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Quasi-normed monoids and quasi-metrics

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Abstract. We present a method to generate quasi-metrics from certain classes of subadditive functions defined on monoids. Several properties of these quasi-metrics are discussed. In particular, bicompletion is explored. Some illustrative examples are given.

1. Introduction and preliminaries

Let $(X,+)$ be a semigroup. A function $f : X \to \mathbb{R}$ is said to be *subadditive* if for each $x,y \in X$, $f(x+y) \leq f(x)+f(y)$.

A *monoid* is a semigroup $(X,+)$ with neutral (or identity) element e.

A *submonoid* of a monoid $(X,+)$ is a subsemigroup of X that contains the neutral element e.

A *prenorm* on a monoid $(X,+)$ is a nonnegative subadditive function p on X such that $p(e)=0$.

A *quasi-norm* on $(X,+)$ is a prenorm p on X such that $x=e$ if and only if $-x \in X$ and $p(x)=p(-x)=0$.

A monoid $(X,+)$ is called *left cancellative* if for all $x,y,z \in X$, $z+x = z+y$ implies $x=y$, and it is called *right cancellative* if $x+z=y+z$ implies $x=y$. $(X,+)$ is said to be *cancellative* if it is both left cancellative and right cancellative.

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The main purpose of this paper is to show that it is possible to generate in a natural way (extended) quasi-metrics from quasi-norms on cancellative monoids. Several properties of these quasi-metrics, including bicompletion, are discussed. We observe that the classical Sorgenfrey quasi-metric on $\mathbb{R}^+$ is an example of this kind of structure. Our construction is also applied to the domain of words and the space of complexity functions, two interesting instances of spaces which appear in some fields of Theoretical Computer Science.

In the sequel, the letters $\mathbb{R}^+$, ω and $\mathbb{N}$ will denote the set of nonnegative real numbers, the set of nonnegative integer numbers and the set of positive integer numbers, respectively.

Our main reference for quasi-pseudo-metric spaces is [3].

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real-valued function d defined on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$, and (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In our context by a quasi-metric we mean a quasi-pseudo-metric d on X such that $d(x, y) = d(y, x) = 0$ if and only if $x = y$.

A quasi-(pseudo-)metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-(pseudo-)metric on X.

Each quasi-pseudo-metric d on a set X induces a topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

If d is a quasi-(pseudo-)metric on a set X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a (pseudo-)metric on X.

A quasi-metric d on a set X is said to be *bicomplete* if d^s is a complete metric on X.

2. Generating quasi-metrics on monoids

Following [5], we say that a quasi-pseudo-metric d on a monoid $(X, +)$ is *left subinvariant* provided that for each $x, y, z \in X$, $d(z + x, z + y) \leq d(x, y)$ and it is *right subinvariant* provided that $d(x + z, y + z) \leq d(x, y)$. If d is both left and right subinvariant, d is said to be subinvariant.

It is well known and easy to see that d is subinvariant if and only if for each $a, b, x, y \in X$, $d(a + b, x + y) \leq d(a, x) + d(b, y)$. Furthermore if d is a subinvariant quasi-pseudo-metric on a monoid $(X, +)$, then $(X, \mathcal{T}(d))$ is a topological monoid. (Let us recall that a *topological monoid* is a triple $(X, +, \mathcal{T})$ such that $(X, +)$ is a monoid and $\mathcal{T}$ is a topology on X for which the operation $+$ is continuous.)

A quasi-pseudo-metric d on a monoid $(X,+)$ is *left invariant* provided that $d(z+x,z+y)=d(x,y)$ for all $x,y,z\in X$ and it is *right invariant* provided that $d(x+z,y+z)=d(x,y)$. d is said to be *invariant* if it is both left invariant and right invariant.

Let $(X,+)$ be a monoid. For each $x\in X$ define $x+X=\{x+y:y\in X\}$.

Proposition 1. *Let p be a prenorm on a monoid $(X,+)$. Then the real-valued function d_p defined on $X\times X$ by*

$d_p(x,y)=\inf\{p(a):y=x+a\}\wedge 1$ *if* $x\in X$ *and* $y\in x+X$,

$d_p(x,y)=1$ *if* $x\in X$ *and* $y\notin x+X$

is a left subinvariant quasi-pseudo-metric on X.

Furthermore for each $x\in X$ and each $\varepsilon\in(0,1)$, $B_{d_p}(x,\varepsilon)=x+\{y\in X:p(y)<\varepsilon\}$, and the left translations are $\mathcal{T}(d_p)$-open.

Proof. Since $p(e)=0$, it follows that for each $x\in X$,

$$\inf\{p(a):x=x+a\}=0.$$

Thus $d_p(x,x)=0$ for all $x\in X$.

Next we show that for all $x,y,z\in X$, $d_p(x,z)\le d_p(x,y)+d_p(y,z)$.

We only consider the case that $y\in x+X$ and $z\in y+X$, with $d_p(x,y)=\inf\{p(a):y=x+a\}$ and $d_p(y,z)=\inf\{p(a):z=y+a\}$, since the triangle inequality is obviously satisfied otherwise.

Choose an arbitrary $\varepsilon\in(0,1)$. There exist $a,b\in X$ such that $y=x+a$, $z=y+b$, $p(a)<d_p(x,y)+\varepsilon$ and $p(b)<d_p(y,z)+\varepsilon$. Since $z=x+a+b$, $z\in x+X$, and thus

$$d_p(x,z)\le p(a+b)\le p(a)+p(b)<d_p(x,y)+d_p(y,z)+2\varepsilon.$$

Consequently $d_p(x,z)\le d_p(x,y)+d_p(y,z)$, for all $x,y,z\in X$, and hence d_p is a quasi-pseudo-metric on X.

Now we show that for all $x,y,z\in X$, $d_p(z+x,z+y)\le d_p(x,y)$.

We only consider the case that $d_p(x,y)=\inf\{p(a):y=x+a\}$. Choose an arbitrary $\varepsilon\in(0,1)$. There exists $a\in X$ such that $y=x+a$ and $p(a)<d_p(x,y)+\varepsilon$. Hence $z+y=z+x+a$ and thus

$$d_p(z+x,z+y)\le p(a)<d_p(x,y)+\varepsilon.$$

So $d_p(z+x,z+y)\le d_p(x,y)$. Therefore d_p is left subinvariant.

Now note that for each $x\in X$, $d_p(e,x)=p(x)\wedge 1$, so for each $\varepsilon\in(0,1)$, we have $B_{d_p}(e,\varepsilon)=\{x\in X:p(x)<\varepsilon\}$. It immediately follows that for each $x\in X$ and each $\varepsilon\in(0,1)$,

$$B_{d_p}(x,\varepsilon)=x+B_{d_p}(e,\varepsilon),$$

and thus the left translations with respect to $+$ are $\mathcal{T}(d_p)$-open. □

Corollary 1. *Let p be a prenorm on a left cancellative monoid $(X,+)$. Then the real-valued function d_p defined on $X \times X$ by*

$d_p(x,y) = p(a) \wedge 1$ if $x \in X$ and $y \in x + X$ with $y = x + a$,

$d_p(x,y) = 1$ if $x \in X$ and $y \notin x + X$

is a left invariant quasi-pseudo-metric on X.

Furthermore for each $x \in X$ and each $\varepsilon \in (0,1)$, $B_{d_p}(x,\varepsilon) = x + \{y \in X : p(y) < \varepsilon\}$, and the left translations are $\mathcal{T}(d_p)$-open.

Proof. We only show that given $x,y,z \in X$ one has $d_p(z+x,z+y) = d_p(x,y)$. Indeed, suppose $d_p(x,y) = p(a) \wedge 1$ for some $a \in X$. Then $y = x + a$, so $z + y = z + x + a$, and hence $d_p(z+x,z+y) = p(a) \wedge 1$. Otherwise $y \notin x + X$, and it immediately follows that $z + y \notin z + x + X$, so $d_p(z+x,z+y) = d_p(x,y) = 1$. We conclude that d_p is left invariant. □

Remark 1. Let p be a prenorm on a monoid $(X,+)$. If d_p is a quasi-metric, then p is a quasi-norm on X.

Proof. Let $x \in X$ be such that $-x \in X$ and $p(x) = p(-x) = 0$. Then $d_p(e,x) = 0$ because $x = x + e$ and $p(x) = 0$. Furthermore $d_p(x,e) = 0$ because $e = x - x$ and $p(-x) = 0$. Since d_p is a quasi-metric, it follows that $x = e$. We conclude that p is a quasi-norm on X. □

Example 1. On $\mathbb{R}^+$, endowed with the usual addition, define a quasi-norm p by $p(x) = 0$ for all $x \in \mathbb{R}^+$. Then the quasi-metric d_p, is exactly the Alexandroff quasi-metric on $\mathbb{R}^+$, i.e. $d_p(x,y) = 0$ if $x \leq y$ and $d_p(x,y) = 1$ otherwise.

Example 2. Let $p : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $p(x) = x$ for all $x \in \mathbb{R}^+$. Clearly p is a quasi-norm on $\mathbb{R}^+$ and $d_p(x,y) = \min\{y - x, 1\}$ if $x \leq y$ and $d_p(x,y) = 1$ if $x > y$. So d_p is the Sorgenfrey quasi-metric on $\mathbb{R}^+$ and hence the topology generated by d_p is exactly the Sorgenfrey topology on $\mathbb{R}^+$.

The following result will be useful later on.

Proposition 2. *Let p be a prenorm on a monoid $(X,+)$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to a point $x \in X$ in $(X,(d_p)^s)$, then $(p(x_n))_{n\in\mathbb{N}}$ converges to $p(x)$ with respect to the Euclidean metric.*

Proof. Since $(x_n)_{n\in\mathbb{N}}$ converges to x in $(X,(d_p)^s)$, for each $\varepsilon \in (0,1)$ there is $n_0 \in \mathbb{N}$ such that $(d_p)^s(x,x_n) < \varepsilon$ for all $n \geq n_0$. Hence there exist two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X such that $x = x_n + a_n$, $x_n = x + b_n$, $p(a_n) < \varepsilon$ and $p(b_n) < \varepsilon$ for all $n \geq n_0$. Then $p(x_n) -$

$p(x) = p(x + b_n) - p(x) \leq p(x) + p(b_n) - p(x) = p(b_n) < \varepsilon$, and, similarly, $p(x) - p(x_n) \leq p(a_n) < \varepsilon$ for all $n \geq n_0$. Therefore $|p(x_n) - p(x)| < \varepsilon$ for all $n \geq n_0$. It follows that $(p(x_n))_{n\in\mathbb{N}}$ converges to $p(x)$ with respect to the Euclidean metric. □

The following is an example of a quasi-norm on a monoid $(X, +)$ such that $(X, +, \mathcal{T}(d_p))$ is not a topological monoid.

Example 3. Let $X = \omega \cup \{\infty\}$. Define $x + y = x$ for all $x, y \in X$ with $x \neq 0$ and $0 + x = x$ for all $x \in X$.

Set $p : X \to \mathbb{R}^+$ given by $p(0) = 0$, $p(n) = 1/n$ for all $n \in \mathbb{N}$ and $p(\infty) = 1$. It is easily seen that p is a quasi-norm on X. By Proposition 1, $d_p(0, n) = 1/n$ for all $n \in \mathbb{N}$, and $d_p(x, y) = 1$ for all other $x, y \in X$ with $x \neq y$. Therefore $n \to 0$ with respect to $\mathcal{T}(d_p)$ but $n + \infty \nrightarrow 0 + \infty$ because $n + \infty = n$ for all $n \in \mathbb{N}$, $0 + \infty = \infty$ and $\{\infty\}$ is $\mathcal{T}(d_p)$-isolated.

The preceding example suggests the question of obtaining conditions under which the quasi-pseudo-metric d_p constructed in Proposition 1 is subinvariant and thus $(X, +, \mathcal{T}(d_p))$ is a topological monoid.

Next we give a reasonable and easy solution to this question.

Remark 2. Let p be a prenorm on an Abelian monoid $(X, +)$. Then the quasi-pseudo-metric d_p constructed in Proposition 1 is subinvariant. Hence $(X, +, \mathcal{T}(d_p))$ is a topological monoid.

Proof. By Proposition 1, d_p is left subinvariant and hence it is also right subinvariant since the monoid X is Abelian. Therefore d_p is subinvariant. □

Proposition 3. *Let p be a quasi-norm on a left cancellative monoid $(X, +)$. Then d_p is a quasi-metric on X.*

Proof. Let $x, y \in X$ be such that $d_p(x, y) = d_p(y, x) = 0$. Then there exist $a, b \in X$ such that $x = y + a$, $y = x + b$ and $p(a) = p(b) = 0$. Since X is left cancellative and $x = x + a + b$, we have $a + b = 0$. Consequently $b = -a$ and thus $p(a) = p(-a) = 0$. Since p is a quasi-norm on X, $a = 0$, so $x = y$. The proof is complete. □

Remark 3. Note that if p is a quasi-norm on a group $(X, +)$, then the quasi-metric d_p is defined by

$$d_p(x, y) = p(y - x) \wedge 1$$

for all $x, y \in X$.

Since some examples of spaces which appear in a natural way by modelling certain processes in Theoretical Computer Science can be considered as Abelian monoids endowed with subinvariant quasi-metrics (see Examples 4 and 5 below), and, on the other hand, the problem of bicompletion has a satisfactory solution in the setting of quasi-metric spaces ([12]), we will focus our attention to quasi-norms which induce quasi-metrics. Thus, in the light of Remark 2 and Proposition 3, we propose the following notion.

Definition 1. A *quasi-normed monoid* is a pair (X, p) such that X is a cancellative Abelian monoid and p is a quasi-norm on X.

At the end of this section we observe that a slight modification of the construction given in Proposition 1 permits us to obtain, as a particular case, the classical (quasi-)metric induced by a (quasi-)norm on a group (see Remark 4 below). To this end we need the use of extended quasi-pseudo-metrics (they satisfy the usual axioms for a quasi-pseudo-metric, except that we allow $d(x, y) = +\infty$).

Proposition 4. *Let p be a prenorm on a monoid $(X, +)$. Then the function e_p defined on $X \times X$ by*
$e_p(x, y) = \inf\{p(a) : y = x + a\}$ *if* $x \in X$ *and* $y \in x + X$,
$e_p(x, y) = +\infty$ *if* $x \in X$ *and* $y \notin x + X$
is a left subinvariant extended quasi-pseudo-metric on X.

Furthermore for each $x \in X$ and each $\varepsilon \in (0, 1)$, $B_{e_p}(x, \varepsilon) = x + \{y \in X : p(y) < \varepsilon\}$, and the left translations are $\mathcal{T}(e_p)$-open.

Remark 4. Note that if p is a (quasi-)norm on a group $(X, +)$, then the extended (quasi-)pseudo-metric e_p of Proposition 4, is the classical (quasi-)metric on X induced by p, i.e.

$$e_p(x, y) = p(y - x),$$

for all $x, y \in X$.

3. The bicompletion of a quasi-normed monoid

Let us recall that a *homomorphism* from a monoid $(X, +)$ to a monoid $(Y, \oplus)$ is a mapping $f : X \to Y$ such that $f(x + y) = f(x) \oplus f(y)$.

In the following we shall denote by $+$ the operation in both the monoids X and Y if no confusion arises.

Definition 2. An *isometry* from a quasi-normed monoid (X, p) to a quasi-normed monoid (Y, q) is a homomorphism $f : X \longrightarrow Y$ such that $q(f(x)) = p(x)$ for all $x \in X$.

Contrary to the quasi-metric case there are isometries on quasi-normed monoids which are not injective mappings (see Example 5 below).

Definition 3. Two quasi-normed monoids (X, p) and (Y, q) are said to be *isometric* if there is a bijective isometry $f : X \to Y$.

Proposition 5. *If (X, p) and (Y, q) are isometric quasi-normed monoids by an (bijective) isometry f, then the quasi-metric spaces (X, d_p) and (Y, d_q) are isometric by f.*

Proof. Let $x, y \in X$. If $f(y) \notin f(x) + Y$, then $d_q(f(x), f(y)) = d_p(x, y) = 1$. Otherwise, we have $d_q(f(x), f(y)) = q(z) \wedge 1$, where $f(y) = f(x)+z$. Since f is a bijection, there is a unique $a_z \in X$ such that $f(a_z) = z$. Hence $f(y) = f(x + a_z)$, so $y = x + a_z$. Therefore $d_q(f(x), f(y)) = q(f(a_z)) \wedge 1 = p(a_z) \wedge 1 = d_p(x, y)$. □

Definition 4. A quasi-normed monoid (X, p) is called bicomplete if d_p is a bicomplete quasi-metric on X.

Observe that the spaces of Examples 1 and 2 are bicomplete quasi-normed monoids.

By a subspace of a quasi-normed monoid (X, p) we mean a submonoid Y of X endowed with the restriction of p to Y.

Definition 5. Let (X, p) be a quasi-normed monoid. We say that a bicomplete quasi-normed monoid (Y, q) is a *bicompletion* of (X, p) if (X, p) is isometric to a subspace of (Y, q) that is dense in the metric space $(Y, (d_q)^s)$.

We shall prove that each quasi-normed monoid (X, p) has a bicompletion $(\widetilde{X}, \widetilde{p})$ such that any bicompletion of (X, p) is isometric to $(\widetilde{X}, \widetilde{p})$.

Let (X, d_p) be the quasi-metric space induced by (X, p). Denote by $\widehat{X}$ the set of all Cauchy sequences in the metric space $(X, (d_p)^s)$. Note that if $x := (x_n)_{n\in\mathbb{N}} \in \widehat{X}$, then for each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $(d_p)^s(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$, so $x_m \in x_n + X$ for all $n, m \geq n_0$.

Define a relation R on $\widehat{X}$ as follows: For each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ put $xRy \iff \lim_{n\to\infty}(d_p)^s(x_n, y_n) = 0$. Then R is in fact an equivalence relation on $\widehat{X}$.

Denote by $\widetilde{X}$ the quotient $\widehat{X}/R$. Thus $\widetilde{X} = \{[x] : x \in \widehat{X}\}$, where $[x] = \{y \in \widehat{X} : xRy\}$ for all $x \in \widehat{X}$.

For each $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in $\widehat{X}$ put $[x] + [y] = [x + y]$ where $x + y = (x_n + y_n)_{n\in\mathbb{N}}$. It is easy to see that these operations are well-defined. Then we have the following result.

Lemma 1. *Let (X, p) be a quasi-normed monoid. Then $(\widetilde{X}, +)$ is a cancellative Abelian monoid.*

Proof. Since $(X, +)$ is a cancellative Abelian monoid we have immediately that $(\widetilde{X}, +)$ is a cancellative Abelian semigroup with neutral element $[0] \in \widetilde{X}$. $\square$

Lemma 2. *Let (X, p) be a quasi-normed monoid and $x := (x_n)_{n\in\mathbb{N}} \in \widehat{X}$. Then:*

(1) $\lim_{n\to\infty} p(x_n)$ *exists and is finite.*

(2) $\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} p(y_n)$ *for all* $y \in [x]$.

Proof. (1) Given $x := (x_n)_{n\in\mathbb{N}} \in \widehat{X}$, then for each $\varepsilon \in (0, 1)$ there is $n_0 \in \mathbb{N}$ such that $(d_p)^s(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$. Hence, for each $n, m \geq n_0$ there exist a_{nm}, $b_{nm} \in X$ such that $x_m = x_n + a_{nm}$, $x_n = x_m + b_{nm}$, $p(a_{nm}) < \varepsilon$ and $p(b_{nm}) < \varepsilon$. Therefore $p(x_n) - p(x_m) = p(x_m + b_{nm}) - p(x_m) \leq p(x_m) + p(b_{nm}) - p(x_m) = p(b_{nm}) < \varepsilon$, and, similarly, $p(x_m) - p(x_n) \leq p(a_{nm}) < \varepsilon$.

Thus $|p(x_n) - p(x_m)| < \varepsilon$ for all $m, n \geq n_0$, so $(p(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers which is convergent with respect to the Euclidean metric, of course.

(2) Let $x := (x_n)_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}}$ be two elements of $\widehat{X}$ such that $y \in [x]$. We may asume that for each $n \in \mathbb{N}$, $y_n = x_n + a_n$ with $p(a_n) \to 0$. Then $p(y_n) = p(x_n + a_n) \leq p(x_n) + p(a_n)$. So $\lim_{n\to\infty} p(y_n) \leq \lim_{n\to\infty} p(x_n)$. Similarly we show that $\lim_{n\to\infty} p(x_n) \leq \lim_{n\to\infty} p(y_n)$. $\square$

In the light of the preceding lemma we may define a function $\widetilde{p} : \widetilde{X} \to \mathbb{R}^+$ given by $\widetilde{p}([x]) = \lim_{n\to\infty} p(x_n)$ for all $x \in \widehat{X}$.

In Lemma 5 below we shall show that $(\widetilde{X}, \widetilde{p})$ is a bicomplete quasi-normed monoid.

Lemma 3. *Let X and Y be two Abelian monoids. If A is a submonoid of X and $f : A \to Y$ is a homomorphism, then $f(A)$ is a submonoid of Y.*

Lemma 4. *Let (X, p) be a quasi-normed monoid and let $x := (x_n)_{n\in\mathbb{N}}$ and $a := (a_n)_{n\in\mathbb{N}}$ be two elements of $\widehat{X}$. Then for each $y \in [x] + [a]$, $\lim_{n\to\infty} d_p(x_n, y_n) = \widetilde{p}([a]) \wedge 1$, where $y := (y_n)_{n\in\mathbb{N}}$.*

PROOF. Let $y \in [x] + [a]$. Then $\lim_{n\to\infty}(d_p)^s(y_n, x_n + a_n) = 0$. Hence there exist two sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ in X and an $n_0 \in \mathbb{N}$ such that $x_n + a_n = y_n + b_n$ and $y_n = x_n + a_n + c_n$ for all $n \geq n_0$, and $p(b_n) \to 0$ and $p(c_n) \to 0$.

Since X is cancellative, we deduce that $b_n = -c_n$ for all $n \geq n_0$. So

$$p(a_n) \leq p(a_n + c_n) + p(-c_n) \leq p(a_n) + p(c_n) + p(-c_n)$$

for all $n \geq n_0$. Consequently $\lim_{n\to\infty} p(a_n) = \lim_{n\to\infty} p(a_n + c_n)$.

Finally, since $d_p(x_n, y_n) = p(a_n + c_n) \wedge 1$ for all $n \geq n_0$, we obtain

$$\lim_{n\to\infty} d_p(x_n, y_n) = (\lim_{n\to\infty} p(a_n + c_n)) \wedge 1 = (\lim_{n\to\infty} p(a_n)) \wedge 1 = \widetilde{p}([a]) \wedge 1. \quad \square$$

Lemma 5. *Let (X, p) be a quasi-normed monoid. Then the following statements hold:*

(1) *$(\widetilde{X}, \widetilde{p})$ is a bicomplete quasi-normed monoid.*

(2) *(X, p) is isometric to a subspace of $(\widetilde{X}, \widetilde{p})$ that is dense in the metric space $(\widetilde{X}, (d_{\widetilde{p}})^s)$.*

PROOF. (1) The cancellative Abelian monoid condition of $(\widetilde{X}, \widetilde{p})$ follows from Lemma 1.

Let $x := (x_n)_{n\in\mathbb{N}}$ be an element of $\widehat{X}$ such that $-[x] \in \widetilde{X}$ and $\widetilde{p}([x]) = \widetilde{p}(-[x]) = 0$. Then $\lim_{n\to\infty} p(x_n) = 0 = \lim_{n\to\infty} p(-x_n)$.

Since $d_p(0, x_n) = p(x_n)$ and $d_p(x_n, 0) = p(-x_n)$ eventually, it follows that $\lim_{n\to\infty}(d_p)^s(0, x_n) = 0$. Thus $[x] = [0]$.

Now let $x := (x_n)_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}}$ be two elements of $\widehat{X}$. In order to show the triangle inequality we consider $p(x_n + y_n) \leq p(x_n) + p(y_n)$, so $\lim_{n\to\infty} p(x_n + y_n) \leq \lim_{n\to\infty} p(x_n) + \lim_{n\to\infty} p(y_n)$. Therefore $\widetilde{p}([x] + [y]) \leq \widetilde{p}([x]) + \widetilde{p}([y])$.

Hence $\widetilde{p}$ is a quasi-norm on $\widetilde{X}$. By Proposition 3, $d_{\widetilde{p}}$ is a quasi-metric on X.

It is well known ([1], [12]) that the bicompletion of the quasi-metric space (X, d_p) is a quasi-metric space $(X^b, (d_p)^b)$, where $X^b = \{[x] : x$ is a Cauchy sequence in the metric space $(X, (d_p)^s)\}$, $(d_p)^b([x], [y]) = \lim_{n\to\infty} d_p(x_n, y_n)$ for all $[x], [y] \in X^b$, and for each Cauchy sequence $x := (x_n)_{n\in\mathbb{N}}$ in $(X, (d_p)^s)$, $[x] = \{y := (y_n) : y$ is a Cauchy sequence in $(X, (d_p)^s)$ and $\lim_{n\to\infty}(d_p)^s(x_n, y_n) = 0\}$.

It is clear that $X^b = \widetilde{X}$.

Next we prove that $d_{\widetilde{p}} = (d_p)^b$ on $\widetilde{X} \times \widetilde{X}$.

Indeed, let $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ be two elements of $\widehat{X}$. We shall distinguish two cases.

Case 1. $[y] \in [x] + \widetilde{X}$. Then there is $a := (a_n)_{n\in\mathbb{N}} \in \widehat{X}$ such that $[y] = [x] + [a]$ and thus $d_{\widetilde{p}}([x],[y])) = \widetilde{p}([a]) \wedge 1$. From Lemma 4 it follows that $d_{\widetilde{p}}([x],[y])) = (d_p)^b([x],[y])$.

Case 2. $[y] \notin [x] + \widetilde{X}$. Then $d_{\widetilde{p}}([x],[y]) = 1$. Suppose that $(d_p)^b([x],[y]) < 1$. Therefore, there exist a sequence $(a_n)_{n\in\mathbb{N}}$ in X and an $n_0 \in \mathbb{N}$ such that $y_n = x_n + a_n$ for all $n \geq n_0$.

We shall show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,(d_p)^s)$: Let $\varepsilon \in (0,1)$. Then there is $n_\varepsilon \geq n_0$ such that $(d_p)^s(x_n,x_m) < \varepsilon/2$ and $(d_p)^s(y_n,y_m) < \varepsilon/2$ for all $n,m \geq n_\varepsilon$. So for each $n,m \geq n_\varepsilon$ there exist two elements t_{nm} and t_{mn} of X such that $x_m = x_n + t_{nm}$, $x_n = x_m + t_{mn}$, $p(t_{nm}) < \varepsilon/2$ and $p(t_{mn}) < \varepsilon/2$. Since X is cancellative, $t_{mn} = -t_{nm}$. Similarly, there exists $s_{nm} \in X$ such that $y_m = y_n + s_{nm}$, $y_n = y_m - s_{nm}$, $p(s_{nm}) < \varepsilon/2$ and $p(-s_{nm}) < \varepsilon/2$. Therefore

$$x_n + t_{nm} + a_m = x_m + a_m = y_m = x_n + a_n + s_{nm},$$

and, hence, $a_m = a_n + s_{nm} - t_{nm}$. Then

$$d_p(a_n,a_m) = p(s_{nm} - t_{nm}) \wedge 1 \leq (p(s_{nm}) + p(-t_{nm})) \wedge 1 < \varepsilon.$$

Similarly, we obtain that $d_p(a_m,a_n) < \varepsilon$, so $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,(d_p)^s)$, and thus $[y] = [x] + [a]$, where $a := (a_n)_{n\in\mathbb{N}}$, which contradicts our assumption. We conclude that $(d_p)^b([x],[y]) = 1$.

Therefore $d_{\widetilde{p}} = (d_p)^b$ and thus $d_{\widetilde{p}}$ is a bicomplete quasi-metric on $\widetilde{X}$. We have shown that $(\widetilde{X},\widetilde{p})$ is a bicomplete quasi-normed monoid.

(2) For each $x \in X$ denote by $\widehat{x}$ the constant sequence $x,x,\ldots,x,\ldots$. Since $(X^b,(d_p)^b)$ is the bicompletion of (X,d_p), $i(X)$ is dense in $(\widetilde{X},(d_{\widetilde{p}})^s)$ where i denotes the one-to-one mapping from X to $\widetilde{X}$ given by $i(x) = [\widehat{x}]$ for all $x \in X$. Note that $[\widehat{x}]$ consists of allxsequences in X which converge to x in the metric space $(X,(d_p)^s)$. It is routine to check i is a homomorphism, so by Lemma 3, $i(X)$ is a (cancellative) submonoid of $\widetilde{X}$. Since $\widetilde{p}(i(x)) = \widetilde{p}([\widehat{x}]) = p(x)$ for all $x \in X$, we deduce that (X,p) and $(i(X),\widetilde{p}_{|i(X)})$ are isometric quasi-normed monoids. The proof is complete. $\square$

Lemma 6. *Let (X,p) be a quasi-normed monoid, let (Y,q) a bicomplete quasi-normed monoid and let f be a one-to-one isometry from a submonoid A of X to Y such that A is dense in $(X,(d_p)^s)$. Then f extends uniquely to a one-to-one isometry from (X,p) to (Y,q).*

Proof. For each $x \in X\backslash A$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty}(d_p)^s(x,x_n)=0$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ (associated with $x \in X\backslash A$), is a Cauchy sequence in the metric space $(X,(d_p)^s)$, for each $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $(d_p)^s(x_n,x_m)<\varepsilon$ for all $m,n \geq n_0$. By Proposition 5, $(d_q)^s(f(x_n),f(x_m))<\varepsilon$ for all $m,n\geq n_0$. Therefore $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(Y,(d_q)^s)$, so it converges to a point $x^* \in Y$ with respect to the metric $(d_q)^s$.

Define $f^*: X \to Y$ by $f^*(x)=f(x)$ for all $x\in A$ and $f^*(x)=x^*$ for all $x \in X\backslash A$.

Observe that the definition of f^* is independent of the choice of sequences $(x_n)_{n\in\mathbb{N}}$. Indeed if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in A that converge to a point $x \in X\backslash A$ with respect to the metric $(d_p)^s$, and denote by x^* and y^* the limit points in $(Y,(d_q)^s)$ of $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ respectively, we deduce that $\lim_{n\to\infty}(d_q)^s(f(x_n),f(y_n))=0$, since $\lim_{n\to\infty}(d_p)^s(x_n,y_n)=0$. Therefore $x^*=y^*$.

Next we show that f^* is an isometry on (X,p). Let $x\in A$, then $q(f^*(x))=q(f(x))=p(x)$. Now let $x\in X\backslash A$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\lim_{n\to\infty}(d_p)^s(x,x_n)=0$. Thus $\lim_{n\to\infty}(d_q)^s(f^*(x),f(x_n))=0$, so by Proposition 2, $\lim_{n\to\infty} q(f(x_n)) = q(f^*(x))$. Hence for each $\varepsilon\in(0,1)$, $q(f^*(x))\leq q(f(x_n))+\varepsilon = p(x_n)+\varepsilon$ eventually. Therefore, for each $\varepsilon \in (0,1)$, $q(f^*(x))\leq p(x)+2\varepsilon$ because $\lim_{n\to\infty}p(x_n)=p(x)$ by Proposition 2. Similarly we show that for each $\varepsilon\in(0,1)$, $p(x)\leq q(f^*(x))+2\varepsilon$. Consequently $q(f^*(x))=p(x)$ for all $x\in X$.

Moreover f^* is a homomorphism on X. Let $x,y\in X$. We only consider the case that $x,y\in X\backslash A$ (recall that f is a homomorphism on A). Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in A that converge to x and y respectively in the metric space $(X,(d_p)^s)$. Since $(X,+)$ is Abelian, $(x_n+y_n)_{n\in\mathbb{N}}$ converges to $x+y$ with respect to $(d_p)^s$, so by definition of f^*, $(f(x_n+y_n))_{n\in\mathbb{N}}$ converges to $f^*(x+y)$ with respect to $(d_q)^s$. Since f is a homomorphism on A, the sequence $(f(x_n)+f(y_n))_{n\in\mathbb{N}}$ converges to

$f^*(x+y)$ with respect to $(d_q)^s$. On the other hand, by definition of f^*, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f^*(x)$ and $(f(y_n))_{n\in\mathbb{N}}$ converges to $f^*(y)$ with respect to $(d_q)^s$. So $(f(x_n)+f(y_n))_{n\in\mathbb{N}}$ converges to $f^*(x)+f^*(y)$ with respect to the metric $(d_q)^s$. Therefore $f^*(x+y)=f^*(x)+f^*(y)$.

Furthermore f^*, is injective on X. Indeed, let $x,y\in X\setminus A$ such that $f^*(x)=f^*(y)$. Then we have that $\lim_{n\to\infty}(d_q)^s(f(x_n),f(y_n))=0$, where $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in A such that $\lim_{n\to\infty}(d_p)^s(x,x_n)=\lim_{n\to\infty}(d_p)^s(y,y_n)=0$. Since f is one-to-one, it follows from Lemma 3 and Proposition 5 that $\lim_{n\to\infty}(d_p)^s(x_n,y_n)=0$. So, by the triangle inequality $(d_p)^s(x,y)\leq \lim_{n\to\infty}(d_p)^s(x,x_n)+\lim_{n\to\infty}(d_p)^s(x_n,y_n)+$ $\lim_{n\to\infty}(d_p)^s(y_n,y)$=0.

Hence $x=y$. So f^* is injective on X. We conclude that f^* is a one-to-one isometry from (X,p) to (Y,q).

Finally, suppose that $\widetilde{f}:X\to Y$ is another one-to-one isometry which is an extension of f to X.

Let $x\in X\setminus A$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to x with respect to $(d_p)^s$.

Then $\lim_{n\to\infty}(d_q)^s(f^*(x),f^*(x_n))=\lim_{n\to\infty}(d_q)^s(\widetilde{f}(x),\widetilde{f}(x_n))=0$. Since $f^*(x_n)=\widetilde{f}(x_n)=f(x_n)$ for all $n\in\mathbb{N}$, it follows that $f^*(x)=\widetilde{f}(x)$. So f^* is unique. □

Lemma 7. *Any bicompletion of a quasi-normed monoid (X,p) is isometric to $(\widetilde{X},\widetilde{p})$.*

Proof. Let (Y,q) be a bicompletion of (X,p). Let f be the one-to-one isometry from (X,p) to $(\widetilde{X},\widetilde{p})$ obtained in Lemma 5. Since X is dense in the metric space $(Y,(d_q)^s)$, it follows from the preceding lemma that f has a unique one-to-one isometry extension f^* to (Y,q). It remains to show that f^* maps Y onto $\widetilde{X}$. Indeed, let x be an arbitrary point of $\widetilde{X}$. Since $f(X)$ is dense in $(\widetilde{X},(d_{\widetilde{p}})^s)$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty}(d_{\widetilde{p}})^s(x,f(x_n))=0$. Thus $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\widetilde{X},(d_{\widetilde{p}})^s)$. Since f^* is an isometry, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y,(d_q)^s)$. Let $y\in Y$ be such that $\lim_{n\to\infty}(d_q)^s(y,x_n)=0$. Then $\lim_{n\to\infty}(d_{\widetilde{p}})^s(f^*(y),f^*(x_n))=0$, and so $f^*(y)=x$. This completes the proof. □

From the above lemmas we immediately deduce the following.

Theorem 1. *Each quasi-normed monoid* (X, p) *has a unique bicompletion (up to bijective isometry).*

4. Further examples

Example 4. Let Σ be a nonempty alphabet and let Σ^F be the set of all finite sequences ("words") over Σ. The elements of Σ are also called letters.

Denote by $\sqsubseteq$ the prefix order on Σ^F, i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y.

The space $(\Sigma^F, \sqsubseteq)$ appears in a natural way by modelling the streams of information in Kahn's model of parallel computation ([4], [15], [8]).

Now, for each $x \in \Sigma^F$ denote by $\ell(x)$ the length of x.

Now suppose that there exists an operation $+$ on Σ for which $(\Sigma, +)$ is an Abelian monoid with neutral element e.

Denote by e the infinite sequence such that $\mathsf{e}(k) = e$ for all $k \in \mathbb{N}$.

Now let $\ell(x)$ be the length of each $x \in \Sigma^F \cup \{\mathsf{e}\}$ (in particular $\ell(\mathsf{e}) = \omega$), and define an operation $\oplus$ on $\Sigma^F \cup \{\mathsf{e}\}$ as follows:

For each $x, y \in \Sigma^F \cup \{\mathsf{e}\}$, let $x \oplus y$ be the element of $\Sigma^F \cup \{\mathsf{e}\}$ of length $\ell(x \oplus y) = \min\{\ell(x), \ell(y)\}$ such that for each $k \leq \ell(x \oplus y)$, $(x \oplus y)(k) = x(k) + y(k)$.

Then $(\Sigma^F \cup \{\mathsf{e}\}, \oplus)$ is an Abelian monoid with neutral element e as it is shown in [11].

Let $p : \Sigma^F \cup \{\mathsf{e}\} \to \mathbb{R}^+$ be defined by $p(x) = 2^{-\ell(x)}$. It is easy to see that p is a quasi-norm on $\Sigma^F \cup \{\mathsf{e}\}$ for which d_p is a quasi-metric.

Example 5. Motivated by the applications to the analysis of complexity of programs and algorithms given in [14], the first author and M. Schellekens have introduced and studied the so-called dual complexity space ([10]), which consists of the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \Big\{ f \in (\mathbb{R}^+)^\omega : \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \Big\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ given by

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0].$$

Several properties of $d_{\mathcal{C}^*}$ are discussed in [10]. In particular observe that the topology induced by $d_{\mathcal{C}^*}$, is not T_1.

On the other hand, $(\mathcal{C}^*, +)$ is clearly a cancellative Abelian monoid with neutral element f_0 given by $f_0(n) = 0$ for all $n \in \omega$, where $+$ is the usual pointwise addition.

Let $w : \mathcal{C}^* \to \mathbb{R}^+$ be defined by $w(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$. It is routine to see that w is a quasi-norm on $\mathcal{C}^*$. Then the induced quasi-metric d_w on $\mathcal{C}^*$ is given by $d_w(f,g) = (\sum_{n=0}^{\infty} 2^{-n}(g(n) - f(n))) \wedge 1$ if $f \leq g$, and $d_w(f,g) = 1$ otherwise.

Some interesting properties of the quasi-metric d_w will be obtained in Propositions 6 and 7 below. In particular, it follows from Proposition 6 that d_w induces a Hausdorff topology on $\mathcal{C}^*$.

Next we show that it is possible to construct a noninjective isometry for quasi-normed monoids (see the comment following Definition 2).

Indeed, let $X = \{f \in \mathcal{C}^* : f(0) > 0\} \cup \{f_0\}$. It is routine to see that X is a submonoid of $\mathcal{C}^*$.

Define $q : X \to \mathbb{R}^+$ by $q(f) = f(0)$. Clearly q is a quasi-norm on X, so (X, q) is a quasi-normed monoid.

Let $F : X \to \mathcal{C}^*$ be defined by $F(f)(0) = f(0)$, and $F(f)(n) = 0$ for all $f \in X$ and $n \in \mathbb{N}$. Obviously F is a homomorphism from $(X,+)$ to $(\mathcal{C}^*,+)$. Moreover $w(F(f)) = \sum_{n=0}^{\infty} 2^{-n} F(f(n)) = f(0) = q(f)$ for all $f \in X$. So F is an isometry from (X,q) to $(\mathcal{C}^*, w)$.

However, if $f, g \in X$ satisfy $f(0) = g(0)$ and $f(1) \neq g(1)$, we obtain $F(f) = F(g)$, and thus F is not injective.

Balanced quasi-metric spaces were introduced by D. Doitchinov ([2]) in order to obtain a satisfactory theory of quasi-metric completion that, contrarily to bicompleteness, preserves (complete) regularity of quasi-metric spaces.

Let us recall that a quasi-metric space (X,d) is called balanced if given $r, s > 0$, $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ sequences in X such that $d(y_m, x_k) \to 0$ as $m, k \to \infty$, and points $x, y \in X$ such that $d(x, x_k) \leq r$ and $d(y_k, y) \leq s$ for all $k \in \mathbb{N}$, then $d(x,y) \leq r + s$.

It is well known that each balanced quasi-metric space is Hausdorff and completely regular ([2]).

The Sorgenfrey quasi-metric space is a paradigmatic example of a balanced quasi-metric space.

Proposition 6. *The quasi-metric space $(\mathcal{C}^*, d_w)$ is balanced.*

Proof. Let $r, s > 0$, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{C}^*$ such that $d_w(g_m, f_k) \to 0$ as $m, k \to \infty$, and $f, g \in \mathcal{C}^*$ such that $d_w(f, f_k) \leq r$ and $d_w(g_k, g) \leq s$ for all $k \in \mathbb{N}$. We may assume without loss of generality that $r + s < 1$. Thus $f \leq f_k$ and $g_k \leq g$ for all $k \in \mathbb{N}$. Moreover $g_m \leq f_k$ eventually.

We first note that $f \leq g$. Indeed, let $n_0 \in \mathbb{N}$. For an arbitrary $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} 2^{-n}(f_k(n) - g_k(n)) < \varepsilon$. Thus $f_k(n_0) - g_k(n_0) < 2^{n_0}\varepsilon$. Hence $f(n_0) \leq f_k(n_0) < 2^{n_0}\varepsilon + g_k(n_0) \leq 2^{n_0}\varepsilon + g(n_0)$. We deduce that $f(n_0) \leq g(n_0)$ for all $n_0 \in \mathbb{N}$, i.e. $f \leq g$.

Finally, choose $k \in \mathbb{N}$ such that $g_k \leq f_k$. Then

$$\begin{aligned} d_w(f, g) &= \sum_{n=0}^{\infty} 2^{-n}(g(n) - f(n)) \\ &\leq \sum_{n=0}^{\infty} 2^{-n}(g(n) - g_k(n)) + \sum_{n=0}^{\infty} 2^{-n}(f_k(n) - f(n)) \\ &= d_w(g_k, g) + d_w(f, f_k) \leq s + r. \end{aligned}$$

We conclude that $(\mathcal{C}^*, d_w)$ is a balanced quasi-metric space. $\square$

Since $(d_w)^s$ is the discrete metric on $\mathcal{C}^*$, $(\mathcal{C}^*, d_w)$ is a bicomplete quasi-metric space. We finish the paper by showing that this space is also right K-sequentially complete in the sense of [9].

Let us recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is right K-Cauchy provided that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m \geq n_0$. (X, d) is said to be right K-sequentially complete if every right K-Cauchy sequence is convergent with respect to $\mathcal{T}(d)$.

Right K-sequential completeness is an appropriate notion of quasi-metric completeness in the study of function spaces and hyperspaces ([6], [7], [13]).

Proposition 7. *The quasi-metric space $(\mathcal{C}^*, d_w)$ is right K-sequentially complete.*

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a right K-Cauchy sequence in $(\mathcal{C}^*, d_w)$. Then, there is $k_0 \in \mathbb{N}$ such that $d_w(f_m, f_k) < 1$ for $m \geq k \geq k_0$. Thus, for each

$k \geq k_0$ there is $h_k \in \mathcal{C}^*$ with $f_k = f_{k+1} + h_k$. Hence $f_{k+1} \leq f_k$ for all $k \geq k_0$.

Define a function $f : \omega \to [0, \infty)$ by

$$f(n) = \inf_{k \geq k_0} f_k(n) \quad \text{for all } n \in \omega.$$

We want to show that $f \in \mathcal{C}^*$ and that $(f_k)_{k \in \mathbb{N}}$ converges to f in $(\mathcal{C}^*, d_w)$. Indeed, let $\varepsilon > 0$. Since $f_{k_0} \in \mathcal{C}^*$ there is $n_\varepsilon \in \omega$ such that $\sum_{n=n_\varepsilon+1}^{\infty} 2^{-n} f_{k_0}(n) < \varepsilon/3$. So $\sum_{n=n_\varepsilon+1}^{\infty} 2^{-n} f(n) < \varepsilon/3$, and, hence, $f \in \mathcal{C}^*$.

Furthermore, since $f_k \leq f_{k_0}$ for all $k \geq k_0$, it follows that $\sum_{n=n_\varepsilon+1}^{\infty} 2^{-n} f_k(n) < \varepsilon/3$ for all $k \geq k_0$. By definition of f and the fact that $f_{k+1} \leq f_k$ for all $k \geq k_0$, there is $k_1 \geq k_0$ such that for each $k \geq k_1$, $2^{-n}(f_k(n) - f(n)) < \varepsilon/3$, $n = 0, 1, \ldots, n_\varepsilon$. Hence for $k \geq k_1$,

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{-n}(f_k(n) - f(n)) &\leq \sum_{n=0}^{n_\varepsilon} 2^{-n}(f_k(n) - f(n)) + \sum_{n=n_\varepsilon+1}^{\infty} 2^{-n} f_k(n) \\ &< \frac{\varepsilon}{3}\Big(\sum_{n=0}^{\infty} 2^{-n}\Big) + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

We have shown that $(f_k)_{k \in \mathbb{N}}$ converges to f in $(\mathcal{C}^*, d_w)$. Consequently $(\mathcal{C}^*, d_w)$ is right K-sequentially complete. □

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