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On p-nilpotency and complemented minimal subgroups of finite groups

By YANG GAOCAI (Shanxi)

Abstract. Let G be a finite group. A subgroup H of G is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In this paper, it is showed that a finite group G is p-nilpotent provided p is the smallest prime number dividing the order of G and every minimal subgroup of the p-focal subgroup of G is complemented in $N_G(P)$, where P is a Sylow p-subgroup of G. As some applications, some interesting results related with complemented minimal subgroups of focal subgroups are obtained.

1. Introduction

Recall that a subgoup H of a finite group G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. We also call the above subgroup K of G a complement of H in G. Complemented subgroups of a finite group plays an important role in the structure theory of finite groups, for instance, P. Hall [8] proved in 1937 that a finite group G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. Also, it is known that a finite group G is solvable if and only if every Sylow subgroup of G is complemented [9]. New criteria for the solvability of finite groups were obtained by Z. Arad and M. B. Ward in 1982. In particular, they have shown that a group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented [1].

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In 1960 Yu. M. Gorchakov studied torsion groups in which all minimal subgroups are complemented; in particular, a class of finite groups of such type is exactly the class of finite supersolvable groups with elementary abelian subgroups (see [16]). In a recent paper, A. Ballester-Bolinches and Xiuyun Guo [4] also investigated the class of finite groups for which every minimal subgroup is complemented. On the other hand, there has been much interest in the past in investigating the influence of minimal subgroups on the structure of finite groups (see, [2], [5], [7], [12], [15]).

Let G be a finite group. Let G' be the derived group of G and P a Sylow p-subgroup of G for a prime number p. Then the subgroup $P \cap G'$ is called the focal subgroup of P with respect to G.

In this paper, we shall continue the investigation on the influence of the existence of complements of minimal subgroups on the structure of finite groups. In particular, we devote ourselves to the minimal subgroups of the focal subgroups. Some interesting results are obtained.

Throughout this paper, all groups considered are finite groups. For notations and terminologies not given in this paper, the reader is referred to the text of D. J. S. Robinson [13].

2. Preliminaries

We first cite the following lemma as it will be useful later on. we also notice that a minimal subgroup of a group is a subgroup of prime order.

Lemma 2.1 ([4], Lemma 1)**.** *Let G be a group and N a normal subgroup of G. Then the following statements hold.*

(1) *If $H \leq K \leq G$ and H is complemented in G, then H is complemented in K.*

(2) *If N is contained in H and H is complemented in G, then H/N is complemented in G/N.*

(3) *Let π be a set of primes. If N is a π'-subgroup and A is a π-subgroup of G, then A is complemented in G if and only if AN/N is complemented in G/N.*

The following lemmas are crucial in proving our main results.

Lemma 2.2. *Let H be a subgroup of a group G. If every minimal subgroup of H is complemented in G, then $\Phi(H) = 1$.*

Proof. If $\Phi(H) \neq 1$, then there exists a minimal subgroup $\langle a \rangle$ of H such that $\langle a \rangle \leq \Phi(H)$. Since $\langle a \rangle$ is complemented in G, by definition, there exists a subgroup K of G such that $G = \langle a \rangle K$ and $K \cap \langle a \rangle = 1$. This leads to $H = \langle a \rangle (H \cap K)$. Also since $\langle a \rangle \leq \Phi(H)$, we have $H = H \cap K$. This implies that $H \leq K$, which contradicts to $K \cap \langle a \rangle = 1$. Hence $\Phi(H) = 1$. □

Lemma 2.3. *Let G be a group and p the smallest prime number dividing the order of G. If every minimal subgroup of G with order p is complemented in G, then G is p-nilpotent.*

Proof. Let $\langle a \rangle$ be a subgroup of order p in G. By our hypothesis, there is a subgroup K of G such that $G = \langle a \rangle K$ and $\langle a \rangle \cap K = 1$. Since $[G : K] = p$ and p is the smallest prime number dividing the order of G, we know that K is a normal subgroup of G. Observe that every subgroup of K with order p must be a minimal subgroup of G. Then, by Lemma 2.1 (1), every subgroup of K with order p has a complement in K. Using induction, we deduce that K has a normal p-complement T. It is clear that T is a Hall p'-subgroup of G and T is normal in G as well. Hence, G is p-nilpotent. The proof is completed. □

Note 2.4. The assumption that p is the smallest prime number dividing the order of G in Lemma 2.3 can not be removed. In fact, if we let $G = PSL(2, 7)$ with $p = 7$. Then it can be easily seen that G has subgroups with order 24 and therefore every subgroup of G with order 7 must have a complement in G. However G is not a 7-nilpotent group.

Lemma 2.3 can be strengthened by the following corollary.

Corollary 2.5. *Let G be a group and p be the smallest prime number dividing the order of G. If every minimal subgroup of P is complemented in $N_G(P)$, then G is p-nilpotent, where P is a Sylow p-subgroup of G.*

Proof. By Lemma 2.3, we know that $N_G(P)$ is p-nilpotent. This means that there exists a subgroup C of $N_G(P)$ such that $N_G(P) = P \times C$. It follows that $N_G(P) = C_G(P)$ since P is an elementary abelian group by Lemma 2.2. Hence, applying the well known Burnside Theorem [13, Theorem 10.1.8], we know that G is p-nilpotent. □

Note 2.6. Same as Note 2.4, we point out here that the assumption that p is the smallest prime number dividing the order of G in Corollary 2.5 can not be removed as well. In fact, if we let $G = A_5$, the alternating group of degree 5, then it is easy to see that $N_G(P)$ is a subgroup of G with order 10 for every Sylow 5-subgroup P of G. Hence every minimal subgroup of order 5 in P has a complement in $N_G(P)$ for Sylow 5-subgroup P of G. However, $G = A_5$ is simple, this leads to a contradiction.

We call a class of groups $\mathcal{F}$ is a formation provided that the following conditions are satisfed:

(1) $\mathcal{F}$ contains all homomorphic images of a group G in $\mathcal{F}$,

(2) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is also in $\mathcal{F}$, where M and N are normal subgroups of G.

Now, we let P be the set of all prime numbers. By a formation function f, we mean a function f defined on P such that $f(p)$, possibly empty, is a formation for any prime p. A principal factor H/K of a group G is called f-central in G if $G/C_G(H/K) \in f(p)$ for all prime numbers p dividing $|H/K|$. A formation $\mathcal{F}$ is said to be a local formation if there exists a formation function f such that $\mathcal{F}$ is the class of all groups G for which every principal factor of G is f-central in G. If $\mathcal{F}$ is a local formation defined by a formation function f, then we write $\mathcal{F} = LF(f)$ and call f a local definition of $\mathcal{F}$.

Among all possible local definitions for a local formation $\mathcal{F}$, there exists exactly one of them, denoted it by F, such that F is both integrated (i.e. $F(p) \subseteq \mathcal{F}$ for all $p \in P$) and full (i.e. $\mathcal{N}_p F(p) = F(p)$ for all $p \in P$).

A formation $\mathcal{F}$ is called saturated if $G/\Phi(G) \in \mathcal{F}$ implies that G belongs to $\mathcal{F}$. It is well known that a formation $\mathcal{F}$ is saturated if and only if $\mathcal{F}$ is a local formation [6].

Lemma 2.7 ([6], Proposition IV. 3.11)**.** *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where each F_i is both an integrated and full formation function of $\mathcal{F}_i$ $(i = 1, 2)$. Then the following statements are equivalent:*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,

(2) $F_1(p) \subseteq F_2(p)$ *for all* $p \in P$.

3. Main results

In this section, we concentrate on the structure of a finite group under the assumption that some minimal subgroups of focal subgroups are complemented.

First we prove the the following result about p-nilpotency.

Theorem 3.1. *Let G be a group and p the smallest prime number dividing the order of G. If every minimal subgroup of the focal subgroup $P \cap G'$ is complemented in $N_G(P)$, then G is p-nilpotent, where P is a Sylow p-subgroup of G.*

Proof. Assume that the theorem is not true and let G be a counterexample of the smallest order. Then we prove the theorem through the following steps.

Step (i) we first claim that $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Then we let $\overline{G} = G/O_{p'}(G)$ and $\overline{P} = (PO_{p'}(G))/\, O_{p'}(G)$. It is easy to see that $N_{\overline{G}}(\overline{P}) {=} (N_G(P)O_{p'}(G))/O_{p'}(G)$ and $(\overline{G})' = (G'O_{p'}(G))/O_{p'}(G)$, and therefore $\overline{P} \cap (\overline{G})' = (P \cap G')O_{p'}(G)/ O_{p'}(G)$. It is also clear that for any minimal subgroup $\overline{A}$ of $\overline{P} \cap (\overline{G})'$, there exists a minimal subgroup A of $(P \cap G')$ such that $\overline{A} = (AO_{p'}(G))/O_{p'}(G)$. However, by our hypothesis, A is complemented in $N_G(P)$. Hence, there exists a subgroup K of $N_G(P)$ such that $N_G(P) = AK$ and $A \cap K = 1$. It is clear that $(N_G(P)O_{p'}(G))/O_{p'}(G) {=} (AO_{p'}(G)/O_{p'}(G))(KO_{p'}(G)/O_{p'}(G))$. If $A \cap (KO_{p'}(G) \neq 1$, then $A \leq KO_{p'}(G)$ and therefore $N_G(P)O_{p'}(G) = KO_{p'}(G)$. But since $|P| \mid |N_G(P)O_{p'}(G)|$ and $|P| \nmid |KO_{p'}(G)|$, we obtain a contradiction. Hence $\overline{A} \cap \overline{K} = 1$ and so the hypothesis of the theorem is true for $\overline{G}$. The minimality of G implies that $\overline{G}$ is p-nilpotent and therefore G is p-nilpotent, a contradiction. Thus our claim is established.

Step (ii) We next prove that $N_G(P)$ is p-nilpotent.

In fact, if $N_G(P) = G$, then $P \lhd G$. By applying the well known Schur–Zassenhaus Theorem, there exists a Hall p'-subgroup K of G such that $G = PK$. For any prime $q \in \pi(K)$ and $Q \in \mathrm{Syl}_q(K)$, it is easy to know that the group $G_1 = PQ$ satisfies the hypothesis of our theorem. Hence, if $G_1 < G$, then by the minimality of G, we know that G_1 is p-nilpotent. Consequently, K is a normal p-complement of G, which is a contradiction. This shows that K is a q-group for some prime q. Now, since G is solvable, this implies that $G' < G$. Let T/G' be a Sylow q-subgroup of G/G'. Then $P \cap G'$ is a Sylow p-subgroup of T and every minimal subgroup of $P \cap G'$ is complemented in T by Lemma 2.1 (1). If $P \cap G' = 1$, then T is a normal p-complement of G. On the other hand, if $P \cap G' \neq 1$, then by Lemma 2.3, T has a normal p-complement N. Since $T/G' \lhd G/G'$, it is easy to know that N is a normal p-complement of G, a contradiction. Thus, we conclude that $N_G(P) < G$. As $N_G(P)$ satisfies the hypothesis of theorem. The minimality of G implies that $N_G(P)$ is p-nilpotent and thereby step (ii) is true.

Step (iii) We now claim that $P \cap G' \leq Z(N_G(P))$, where $Z(N_G(P))$ is the center of $N_G(P)$.

In fact, by the hypothesis of theorem and Lemma 2.2, we know that $P \cap G'$ is an elementary abelian group. If $P \cap G' = 1$, then there is nothing to be proved. Now, since $P \cap G' \lhd P$, we assume that N_1 is a minimal normal subgroup of P and $N_1 \leq P \cap G'$. By the properties of nilpotent groups, we have that $N_1 \leq Z(P)$ and $|N_1| = p$. Also by our hypothesis and Lemma 2.1 (1), there is a subgroup K of P such that $P = N_1K$ and $N_1 \cap K = 1$. Noticing that $(P \cap G') \cap K$ is still a normal subgroup of P, therefore, by using similar arguments, we can prove that $P \cap G' = N_1 \times N_2 \times \cdots \times N_s$ and $N_i \leq Z(P)$. This shows that $P \cap G' \leq Z(P)$. By step(ii), $N_G(P)$ is p-nilpotent, and therefore $P \cap G' \leq Z(N_G(P))$. This establishes our claim.

Step (iv) Our final step is to prove our theorem.

In fact, since G is not p-nilpotent, G has a subgroup H such that H is a minimal non-p-nilpotent group (that is, H is not p-nipotent but every proper subgroup of H is p-nilpotent). By a result of Itô [13, Theorem 10.3.3], we know that H is a minimal non-nilpotent group. According to a result due to Schmidt [13, Theorem 9.1.9 and Exercises 9.1.11], H has a normal Sylow p-subgroup H_p such that $H = H_pH_q$ for a Sylow q-subgroup H_q in $H(q \neq p)$. Moreover, $H_p = [H_p, H_q]$. Hence, it follows that $H_p \leq H' \leq G'$. On the other hand, without loss of generality, we may assume that H_p is contained in P. Hence $H_p \leq P \cap G'$.

Let $A = N_G(H_p)$. Since $H_p \leq P \cap G'$ and $P \cap G' \leq Z(N_G(P))$ (Step (iii)), we have H_p is centralized by $N_G(P)$. In particular, $P \leq C_G(H_p)$. As $C_G(H_p) \lhd N_G(H_p) = A$ and $P \in \mathrm{Syl}_p(C_G(H_p))$, we have, by the Frattini argument,

$$A = N_G(H_p) = C_G(H_p)N_A(P).$$

Since $H_p \leq Z(N_G(P))$ and $N_A(P) \leq N_G(P)$, we have $N_A(P) \leq C_G(H_p)$. It follows that $N_G(H_p) = C_G(H_p)$ and therefore $H = H_p \times H_q$, which is a contradiction. This proves the theorem. $\square$

Remark 3.2. By our Note 2.6, we notice that the requirement that p is the smallest prime dividing the order of G in Theorem 3.1 can not be removed.

Corollary 3.3. *Let G be a group. If every minimal subgroup of $P \cap G'$ is complemented in $N_G(P)$ for every Sylow subgroup P of G, then G has a Sylow tower of supersolvable type.*

As an application of Theorem 3.1, we prove the following

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of supersolvable groups. Let H be a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If for every Sylow subgroup P of H, every minimal subgroup of $P \cap G'$ is complemented in $N_G(P)$, then G is in $\mathcal{F}$.*

Proof. Since $\mathcal{U}$ and $\mathcal{F}$ are saturated formations, we can let F_i $(i = 1, 2)$ be the full and integrated formation function such that $\mathcal{U} = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively. If the theorem is false, then we can let G be a minimal counterexample. Then by Lemma 2.1 and Corollary 3.3, the normal subgroup H of G has a Sylow tower of supersolvable type. Let p be the largest prime number in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Then P must be a normal subgroup of G. Now let $\overline{G} = G/P$ and $\overline{H} = H/P$. Clearly, $\overline{G}/\overline{H} \simeq G/H \in \mathcal{F}$. Observe that $N_{\overline{G}}(\overline{Q}) = N_G(Q)P/P$ for every Sylow q-subgroup $\overline{Q} = QP/P$ of $\overline{H}$, where $Q \in \mathrm{Syl}_q(H)(q \neq p)$, and $(\overline{G})' = G'P/P$, we know that, for every element $\overline{x}$ of order q in $\overline{Q} \cap (\overline{G})'$, $\overline{x} = xP$ for some element $x \in Q \cap G'$. Thus, by our hypothesis, there exists a subgroup K of $N_G(Q)$ such that $N_G(Q) = \langle x \rangle K$ and $\langle x \rangle \cap K = 1$. It is clear that $N_{\overline{G}}(\overline{Q}) = \overline{\langle x \rangle K}$. If $\langle x \rangle \cap KP \neq 1$, then $\langle x \rangle \leq KP$ and therefore $N_G(Q)P = KP$. It follows that $|N_G(Q)|\,|P|/|N_G(Q) \cap P| = |K|\,|P|/|K \cap P|$. But $|Q| \mid (|N_G(Q)|\,|P|/|N_G(Q) \cap P|)$ and $|Q| \nmid (|K|\,|P|/|K \cap P|)$, which is a contradiction. Hence $\langle x \rangle \cap KP = 1$, and so $\overline{\langle x \rangle} \cap \overline{K} = 1$. Now we have proved that G/P satifies the hypothesis of the theorem. Thereby, by the minimality of G, we have $G/P \in \mathcal{F}$.

Since G/G' is abelian and $\mathcal{U}$ is contained in $\mathcal{F}$, we have $G/G' \in \mathcal{F}$. It follows that $G/(G' \cap P) \in \mathcal{F}$ and, by our hypothesis, we know that every minimal subgroup of $G' \cap P$ is complemented in G since P is normal in G. By Lemma 2.2, $G' \cap P$ is an elementary abelian subgroup. Now, let N be a minimal normal subgroup of G such that $N \leq G' \cap P$. Then, it can be easily proved that N is a cyclic group of order p since every minimal subgroup of N is complemented in G. We now denote with bars the images in $\overline{G} = G/N$. Then, $\overline{G}$ has a normal subgroup $\overline{G' \cap P}$ such that $\overline{G}/\overline{G' \cap P}$ belongs to $\mathcal{F}$. Obviously, $(\overline{G})' \cap \overline{G' \cap P} = (G' \cap P)/N$ and $\overline{G' \cap P} \lhd \overline{G}$. We now proceed to prove that every minimal subgroup of $(G' \cap P)/N$ is complemented in $\overline{G}$. For this purpose, we let $\overline{\langle x \rangle}$ be a minimal subgroup of $\overline{G' \cap P}$. Since $G' \cap P$ is an elementary abelian group, we know that there is an element $x \in G' \cap P$ with order p such that $\overline{\langle x \rangle} = \langle x \rangle N/N$. Since $\langle x \rangle$ is minimal in G and so by the hypothesis, there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = 1$. If $N \leq K$, then it is clear that $\overline{G} = \overline{\langle x \rangle}\,\overline{K}$ and $\overline{\langle x \rangle} \cap \overline{K} = 1$. If $N \nleq K$, then $G = NK$ and $N \cap K = 1$. It follows that $|(\langle x \rangle N) \cap K| = p$. Denote $(\langle x \rangle N) \cap K = A$. Then A is a minimal subgroup of $G' \cap P$ and $A \leq K$. By Lemma 2.1 (1), there is a subgroup K_1 of K such that $K = AK_1$ and $A \cap K_1 = 1$. It is clear that $AN = \langle x \rangle N$ and therefore $\overline{G} = \overline{\langle x \rangle}\,\overline{K_1}$. We now claim that $\overline{\langle x \rangle} \cap \overline{K_1} = 1$. For if not, then we have $|(\langle x \rangle N) \cap K_1| \geq p$. This implies that $|N|\,|K| = |G| = |\langle x \rangle N K_1| \leq (|\langle x \rangle N|\,|K_1|)/p = (|N|\,|K|)/p$, a contradiction. Hence, $\overline{G}$ satisfies the hypothesis of the theorem. By minimality of G, we have that $\overline{G} = G/N \in \mathcal{F}$.

Now, since N is a cyclic group of order p, $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$. Also, since $G/C_G(N) \leq \mathrm{Aut}(N)$, by Lemma 2.7, we have $G/C_G(N) \in F_1(p) \subseteq F_2(p)$ and therefore $G \in \mathcal{F}$, a contradiction. The proof of the theorem is now completed. $\square$

Remark 3.5. Let $\mathcal{F}$ be the class of groups G with G' nilpotent (or $\mathcal{U}$, the class of supersolvable groups). It is easy to see that $\mathcal{F}$ is a saturated formation containing the class $\mathcal{U}$. Thus, by Theorem 3.4, we can see that $G \in \mathcal{F}$ if $G/H \in \mathcal{F}$ and all minimal subgroups of $P \cap G'$ are comlemented in $N_G(P)$ for every Sylow subgroup P of H.

Remark 3.6. We remark here that Theorem 3.4 is not true if the saturated formations $\mathcal{F}$ does not contain $\mathcal{U}$ (the class of supersolvable groups). For example, if we let $\mathcal{F}$ be the saturated formation of all niplotent groups, then the symmetric group of degree three is a counterexample.

If G is assumed to be a solvable group, then the number of complemented minimal subgroups in Theorem 3.4 can be further reduced. In fact, we have the following theorem.

Theorem 3.7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of supersolvable groups. Let H be a normal subgroup of a solvable group G such that $G/H \in \mathcal{F}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ has a complement in G, then G belongs to $\mathcal{F}$.*

Remark 3.8. Since $F(G' \cap H) = G' \cap F(H) = (G' \cap P_1) \times (G' \cap P_2) \times \cdots \times (G' \cap P_k)$, we know that every minimal subgroup of $F(G' \cap H)$ in Theorem 3.7 is still a minimal subgroup of some focal subgroup $G' \cap P_i$, where P_i is the Sylow p_i-subgroup of $F(H)$ for some prime p_i.

Proof of Theorem 3.7. Assume that the theorem is false and let G be a counterexample of the smallest order. Since G/G' is abelian, we have that $G/G' \in \mathcal{F}$ and so $G/(H \cap G') \in \mathcal{F}$. Hence, we can prove our theorem by replacing $G' \cap H$ by H and assume that $H \leq G'$.

We first prove that $\Phi(G) = 1$. If $\Phi(G) \neq 1$, then there is a prime number q dividing the order of $\Phi(G)$ and $Q \in \mathrm{Syl}_q(\Phi(G))$. Since Q is a characteristic subgroup of $\Phi(G)$ and $\Phi(G) \lhd G$, we know that Q is a normal subgroup of G. Observe that $(G/Q)' = G'Q/Q$, so we still have $HQ/Q \leq (G/Q)'$. Clearly, $(G/Q)/(HQ/Q) \simeq G/HQ \in \mathcal{F}$. By a result [10, Satz 3.5, P270], $F(HQ/Q) = F(HQ)/Q$ and therefore by [3, Lemma 3.1], we have $F(HQ) = F(H)Q$. It follows that $F(HQ/Q) = F(H)Q/Q$. Thus, for any minimal subgroup $\overline{A}$ of $F(HQ/Q)$, we can find a minimal subgroup $A \leq F(H)$ such that $\overline{A} = AQ/Q$. By the hypothesis of the theorem, there exists a subgroup K of G such that $G = AK$ and $A \cap K = 1$. The minimality of A implies that K has a prime index in G and so K is a maximal subgroup of G. It follows that $Q \leq K$ and therefore $(K/Q) \cap (AQ/Q) = 1$. It is clear that $G/Q = (AQ/Q)K/Q$. Thus, we have shown that G/Q satisfies the hypothesis of the theorem. The minimality of G implies that $G/Q \in \mathcal{F}$. Hence $G \in \mathcal{F}$ since $Q \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, a contradiction. Thus $\Phi(G) = 1$.

Next, by applying a result of Deyu Li and Xiuyun Guo in [11, Lemma 2.3], we deduce that

$$F(G) = M_1 \times M_2 \times \cdots \times M_s \times N_1 \times N_2 \times \cdots \times N_t$$

where M_i and N_j $(i = 1, 2, \dots, s,\ j = 1, 2, \dots, t)$ are minimal normal subgroups of G, $M_i \cap H = 1$ and $F(H) = N_1 \times \cdots \times N_t$.

Now let $F_i (i = 1, 2)$ be the full and integrated formation functions such that $\mathcal{U} = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively. Then for any minimal normal subgroup N of G, we have that $N \leq F(G)$. If $N \cap H = 1$, we can assume that $N = M_i$. Since $M_i \simeq M_iH/H$ and $G/H \in \mathcal{F}$, we know that $(G/H)/C_{G/H}(M_iH/H) \in F_2(p)$ if $|M_i| = p^{\alpha}$. By [14, Appendix B: Theorem 2 and Theorem 3], we have $G/C_G(M_i) \simeq (G/H)/C_{G/H}(M_iH/H)$. Hence $G/C_G(M_i) \in F_2(p)$. If $N \leq H$, then $N \leq F(H)$ and without loss of generality, we may assume that $N = N_j$. Let A be a minmal subgroup of N_j. Then, by our hypothesis, there exists a subgroup K of G such that $G = AK$ and $A \cap K = 1$. As $N_j = A(N_j \cap K)$ and $N_j \cap K \lhd K$, we have that $N_j \cap K$ is a normal subgroup of G since N_j is abelian. Thus, the minmality of N_j implies that $N_j \cap K = 1$ and therefore $N_j = A$ is a cyclic group of order prime. Since $G/C_G(N_j)$ is isomorphic to a subgroup of $\mathrm{Aut}(N_j)$ and $\mathrm{Aut}(N_j)$ itself is a cyclic group, we have $G/C_G(N_j) \in F_1(p)$, where $p = |N_j|$. Now, by Lemma 2.7, we have $G/C_G(N_j) \in F_2(p)$. Hence, for every minimal normal subgroup N of G, if $|N| = p^{\alpha}$, we have $G/C_G(N) \in F_2(p)$ and thereby $G \in \mathcal{F}$ by Lemma 5.1.13 in [14]. The proof of the theorem is now completed. $\square$

Remark 3.9. A subgroup H of a finite group G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G = core_G(H)$. According to the referee's suggestion, we may give the following example. Let $G = S_4$ and P a Sylow 2-subgroup of G. It is clear that $N_G(P) = P$ and every minimal subgroup of $P \cap G'$ is c-supplemented in $N_G(P) = P$. But G is not 2-nilpotent. Also if Q is a Sylow 3-subgroup of G, then it is easy to know that every minimal subgroup of $Q \cap G'$ is c-supplemented in $N_G(Q)$. But G is not a Sylow tower group. Hence our results (Theorem 3.1 and Theorem 3.4) are not true if we replace complementation by c-supplementation.

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YANG GAOCAI
DEPARTMENT OF MATHEMATICS
JINZHONG TEACHER'S COLLEGE
YUCI, SHANXI
P. R. CHINA

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