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Constant regression of permanental statistics on the sum of sample elements

By B. GYIRES (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Let n and m be arbitrary positive integers. Let the entries of matrix

$$X = (X_{jk})_{j,k=1}^{n} \tag{1}$$

be identically and independently distributed random variables with moment of order m. Denote by $f(t)$ their common characteristic function, i.e., let

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x), \quad t \in \mathbb{R}, \tag{2}$$

where $F(x)$ is the common distribution function of the random variables (1). Then the polynomial statistics ([2], p.110)

$$P(X) = \operatorname{Per}\left(X_{jk}^{m}\right)_{j,k=1}^{n}$$

of degree m and order nm is said to be permanental ([3], 1.1) statistics of degree m.

The purpose of this paper is to prove two theorems, which contain statements about the constant regression of permanental statistics on the sum of the sample elements.

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Theorem 1. *Let $n \geq 2$ be an integer. Let the entries of matrix X be independently and identically distributed random variables. Suppose that X_{11} has moment of order m, where m is positive integer. Then $P(X)$ has constant regression on*

$$Y = \sum_{j,k=1}^{n} X_{jk} \tag{3}$$

([2], 6.2) *if and only if X_{11} is a degenerated distributed random variable, which has a single jump of magnitude one at the origin.*

Theorem 2. *Let n be an arbitrary positive integer. Let X be a random variable with moment of order m. Then random variable X^m has constant regression on X if and only if the distribution function of X is an arbitrary degenerated distribution function.*

Proof. In order to have the permanental statistics $P(X)$ constant regression on linear statistics (3), it is necessary and sufficient the satisfaction of the identity

$$E(Pe^{itY}) = E(P)E(e^{itY}), \quad t \in \mathbb{R} \tag{4}$$

([2], Theorem 6.1.1), where

$$E(P) = \operatorname{Per}\,(E(X_{jk}^m))_{j,k=1}^n$$

by the idependency of the entries of matrix (1). It is known ([1], 787 (2)) that the left hand side of (4) can be written in the form

$$\sum i^{-nm} f_{1i_1}(t) \dots f_{ni_n}(t) = i^{-nm} \operatorname{Per}\left(f_{jk}^{(m)}(t)\right)_{j,k=1}^n \tag{5}$$

too, where the summation is extended over all permutations without repetition of the elements $1, \dots, n$, and $f^{(m)}(t)$ denotes the m th derivative of $f(t)$. Taking into consideration that the entries of matrix X are identically distributed random variables we get by (4) that the differential equation

$$f^{(m)}(t) = f^{(m)}(0)f^n(t), \quad t \in \mathbb{R} \tag{6}$$

must be satisfied by the characteristic function (2).

$f(t)$ is an entire function ([1], Theorem 4.1). After the derivation of identity (6) we get

$$f^{(m+1)}(t) = f^{(m)}(0)nf^{n-1}(t)f'(t),$$

and the differential equation

$$f^{(m+1)}(t)f(t) = nf^{(m)}(t)f'(t), \quad t \in \mathbb{R}\,, \tag{7}$$

respectively, which is an equivalent form of differential equation (6).

Since $f(t)$ is an entire function, all moments

$$M_k = E(X_{11}^k) = \int_{-\infty}^{\infty} x^k dF(x) \quad (k = 0, 1, \dots) \tag{8}$$

of $F(x)$ exist.

In the following we prove the identity

$$\sum_{j=0}^{\nu} \binom{\nu}{j} M_{m+\nu-j+1} M_j = n \sum_{j=0}^{\nu} \binom{\nu}{j} M_{m+\nu-j} M_{j+1} \qquad (\nu = 0, 1, 2, \dots)\,. \tag{9}$$

Namely, derivating ν time first the right side, then the left side of identity (7), we get

$$(nf^{(m)}(t)f'(t))^{(\nu)} = n \sum_{j=0}^{\nu} \binom{\nu}{j} f^{(m+\nu-j)}(t) f^{(j+1)}(t)\,, \tag{10}$$

$$(f^{(m+1)}(t)f(t))^{(\nu)} = \sum_{j=0}^{\nu} \binom{\nu}{j} f^{(m+\nu-j+1)}(t) f^{(j)}(t)\,. \tag{11}$$

If expressions (10) and (11) will be identified, we have statement (9) by substitution $t = 0$.

Let us write equality (9) in the form

$$M_{m+\nu+1} = n \sum_{j=0}^{\nu} \binom{\nu}{j} M_{m+\nu-j} M_{j+1} - \sum_{j=1}^{\nu} \binom{\nu}{j} M_{m+\nu-j+1} M_j\,,$$

or in the form

$$M_{m+\nu+1} = \sum_{j=1}^{\nu+1} \left[n \binom{\nu}{j-1} - \binom{\nu}{j} \right] M_{m+\nu-j+1} M_j \qquad (\nu = 1, 2, \dots)\,. \tag{12}$$

Our next duty is the solution of the difference equation system (12) under the condition that quantities

$$M_j \ (j = 0, 1, \dots)$$

are moments of a distribution function with entire characteristic function.

Let us denote by $\bar{M}_j$ the j-th absolute moment of the distribution function $F(x)$, i.e. let

$$\bar{M}_j = \int_{-\infty}^{\infty} |x|^j dF(x) \quad (j = 0, 1, \dots).$$

First we give the solution of the difference equation (12) if ν with $1 \le \nu \le n$ is a fixed integer.

In the following we verify that the only solution of the difference equation (12) under the given conditions is the following:

For $1 \le \nu \le n$ we have

$$\bar{M}_j = c_\nu \lambda^j \quad (j = 1, 2, \dots, n + \nu + 1) \tag{13}$$

where $\lambda \ge 0$, and

$$c_\nu^{-1} = \sum_{j=1}^{\nu+1} \left[n\binom{\nu}{j-1} - \binom{\nu}{j} \right] = (n-1)2^\nu + 1. \tag{14}$$

Namely, if $1 \le \nu \le n$ then

$$\left.\begin{matrix} n\binom{\nu}{j-1} - \binom{\nu}{j} \ge 0 \\ > 0 \end{matrix}\right\} \quad \text{for} \quad \begin{matrix} j = 1 \\ j = 2, 3, \dots, \nu + 1. \end{matrix}$$

Thus

$$M_{m+\nu+1} \le |M_{m+\nu+1}| = \sum_{j=1}^{\nu+1} \left[n\binom{\nu}{j-1} - \binom{\nu}{j} \right] |M_{m+\nu-j+1}|\,|M_j| \le$$

$$\le \sum_{j=1}^{\nu+1} \left[n\binom{\nu}{j-1} - \binom{\nu}{j} \right] \bar{M}_{m+\nu-j+1}\bar{M}_j \le c_\nu^{-1}\bar{M}_{m+\nu+1}$$

by (12) using the well-known inequalities

$$\bar{M}_{m+\nu-j+1}\bar{M}_j \le \bar{M}_{m+\nu+1} \quad (j = 1, \dots, \nu + 1)$$

among the absolute moments of the distribution function $F(x)$, with equality if and only if $\bar{M}_j$ is a function of exponential type. Consequently difference equation

$$\bar{M}_{m+\nu+1} = \sum_{j=1}^{\nu+1} \left[n\binom{\nu}{j-1} - \binom{\nu}{j} \right] \bar{M}_{m+\nu-j+1}\bar{M}_j$$

has the only solution (13).

Now we return to our basic problem, i.e. to the solution of difference equation system (12) under the condition that the unknown functions are moments of a distribution function.

First we assume that $n \geq 2$. In this case the sequence

$$\{c_\nu^{-1}\}_1^\infty$$

is strictly increasing by (14).

Let us suppose that (13) is the solution of the difference equation (12), where $n > \nu \geq 1$, and

$$\bar{M}_j = c_{\nu+1}\lambda_1^j \tag{15}$$

is the solution of this difference equation in the case of $\nu + 1$, with $\lambda > 0$, $\lambda_1 > 0$. Comparing (15) with (13) and (14), we get that

$$\frac{c_\nu}{c_{\nu+1}} = \left(\frac{\lambda_1}{\lambda}\right)^j$$

for all considerable j, contradicting to the fact, that sequence

$$\left\{\left(\frac{\lambda_1}{\lambda}\right)^j\right\}$$

is strictly increasing by $\frac{\lambda_1}{\lambda} > 1$. Thus at least one is equal to zero among the moments of order even, i.e. distribution function $F(x)$ is degenerated with jump at the origin. (Consequently all moments are equal to zero, i.e. difference equation system (12) has the only solution $\lambda = 0$.

The converse statement that differential equation (6) is satisfied by $f(t) \equiv 1,\ t \in \mathbb{R}$, is a triviality.

Thus the proof of Theorem 1 is finished.

If $n = 1$, then the solution of the difference equation (12) with $\nu = 1$ is equal to

$$\bar{M}_j = \lambda^j \quad (j = 0, 1, 2)\,.$$

by (14), where λ is an arbitrary non-negative number. In this case

$$M_2 - \bar{M}_1^2 = 0\,,$$

i.e. the variance of random variable $|X_{11}|$ is equal to zero. This shows that the distribution function $F(x)$ is degenerated with jump at $t = \lambda$, where λ is an arbitrary real number. Conversely, differential equation (6) is satisfied by the characteristic function $f(t) = e^{i\lambda t},\ t \in \mathbb{R}$ with arbitrary real number λ if $n = 1$.

This completes the proof of Theorem 2.

References

[1] B. Gyires, On an extension of a theorem of Linnik and Zinger, *Adv. Appl. Prob.* **7** (1975), 786–800.
[2] E. Lukács and R. G. Laha, Applications of characteristic functions, *Charles Griffin and Co., Ltd.*, *London*, 1961.
[3] H. Minc, Permanents, *Addison–Wesley Publ. Co.*, 1978.

B. GYIRES
KOSSUTH L. UNIVERSITY OF DEBRECEN
HUNGARY

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