^{\mathrm{t}}T'; \tag{27}$$

$$B_2 = X_1 B_1 S'' + X_1 B_2 {}^{\mathrm{t}}T'' + \eta^2 Y_{11} {}^{\mathrm{t}}T'' - \eta^2 Y_{12} S'' - Y_{21}\mathbf{J}_p {}^{\mathrm{t}}T'' + Y_{22} {}^{\mathrm{t}}\mathbf{J}_p S''; \tag{28}$$

$$\mathbf{0} = X_2 B_1 S'' + X_2 B_2 {}^{\mathrm{t}}T'' - Y_{11}\mathbf{J}_p {}^{\mathrm{t}}T'' + Y_{12} {}^{\mathrm{t}}\mathbf{J}_p S'' + Y_{21} {}^{\mathrm{t}}T'' - Y_{22} S''. \tag{29}$$

Multiply on the right both the sides of (26) and (27) (resp. (28) and (29)) by $T''$ (resp. $S'$). Then, with the aid of the identities ${}^{\mathrm{t}}T'T'' - S''S' = \mathbf{I}_p$ and $S'T'' - {}^{\mathrm{t}}T''S' = \mathbf{0}$, from (26) and (28) we obtain

$$B_1 T'' - B_2 S' = X_1 B_1 - \eta^2 Y_{12} + Y_{22} {}^{\mathrm{t}}\mathbf{J}_p \tag{30}$$

and from (27) and (29)

$$Y_{22} = Y_{12} {}^{\mathrm{t}}\mathbf{J}_p + X_2 B_1. \tag{31}$$

Now, (30) and (31) give

$$B_1 T'' - B_2 S' = X_1 B_1 + X_2 B_1 {}^{\mathrm{t}}\mathbf{J}_p + Y_{12}({}^{\mathrm{t}}\mathbf{J}_p^2 - \eta^2 \mathbf{I}_p). \tag{32}$$

Likewise, we obtain from (26)–(29)

$$Y_{21} = Y_{11}\mathbf{J}_p - X_2 B_2; \tag{33}$$

$$B_2 T' - B_1 S'' = X_1 B_2 + X_2 B_2 \mathbf{J}_p - Y_{11}(\mathbf{J}_p^2 - \eta^2 \mathbf{I}_p). \tag{34}$$

Write the entries of $T'$, $S'$, $S''$, $T''$ as in Theorem 4.5, then, (34) gives in particular

$$ {}^{\mathrm{t}}\mathbf{B}\, T_1' - \mathbf{B}S_1'' = X_1 {}^{\mathrm{t}}\mathbf{B} + X_2 {}^{\mathrm{t}}\mathbf{B}\mathbf{J}_2. \tag{35} $$

Also, (32) gives

$$ \mathbf{B}T_1'' - {}^{\mathrm{t}}\mathbf{B}S_1' = X_1\mathbf{B} + X_2\mathbf{B}\,{}^{\mathrm{t}}\mathbf{J}_2. \tag{36} $$

Now, from (35) and (36) it follows

$$ X_1 = \begin{pmatrix} t'_{11} & -s'_{12} \\ -s''_{12} & t''_{11} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} t'_{12} & -s'_{11} \\ -s''_{11} & t''_{12} \end{pmatrix}, \tag{37} $$

so, looking at Remark 4.6, we see that $M_\sigma \mapsto X = X_1 + \eta X_2$ is a group epimorphism $\varphi : \mathbf{Iso}(W) \to SL_2(L)$ the kernel of which is the subgroup of $\mathbf{Iso}(W)$ leaving the $L$-component of $W$ pointwise fixed. From (37) it follows, as well, that such a kernel is just the unipotent radical $R_u$ of $\mathbf{Iso}(W)$. As a consequence, $\mathbf{Iso}(W)$ has *Levi factors* (see [6], p. 184) isomorphic to $\mathbf{SL}_2(L)$. More precisely, if we represent isometries using the basis

$$ \tilde{\mathcal{B}} = \{\varepsilon_1, \varepsilon_2, u_1', \dots, u_p', e_1', e_2', e_1'', e_2'', u_1'', \dots, u_p''\} $$

of $W$ over $L$, a Levi factor of $\mathbf{Iso}(W)$ is given by the group of matrices

**4.9.**

$$ \begin{pmatrix}
{}^{\mathrm{t}}X^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
L_1' & T_1' & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & S_1' \\
L_2' & \mathbf{0} & \ddots & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \cdot^{\cdot^{\cdot}} & \mathbf{0} \\
\vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\
L'_{\frac{p}{2}} & \mathbf{0} & \cdots & \mathbf{0} & T_1' & \mathbf{0} & \mathbf{0} & S_1' & \mathbf{0} & \cdots & \mathbf{0} \\
L'_{\frac{p}{2}+1} & Y_1' & \cdots & Y'_{\frac{p}{2}-1} & Y'_{\frac{p}{2}} & X_1 & X_2 & Z_1' & Z_2' & \cdots & Z'_{\frac{p}{2}} \\
L''_{\frac{p}{2}+1} & Y_1'' & \cdots & Y''_{\frac{p}{2}-1} & Y''_{\frac{p}{2}} & \eta^2 X_2 & X_1 & Z_1'' & Z_2'' & \cdots & Z''_{\frac{p}{2}} \\
L''_{\frac{p}{2}} & \mathbf{0} & \cdots & \mathbf{0} & S_1'' & \mathbf{0} & \mathbf{0} & T_1'' & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \vdots & \vdots & & \ddots & \ddots & \vdots \\
L_2'' & \mathbf{0} & \cdot^{\cdot^{\cdot}} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \ddots & \mathbf{0} \\
L_1'' & S_1'' & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & T_1''
\end{pmatrix} $$

*with*

$$T_1' = \begin{pmatrix} t_1' & \eta^2 t_2' \\ t_2' & t_1' \end{pmatrix} \in \mathcal{T}', \qquad S_1' = \begin{pmatrix} \eta^2 s_1' & s_2' \\ s_2' & s_1' \end{pmatrix} \in \mathcal{S}',$$

$$S_1'' = \begin{pmatrix} s_1'' & s_2'' \\ s_2'' & \eta^2 s_1'' \end{pmatrix} \in \mathcal{S}'', \qquad T_1'' = \begin{pmatrix} t_1'' & t_2'' \\ \eta^2 t_2'' & t_1'' \end{pmatrix} \in \mathcal{T}'',$$

*such that* ${}^{\mathrm{t}}T_1' T_1'' - S_1'' S_1' = \mathbf{I}_2$ *and*

$$X_1 = \begin{pmatrix} t_1' & -s_2' \\ -s_2'' & t_1'' \end{pmatrix}, \qquad X_2 = \begin{pmatrix} t_2' & -s_1' \\ -s_1'' & t_2'' \end{pmatrix},$$

$$L'_{\frac{p}{2}} + 1 = \begin{cases} -\dfrac{p+4}{2(2\eta)^{\frac{p}{2}+2}} X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B}) & \text{if } \dfrac{p}{2} \text{ is odd;} \\[2ex] -\dfrac{p}{2(2\eta)^{\frac{p}{2}+2}} X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B}) & \text{if } \dfrac{p}{2} \text{ is even;} \end{cases}$$

$$L''_{\frac{p}{2}+1} = \begin{cases} \dfrac{p-4}{4(2\eta)^{\frac{p}{2}+1}} (X_1 - \eta X_2)({}^{\mathrm{t}}\mathbf{B} - \mathbf{B})\, {}^{\mathrm{t}}X_2 {}^{\mathrm{t}}(X_1 + \eta X_2)^{-1} & \text{if } \dfrac{p}{2} \text{ is odd;} \\[2ex] \dfrac{p}{4(2\eta)^{\frac{p}{2}+1}} (X_1 - \eta X_2)({}^{\mathrm{t}}\mathbf{B} - \mathbf{B})\, {}^{\mathrm{t}}X_2 {}^{\mathrm{t}}(X_1 + \eta X_2)^{-1} & \text{if } \dfrac{p}{2} \text{ is even;} \end{cases}$$

*and for* $k = 1, \ldots, \frac{p}{2}$,

$$Y_k' = -\left(-\frac{1}{2}\right)^{\frac{p}{2}-k+1} X_2 {}^{\mathrm{t}}\mathbf{B} \mathbf{J}_2^{-\frac{p}{2}+k-1}, \quad Z_k' = \left(-\frac{1}{2}\right)^{k} X_2 \mathbf{B}\, {}^{\mathrm{t}}\mathbf{J}_2^{-k},$$

$$Y_k'' = \left(-\frac{1}{2}\right)^{\frac{p}{2}-k+1} X_2 {}^{\mathrm{t}}\mathbf{B} \mathbf{J}_2^{-\frac{p}{2}+k}, \qquad Z_k'' = -\left(-\frac{1}{2}\right)^{k} X_2 \mathbf{B}\, {}^{\mathrm{t}}\mathbf{J}_2^{-k+1},$$

$$L_k' = \left(-\frac{1}{2}\right)^{k} (\mathbf{J}_2 - \eta \mathbf{I}_2) \mathbf{J}_2^{-k} (T_1' \, {}^{\mathrm{t}}\mathbf{B} + S_1' \mathbf{B})\, {}^{\mathrm{t}}X_2 {}^{\mathrm{t}}X^{-1},$$

$$L_k'' = \left(-\frac{1}{2}\right)^{\frac{p}{2}-k+1} {}^{\mathrm{t}}(\mathbf{J}_2 - \eta \mathbf{I}_2)\, {}^{\mathrm{t}}\mathbf{J}_2^{-\frac{p}{2}+k-1} (T_1'' \, {}^{\mathrm{t}}\mathbf{B} + S_1'' \mathbf{B})\, {}^{\mathrm{t}}X_2 {}^{\mathrm{t}}X^{-1}.$$

(Keep in mind the equivalence

$$X_1({}^{\mathrm{t}}\mathbf{B}-\mathbf{B})\,{}^{\mathrm{t}}X_1+\eta^2X_2({}^{\mathrm{t}}\mathbf{B}-\mathbf{B})\,{}^{\mathrm{t}}X_2={}^{\mathrm{t}}\mathbf{B}-\mathbf{B}\Longleftrightarrow \det(X_1+\eta X_2)=1$$

to verify that 4.9 is a group.)

Using the basis $\tilde{\mathcal{B}}$, isometries in the unipotent radical take the shape

**4.10.**

$$\begin{pmatrix}
\mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
L'_1 & \mathbf{I}_2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
L'_2 & T'_2 & \mathbf{I}_2 & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \iddots & \mathbf{0} & S'_2 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\
L'_{\frac{p}{2}-1} & T'_{\frac{p}{2}-1} & & \ddots & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \iddots & & S'_{\frac{p}{2}-1} \\
L'_{\frac{p}{2}} & T'_{\frac{p}{2}} & T'_{\frac{p}{2}-1} & \dots & T'_2 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & S'_2 & \dots & S'_{\frac{p}{2}-1} & S'_{\frac{p}{2}} \\
L'_{\frac{p}{2}+1} & Y'_1 & Y'_2 & \dots & Y'_{\frac{p}{2}-1} & Y'_{\frac{p}{2}} & \mathbf{I}_2 & \mathbf{0} & Z'_1 & Z'_2 & \dots & Z'_{\frac{p}{2}-1} & Z'_{\frac{p}{2}} \\
L''_{\frac{p}{2}+1} & Y''_1 & Y''_2 & \dots & Y''_{\frac{p}{2}-1} & Y''_{\frac{p}{2}} & \mathbf{0} & \mathbf{I}_2 & Z''_1 & Z''_2 & \dots & Z''_{\frac{p}{2}-1} & Z''_{\frac{p}{2}} \\
L''_{\frac{p}{2}} & S''_{\frac{p}{2}} & S''_{\frac{p}{2}-1} & \dots & S''_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & T''_2 & \dots & T''_{\frac{p}{2}-1} & T''_{\frac{p}{2}} \\
L''_{\frac{p}{2}-1} & S''_{\frac{p}{2}-1} & & \iddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \ddots & & T''_{\frac{p}{2}-1} \\
\vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
L''_2 & S''_2 & \mathbf{0} & \iddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{I}_2 & T''_2 \\
L''_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}_2
\end{pmatrix}$$

*with* $L'_i, L''_i \in \mathbf{Mat}_{2\times 2}(L)$, $Y'_j, Y''_j, Z'_j, Z''_j \in \mathbf{Mat}_{2\times 2}(K)$, $T'_h \in \mathcal{T}'$, $S'_h \in \mathcal{S}'$, $S''_h \in \mathcal{S}''$, $T''_h \in \mathcal{T}''$ *subjected to the conditions:*

a) $\quad {}^{\mathrm{t}}L'_1 = -Z'_1\,{}^{\mathrm{t}}A_2, \quad {}^{\mathrm{t}}L''_1 = Y'_{\frac{p}{2}}A_2,$

b) $\quad {}^{\mathrm{t}}L'_{\frac{p}{2}+1} - L'_{\frac{p}{2}+1} = \sum_{h=1}^{\frac{p}{2}} (Y'_h A_2 {}^{\mathrm{t}}Z'_h - Z'_h\,{}^{\mathrm{t}}A_2 {}^{\mathrm{t}}Y'_h) + \sum_{h=1}^{\frac{p}{2}-1} (Y'_{h+1}\,{}^{\mathrm{t}}Z'_h - Z'_h\,{}^{\mathrm{t}}Y'_{h+1}),$

c) $\quad {}^{\mathrm{t}}L''_{\frac{p}{2}+1} - \eta L'_{\frac{p}{2}+1} = \sum_{h=1}^{\frac{p}{2}} (Y'_h A_2 {}^{\mathrm{t}}Z''_h - Z'_h\,{}^{\mathrm{t}}A_2 {}^{\mathrm{t}}Y''_h)$

$\qquad + \sum_{h=1}^{\frac{p}{2}-1} (Y'_{h+1}\,{}^{\mathrm{t}}Z''_h - Z'_h\,{}^{\mathrm{t}}Y''_{h+1}) + Y'_1\,{}^{\mathrm{t}}\mathbf{B} + Z'_{\frac{p}{2}}\mathbf{B},$

d) $\eta{}^{\mathrm{t}}L''_{\frac{p}{2}+1} - \eta L''_{\frac{p}{2}+1} = \sum_{h=1}^{\frac{p}{2}} (Y''_h A_2 {}^{\mathrm{t}}Z''_h - Z''_h {}^{\mathrm{t}}A_2 {}^{\mathrm{t}}Y''_h)$
$+ \sum_{h=1}^{\frac{p}{2}-1} (Y''_{h+1} {}^{\mathrm{t}}Z''_h - Z''_h {}^{\mathrm{t}}Y''_{h+1}) + Y''_1 {}^{\mathrm{t}}\mathbf{B} - \mathbf{B} {}^{\mathrm{t}}Y''_1 + Z''_{\frac{p}{2}}\mathbf{B} - {}^{\mathrm{t}}\mathbf{B} {}^{\mathrm{t}}Z''_{\frac{p}{2}},$

*and for* $p > 2$,

e) $\sum_{\substack{i+j=k+1\\ i,j>0}} {}^{\mathrm{t}}T'_i T''_j - S''_i S'_j = \mathbf{0}$ $\quad (k = 2, \ldots, \frac{p}{2}),$
$(T'_1 = T''_1 = \mathbf{I}_2,\ S'_1 = S''_1 = \mathbf{0}),$

f) $Y'_k = -\sum_{h=1}^{\frac{p}{2}-k+1} \left(-\frac{1}{2}\right)^h (\mathbf{B}S''_{\frac{p}{2}-k-h+3} - {}^{\mathrm{t}}\mathbf{B}T'_{\frac{p}{2}-k-h+3})\mathbf{J}_2^{-h}$
$(k = 2, \ldots, \frac{p}{2}),$

g) $Z'_k = -\sum_{h=1}^{k} \left(-\frac{1}{2}\right)^h (\mathbf{B}T''_{k-h+2} - {}^{\mathrm{t}}\mathbf{B}S'_{k-h+2}) {}^{\mathrm{t}}\mathbf{J}_2^{-h}$ $\quad (k = 1, \ldots, \frac{p}{2}-1),$

h) $Y''_k = \frac{1}{2}(\mathbf{B}S''_{\frac{p}{2}-k+2} - {}^{\mathrm{t}}\mathbf{B}T'_{\frac{p}{2}-k+2})$
$+ \sum_{h=1}^{\frac{p}{2}-k} \left(-\frac{1}{2}\right)^{h+1} (\mathbf{B}S''_{\frac{p}{2}-k-h+2} - {}^{\mathrm{t}}\mathbf{B}T'_{\frac{p}{2}-k-h+2})\mathbf{J}_2^{-h}$ $\quad (k{=}2, \ldots, \frac{p}{2}),$

i) $Z''_k = \frac{1}{2}(\mathbf{B}T''_{k+1} - {}^{\mathrm{t}}\mathbf{B}S'_{k+1})$
$+ \sum_{h=1}^{k} \left(-\frac{1}{2}\right)^{h+1} (\mathbf{B}T''_{k-h+1} - {}^{\mathrm{t}}\mathbf{B}S'_{k-h+1}) {}^{\mathrm{t}}\mathbf{J}_2^{-h}$ $\quad (k{=}1, \ldots, \frac{p}{2}-1),$

j) ${}^{\mathrm{t}}L'_k = \sum_{h=2}^{k} Y'_{\frac{p}{2}+h-k} A_2 S'_h + \sum_{h=2}^{k} Y'_{\frac{p}{2}+h-k} S'_{h-1}$ $\quad (k = 2, \ldots, \frac{p}{2}),$
$- \sum_{\substack{i+j=k+1\\ i,j>0}} Z'_i {}^{\mathrm{t}}A_2 {}^{\mathrm{t}}T'_j - \sum_{\substack{i+j=k\\ i,j>0}} Z'_i {}^{\mathrm{t}}T'_j$ $\quad (T'_1 = \mathbf{I}_2,\ S'_1 = \mathbf{0}),$

k) ${}^{\mathrm{t}}L''_k = \sum_{h=1}^{k} Y'_{\frac{p}{2}+h-k} A_2 {}^{\mathrm{t}}T''_h + \sum_{h=2}^{k} Y'_{\frac{p}{2}+h-k} {}^{\mathrm{t}}T''_{h-1}$ $\quad (k = 2, \ldots, \frac{p}{2})$
$- \sum_{\substack{i+j=k+1\\ i,j>0}} Z'_i {}^{\mathrm{t}}A_2 S''_j - \sum_{\substack{i+j=k\\ i,j>0}} Z'_i S''_j$ $\quad (T''_1 = \mathbf{I}_2,\ S''_1 = \mathbf{0}).$

Notice that the above conditions a)–k), translating the fundamental equations (5)–(11) of an isometry, give $17p+2$ independent algebraic conditions over $K$. Therefore, $R_u$ is an algebraic group of dimension $3p+6$

over $K$. Notice, too, that $R_u$ is a nonabelian nilpotent group of class $\frac{p}{2}$.

Summing up we have

**Theorem 4.11.** *Let $W$ be of third kind. Every element in $\mathbf{Iso}(W)$ is uniquely represented as the product of a matrix 4.9 by a matrix 4.10. The algebraic group $\mathbf{Iso}(W)$ has Levi decomposition $R_u \rtimes \mathbf{SL}_2(L)$ with $R_u$ a nonabelian nilpotent group of class $\frac{p}{2}$ and dimension $3p+6$ over $K$.*

## 5. Three exceptional cases

As we said in Section 3, there are exactly three cases where $\dim U = 0$, one for each kind. It turns out that Theorems 4.4, 4.7 and 4.11 cover these cases, also. The structure of the corresponding isometry groups can be described as follows:

- $\mathbf{H_{11}} : W = \langle \varepsilon \rangle_L \oplus \langle e \rangle_K$ with

$$f(\varepsilon, e) = 1.$$

A representation of an isometry $\sigma \in \mathbf{Iso(H_{11})}$ has the shape

$$\begin{pmatrix} \alpha & 0 \\ \beta & a \end{pmatrix}$$

with $\alpha, \beta \in L$ and $a \in K$. Also, $\det \sigma = \det \sigma_L = 1$ because the alternating space $(L\mathbf{H_{11}}, f^{L\mathrm{H}_{11}})$ is regular. Hence, $\alpha = a^{-1}$ and we have

$$\mathbf{Iso(H_{11})} \simeq L_+ \rtimes K^\times \simeq K_+^2 \rtimes K^\times.$$

- $\mathbf{H_{12}} : W = \langle \varepsilon \rangle_L \oplus \langle e', e'' \rangle_K$ with

$$f(\varepsilon, e') = 1; \qquad f(\varepsilon, e'') = \eta; \qquad f(e', e'') = 0.$$

Let $\sigma \in \mathbf{Iso(H_{12})}$. Then, a representation of $\sigma$ has the shape

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & a & b \\ \gamma & c & d \end{pmatrix}$$

with $\alpha, \beta, \gamma \in L$ and $a, b, c, d \in K$. As $\sigma_L$ leaves the radical $R = \langle e'' - \eta e' \rangle_L$ of $\mathbf{H_{12}}$ stable, we find $\gamma = \eta\beta$, $a = d$ and $c = \eta^2 b$. Furthermore,

$$x + R \mapsto \sigma_L(x) + R$$

yields an isometry of a regular alternating space and this, in turn, says that $\alpha = a^{-1}$. Also, we have

$$\begin{aligned} 1 = f(\varepsilon, e') = f(\sigma(\varepsilon), \sigma(e')) &= f(a^{-1}\varepsilon, \beta\varepsilon + ae' + be'') \\ &= 1 + a^{-1}b\eta \implies b = 0. \end{aligned}$$

Therefore,

$$\mathbf{Iso}(\mathbf{H_{12}}) \simeq \mathbf{Iso}(\mathbf{H_{11}}) \simeq K_+^2 \rtimes K^\times.$$

• $\mathbf{H_{24}}$ : $W = \langle \varepsilon_1, \varepsilon_2 \rangle_L \oplus \langle e_1', e_2', e_1'', e_2'' \rangle_K$ with

$$f(\varepsilon_i, e_j') = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases} \qquad f(e_1'', e_2'') = 1;$$

$$f(\varepsilon_i, e_j'') = \begin{cases} \eta & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases} \qquad f(e_i', e_j'') = f(\varepsilon_1, \varepsilon_2) = f(e_1', e_2') = 0.$$

A representation of an isometry $\sigma \in \mathbf{Iso}(\mathbf{H_{24}})$ has the shape

$$\begin{pmatrix} L_0 & \mathbf{0} & \mathbf{0} \\ L_1 & X_{11} & X_{12} \\ L_2 & X_{21} & X_{22} \end{pmatrix}$$

with $L_0 \in \mathbf{GL}_2(L)$, $L_i \in \mathbf{Mat}_{2\times 2}(L)$, $X_{ij} \in \mathbf{Mat}_{2\times 2}(K)$ and $(X_{ij}) \in \mathbf{GL}_4(K)$ $(i, j = 1, 2)$. The subspace of vectors in $L\mathbf{H_{24}}$ orthogonal to the $L$-component $C = \langle \varepsilon_1, \varepsilon_2 \rangle_L$ is the subspace

$$C^{\perp_{L\mathbf{H_{24}}}} = \langle \varepsilon_1, \varepsilon_2, e_1'' - \eta e_1', e_2'' - \eta e_2' \rangle_L.$$

Since $C^{\perp_{L\mathbf{H_{24}}}}$ is stable under $\sigma_L$, we infer that $X_{11} = X_{22}$ and $X_{21} = \eta^2 X_{12}$. We shall write $X_1$ and $X_2$ instead of $X_{11}$ and $X_{12}$. Furthermore, setting

$$\bar{f}(x + C, y + C) = f(x, y),$$

we have a well defined nonsingular alternating form on the $L$-space $C^{\perp_L \mathbf{H_{24}}}/C$ given by the matrix

$$ {}^{\mathrm{t}}\mathbf{B} - \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Clearly, $x + C \mapsto \sigma_L(x) + C$ yields an isometry of $(C^{\perp_L \mathbf{H_{24}}}/C, \bar{f})$ which is represented, with respect to the basis $\{e''_1 - \eta e'_1 + C,\ e''_2 - \eta e'_2 + C\}$, by $X_1 - \eta X_2$, i.e.

$$ \begin{aligned} (X_1 - \eta X_2)({}^{\mathrm{t}}\mathbf{B} - \mathbf{B})({}^{\mathrm{t}}X_1 - \eta {}^{\mathrm{t}}X_2) = ({}^{\mathrm{t}}\mathbf{B} - \mathbf{B}) &\iff \det(X_1 - \eta X_2) = 1 \\ &\iff \det(X_1 + \eta X_2) = 1. \end{aligned} $$

Now, to ask that $\sigma$ is an isometry of $\mathbf{H_{24}}$ is equivalent to impose the conditions

$$ f(\sigma(\varepsilon_i), \sigma(e'_j)) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases} \tag{38} $$

$$ f(\sigma(\varepsilon_i), \sigma(e''_j)) = \begin{cases} \eta & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases} \tag{39} $$

$$ f(\sigma(e'_1), \sigma(e'_2)) = 0; \tag{40} $$

$$ f(\sigma(e'_i), \sigma(e''_j)) = 0 \qquad (i, j = 1, 2). \tag{41} $$

Condition (38), as well as condition (39), is equivalent to

$$ {}^{\mathrm{t}}L_0(X_1 + \eta X_2) = \mathbf{I}_2 $$

and, consequently, (40) and (41) turn into

$$ \begin{aligned} (X_1 + \eta X_2)\,{}^{\mathrm{t}}L_1 - L_1{}^{\mathrm{t}}(X_1 + \eta X_2) &= X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B})\,{}^{\mathrm{t}}X_2; \\ (X_1 + \eta X_2)\,{}^{\mathrm{t}}L_2 - \eta L_1{}^{\mathrm{t}}(X_1 + \eta X_2) &= X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B})\,{}^{\mathrm{t}}X_1. \end{aligned} \tag{42} $$

Summing up, we have for $\sigma$ a representation

$$ M_\sigma = \begin{pmatrix} {}^{\mathrm{t}}(X_1 + \eta X_2)^{-1} & \mathbf{0} & \mathbf{0} \\ L_1 & X_1 & X_2 \\ L_2 & \eta^2 X_2 & X_1 \end{pmatrix} $$

with $X_1, X_2 \in \mathbf{Mat}_{2\times 2}(K)$ such that $\det(X_1 + \eta X_2) = 1$ and $L_1, L_2 \in \mathbf{Mat}_{2\times 2}(L)$ fulfilling (42).

Clearly, there is a epimorphism

$$\begin{array}{rcl}\mathbf{Iso}(\mathbf{H_{24}}) & \to & \mathbf{SL}_2(L) \\ M_\sigma & \mapsto & X_1 + \eta X_2\end{array}$$

the kernel of which is the unipotent radical of $\mathbf{Iso}(\mathbf{H_{24}})$:

$$\left\{ \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ S & \mathbf{I}_2 & \mathbf{0} \\ \eta S & \mathbf{0} & \mathbf{I}_2 \end{pmatrix} : S \in \mathbf{Mat}_{2\times 2}(L), {}^{\mathrm{t}}S = S \right\}.$$

A Levi factor of $\mathbf{Iso}(\mathbf{H_{24}})$ is

$$\left\{ \begin{pmatrix} {}^{\mathrm{t}}(X_1 + \eta X_2)^{-1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2\eta} X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B}) & X_1 & X_2 \\ \frac{1}{2} X_2({}^{\mathrm{t}}\mathbf{B} - \mathbf{B}) & \eta^2 X_2 & X_1 \end{pmatrix} : \det(X_1 + \eta X_2) = 1 \right\}.$$

Thus, we conclude that $\mathbf{Iso}(\mathbf{H_{24}})$ is the semidirect product of the 3-dimensional vector group over $L$ by $\mathbf{SL}_2(L)$, hence $\mathbf{Iso}(\mathbf{H_{24}}) \simeq K_+^6 \rtimes \mathbf{SL}_2(L)$.

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*(Received April 17, 2001; revised March 28, 2002)*