Publ. Math. Debrecen
62/1-2 (2003), 83–98

A new class of general variational inclusions involving maximal η-monotone mappings

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Abstract. A new class of maximal η-monotone mappings is introduced and studied in Hilbert spaces and the Lipschitz continuity of the resolvent operator for maximal η-monotone mapping is proved in this paper. We also introduce and study a new class of general variational inclusions involving maximal η-monotone mappings and construct a new algorithm for solving this class of general variational inclusions by using the resolvent operator technique for maximal η-monotone mapping. The results presented in this paper extend and improve many known results in the literature.

1. Introduction and preliminaries

It is known that variational inclusion is an important and useful generalization of variational inequality. Because of the wide applications to optimization and control, economic and tansportation equilibrium, and engineering sciences, variational inequalities and variational inclusions have been studied by many authors (see [1]–[13], [15]–[21] and the references therein). We also know that one of the most important and interesting problems in the theory of variational inequality is the development of an efficient and implementable algorithm for solving variational inequal-

Mathematics Subject Classification: 49J40, 47H19, 47H10.
Key words and phrases: variational inclusion, maximal η-monotone mapping, resolvent operator, iterative algorithm, convergence.
This work was supported by the National Natural Science Foundation of China (10171070) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

ity. For the past years, many numerical methods have been developed for solving various classes of variational inequalities, such as the projection method and its variant forms, linear approximation, descent, and Newton's methods. Recently, Ding and Luo [4] introduced two new concepts of η-subdifferential and η-proximal mapping of a proper function in Hilbert spaces and proved the existence and Lipschitz continuity of η-proximal mapping of a proper function. In terms of these concepts, Ding and Luo [4] developed some novel and innovative perturbed iterative algorithms for a new class of general quasi-variational-like inclusions with nonconvex functionals. Some related works, we refer to Ding [3] and Lee *et al.* [13].

Motivated and inspired by the recent works of [3], [4], [13], in this paper, we introduce a new concept of maximal η-monotone mapping and prove the Lipschitz continuity of the resolvent operator for maximal η-monotone mapping. By using the resolvent operator technique for maximal η-monotone mapping, we construct a new iterative algorithm for solving a new class of general variational inclusions involving maximal η-monotone mappings. The results presented in this paper extend and improve the corresponding results of [3], [4], [7]–[9], [13], [18]–[20].

Throughout this paper, we suppose that H is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. Let 2^H, $\mathrm{CB}(H)$, and $H(\cdot,\cdot)$ denote the family of all the nonempty subsets of H, the family of all the nonempty closed bounded subsets of H, and the Hausdorff metric on $\mathrm{CB}(H)$, respectively. In the sequel, let us recall some concepts.

Definition 1.1. A multivalued mapping $A : H \to 2^H$ is said to be

(i) monotone if

$$\langle x-y, u-v\rangle \geq 0 \quad \text{for all } u,v \in H,\ x\in Au,\ y\in Av;$$

(ii) strictly monotone if

$$\langle x-y, u-v\rangle \geq 0 \quad \text{for all } u,v \in H,\ x\in Au,\ y\in Av$$

and equality holds if and only if $u=v$;

(iii) strongly monotone if there exists a constant $\gamma>0$ such that

$$\langle x-y, u-v\rangle \geq \gamma\|u-v\|^2 \quad \text{for all } u,v \in H,\ x\in Au,\ y\in Av;$$

(iv) maximal monotone if A is monotone and $(I+\lambda A)(H)=H$ for any $\lambda>0$, where I denotes the identity mapping.

Remark 1.1. We note that A is maximal monotone if and only if A is monotone and there is no other monotone mapping whose graph contains strictly the graph Graph(A) of A, where Graph$(A) = \{(u, x) \in H \times H : x \in Au\}$.

Definition 1.2. A multivalued mapping $A : H \to \mathrm{CB}(H)$ is said to be H-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$H(Au, Av) \leq \xi \|u - v\| \quad \text{for all } u, v \in H.$$

Definition 1.3 ([3]). Let $A : H \to 2^H$ be a multivalued mapping. A mapping $N : H \times H \to H$ is said to be

(i) strongly monotone with respect to A in the first argument if, there exists a constant $r > 0$ such that

$$\langle N(x, \cdot\,) - N(y, \cdot\,), u - v\rangle \geq r\|u - v\|^2 \quad \text{for all } u, v \in H,\ x \in Au,\ y \in Av;$$

(ii) Lipschitz continuous in the first argument if, there exists a constant $\alpha > 0$ such that

$$\|N(u, \cdot\,) - N(v, \cdot\,)\| \leq \alpha\|u - v\| \quad \text{for all } u, v \in H.$$

Similarly, we can define the Lipschitz continuity of $N(\cdot\,, \cdot)$ in the second argument.

2. Maximal η-monotone mappings

Definition 2.1. A mapping $\eta : H \times \to H$ is said to be

(i) monotone if

$$\langle u - v, \eta(u, v)\rangle \geq 0 \quad \text{for all } u, v \in H;$$

(ii) strictly monotone if

$$\langle u - v, \eta(u, v)\rangle \geq 0 \quad \text{for all } u, v \in H$$

and equality holds if and only if $u = v$;

(iii) strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle u - v, \eta(u, v)\rangle \geq \delta\|u - v\|^2 \quad \text{for all } u, v \in H;$$

(iv) Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \leq \tau\|u-v\| \quad \text{for all } u,v \in H.$$

We remark that the strong monotonicity of η implies the strict monotonicity of η.

Definition 2.2. Let $\eta : H \times H \to H$ be a single-valued mapping. A multivalued mapping $M : H \to 2^H$ is said to be

(i) η-monotone if

$$\langle x-y, \eta(u,v)\rangle \geq 0 \quad \text{for all } u,v \in H,\ x \in Mu,\ y \in Mv;$$

(ii) strictly η-monotone if

$$\langle x-y, \eta(u,v)\rangle \geq 0 \quad \text{for all } u,v \in H,\ x \in Mu, y \in Mv$$

and equality holds if and only if $u = v$;

(iii) strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle x-y, \eta(u,v)\rangle \geq r\|u-v\|^2 \quad \text{for all } u,v \in H,\ x \in Mu,\ y \in Mv;$$

(iv) maximal η-monotone if M is η-monotone and $(I+\lambda M)(H) = H$ for any $\lambda > 0$.

Remark 2.1. If $\eta(u,v) = u - v$ for all u, v in H, then (i)–(iv) of Definition 2.2 reduce to the classical definitions of monotonicity, strict monotonicity, strong monotonicity, and maximal monotonicity, respectively.

Definition 2.3 ([21]). A function $f : H \times H \to R \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV) in x if, for any finite set $\{x_1, \ldots, x_n\} \subset H$ and for any $y = \sum_{i=0}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=0}^n \lambda_i = 1$,

$$\min_{0 \leq i \leq n} f(x_i, y) \leq 0.$$

Definition 2.4 ([4], [13]). Let $\eta : H \times H \to H$ be a single-valued mapping. A proper function $\phi : H \to R \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in H$ if there exists a point $f^* \in H$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \quad \forall y \in H,$$

where f^* is called an η-subdifferential of ϕ at x. The set of all η-subdifferential of ϕ at x is denoted by $\Delta\phi(x)$. The mapping $\Delta\phi : H \to 2^H$ defined by

$$\Delta\phi(x) = \{f^* \in H : \phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \quad \forall y \in H\} \tag{2.1}$$

is said to be η-subdifferential of ϕ.

Remark 2.2. If $\eta(x,y) = x-y$ for all x,y in H and ϕ is a proper convex lower semicontinuous functional on H, then Definition 2.4 reduces to the usual definitions of subdifferential of a functional ϕ. If ϕ is differentiable at $x \in H$ and satisfies

$$\phi(x + \lambda\eta(y,x)) \leq \lambda\phi(y) + (1-\lambda)\phi(x), \quad y \in H,\ \lambda \in [0,1],$$

then ϕ is η-subdiferentiable at $x \in H$, see [16, p. 424]

Proposition 2.1. *Let $\eta : H \times H \to H$ be Lipschitz continuous and strongly monotone such that $\eta(u,v) = -\eta(v,u)$ for all $u, v \in H$, and for any given $x \in H$, the function $h(y,u) = \langle x - u, \eta(y,u)\rangle$ is 0-*DQCV *in y. Let $\phi : H \to R \cup \{+\infty\}$ be a lower semicontinuous and η-subdifferetiable functional. Then the η- subdifferential $\Delta\phi$ defined by* (2.1) *is maximal η-monotone.*

Proof. The fact $(I+\lambda\Delta\phi)(H) = H$ follows directly from Theorem 2.8 of Ding and Luo [4]. Next, we prove that $\Delta\phi$ is η-monotone. For any given $u_1, u_2 \in H$, $x_1 \in \Delta\phi(u_1)$, $x_2 \in \Delta\phi(u_2)$, it follows from the definition of η-subdifferential that

$$\phi(y) - \phi(u_1) \geq \langle x_1, \eta(y,u_1)\rangle, \quad \forall y \in H \tag{2.2}$$

and

$$\phi(y) - \phi(u_2) \geq \langle x_2, \eta(y,u_2)\rangle, \quad \forall y \in H. \tag{2.3}$$

Setting $y = u_2$ in (2.2) and $y = u_1$ in (2.3) and adding them, then we have

$$\langle x_1 - x_2, \eta(u_1,u_2)\rangle \geq 0 \quad \text{for all } u_1, u_2 \in H,\ x_1 \in \Delta\phi(u_1),\ x_2 \in \Delta\phi(u_2).$$

The proof is complete. $\square$

Remark 2.3. Proposition 2.1 shows that the existence of the maximal η-monotone mapping. We emphasize that the functional ϕ may not be convex in Proposition 2.1. The following example shows that the existence of the mapping $\eta : H \times H \to H$ satisfying all conditions in Proposition 2.1.

Example 2.1 ([13]). Let $H = R = (-\infty, \infty)$ and $\eta : R \times R \to R$ be defined by

$$\eta(x,y) = \begin{cases} x-y, & \text{if } |xy| < 1, \\ |xy|(x-y), & \text{if } 1 \le |xy| < 2, \\ 2(x-y), & \text{if } 2 \le |xy|. \end{cases}$$

Then it is easy to see that

1. $|\eta(x,y)| \le 2|x-y|$ for all $x, y \in R$, i.e., η is Lipschitz continuous;
2. $\langle \eta(x,y), x-y \rangle \ge |x-y|^2$ for all $x, y \in R$, i.e., η is strongly monotone;
3. $\eta(x,y) = -\eta(y,x)$ for all $x, y \in R$;
4. For any given $x \in R$, the function $h(y,u) = \langle x-u, \eta(y,u) \rangle = (x-u)\eta(y,u)$ is 0-DQCV in y. If it is false, then there exist a finite set $\{y_1, \dots, y_n\}$ and $u_0 = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that for each $i = 1, \dots, n$,

$$0 < h(y_i, u_0) = \begin{cases} (x-u_0)(y_i-u_0), & \text{if } |y_i u_0| < 1, \\ (x-u_0)|y_i-u_0|(y_i-u_0), & \text{if } 1 \le |y_i u_0| < 2, \\ 2(x-u_0)(y_i-u_0), & \text{if } 2 \le |y_i u_0|. \end{cases}$$

It follows that $(x-u_0)(y_i-u_0) > 0$ for each $i = 1, \dots, n$ and we have

$$0 < \sum_{i=1}^n \lambda_i (x-u_0)(y_i-u_0) = (x-u_0)(u_0-u_0) = 0$$

which is a contradiction. This prove that for any given $x \in R$, the function $h(y,u)$ is 0-DQCV in y. Thus η satisfies all assumptions in Proposition 2.1.

Theorem 2.1. *Let $\eta : H \times H \to H$ be strictly monotone and $M : H \to 2^H$ be a maximal η-monotone mapping. Then the following conclusions hold:*

(1) *$\langle x-y, \eta(u,v) \rangle \ge 0$ for all $(v,y) \in \mathrm{Graph}(M)$ implies that $(u,x) \in \mathrm{Graph}(M)$, where $\mathrm{Graph}(M) = \{(u,x) \in H \times H : x \in Mu\}$;*

(2) *the inverse mapping $(I + \lambda M)^{-1}$ is single-valued for any $\lambda > 0$.*

Proof. Suppose that (1) is false. Then there exists $(u_0, x_0) \notin \mathrm{Graph}(M)$ such that

$$\langle x_0 - y, \eta(u_0, v)\rangle \geq 0 \quad \text{for all } (v, y) \in \mathrm{Graph}(M). \tag{2.4}$$

Since M is maximal η-monotone, we have $(I + \lambda M)(H) = H$. Then there exists $(u_1, x_1) \in \mathrm{Graph}(M)$ such that

$$u_1 + \lambda x_1 = u_0 + \lambda x_0. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\langle x_0 - x_1, \eta(u_0, u_1)\rangle = \frac{1}{\lambda}\langle u_1 - u_0, \eta(u_0, u_1)\rangle \geq 0.$$

This implies that

$$\langle u_0 - u_1, \eta(u_0, u_1)\rangle \leq 0.$$

Since η is strictly monotone, we have $u_0 = u_1$ and hence from (2.5) we get $x_1 = x_0$. This contradicts the fact $(u_0, x_0) \notin \mathrm{Graph}(M)$. Thus (1) is true.

Now we prove (2). For any given $z \in H$ and a constant $\lambda > 0$, let $u, v \in (I + \lambda M)^{-1}(z)$. Then $\lambda^{-1}(z - u) \in M(u)$ and $\lambda^{-1}(z - v) \in M(v)$. By η-monotonicity of M, we obtain

$$\begin{aligned} 0 = \langle z - z, \eta(u, v)\rangle &= \lambda \left\langle \frac{1}{\lambda}(z - u) - \frac{1}{\lambda}(z - v), \eta(u, v) \right\rangle \\ &+ \langle u - v, \eta(u, v)\rangle \geq \langle u - v, \eta(u, v)\rangle. \end{aligned}$$

Since η is strictly monotone, we have $u = v$. Thus $(I + \lambda M)^{-1}$ is single-valued.

This completes the proof. □

Based on Theorem 2.1, we can define the resolvent operator for a maximal η-monotone mapping M as follows:

$$J_\rho^M(z) = (I + \rho M)^{-1}(z) \quad \text{for all } z \in H, \tag{2.6}$$

where $\rho > 0$ is a constant and $\eta : H \times H \to H$ is a strictly monotone mapping.

Remark 2.4. If $\eta(x, y) = x - y$ for all x, y in H, then the resolvent operator J_ρ^M defined by (2.6) reduces to the usual one for maximal monotone mapping.

Theorem 2.2. *Let $\eta : H \times H \to H$ be strongly monotone and Lipschtiz continuous with constants $\delta > 0$ and $\tau > 0$, respectively. Let $M : H \to 2^H$ be a maximal η-monotone mapping. Then the resolvent operator J_ρ^M for M is Lipschitz continuous with constant τ/δ, i.e.,*

$$\|J_\rho^M(u) - J_\rho^M(v)\| \leq \frac{\tau}{\delta}\|u - v\| \quad \textit{for all } u, v \in H.$$

Proof. Let u, v be any given points in H. From the definition of J_ρ^M, we have

$$J_\rho^M(u) = (I + \rho M)^{-1}(u)$$

and

$$J_\rho^M(v) = (I + \rho M)^{-1}(v).$$

This implies that

$$\frac{1}{\rho}(u - J_\rho^M(u)) \in M(J_\rho^M(u))$$

and

$$\frac{1}{\rho}(v - J_\rho^M(v)) \in M(J_\rho^M(v)).$$

Since M is η-monotone, we obtain

$$\begin{aligned}&\frac{1}{\rho}\left\langle u - J_\rho^M(u) - (v - J_\rho^M(v)), \eta(J_\rho^M(u), J_\rho^M(v))\right\rangle \\ &\quad = \frac{1}{\rho}\left\langle u - v - (J_\rho^M(u) - J_\rho^M(v)), \eta(J_\rho^M(u), J_\rho^M(v))\right\rangle \geq 0.\end{aligned}$$

From the above inequality, we have

$$\begin{aligned}\delta\|J_\rho^M(u) - J_\rho^M(v)\|^2 &\leq \langle J_\rho^M(u) - J_\rho^M(v), \eta(J_\rho^M(u), J_\rho^M(v))\rangle \\ &\leq \langle u - v, \eta(J_\rho^M(u), J_\rho^M(v))\rangle \\ &\leq \tau\|u - v\| \cdot \|J_\rho^M(u) - J_\rho^M(v)\|.\end{aligned}$$

This implies that

$$\|J_\rho^M(u) - J_\rho^M(v)\| \leq \frac{\tau}{\delta}\|u - v\| \quad \text{for all } u, v \in H.$$

The proof is complete. □

Remark 2.5. Theorem 2.2 generalizes Theorem 2.2 of [3] and Theorem 2.10 of [4].

3. Variational inclusions

In this section, by using the new concept of maximal η-monotone mapping and the results obtained in Section 2, we shall study a new class of general variational inclusions involving maximal η-monotone mappings in Hilbert spaces and construct a new iterative algorithm for approximating the solution of this class of general variational inclusions involving maximal η-monotone mappings.

Let $\eta, N : H \times H \to H$ be two single-valued mappings with two variables. Let $S, T, G : H \to \mathrm{CB}(H)$ be three multivalued mappings and $M : H \times H \to 2^H$ be a multivalued mapping such that for each $t \in H$, $M(\cdot, t)$ is maximal η-monotone with $\mathrm{Range}(G) \cap \mathrm{dom}\, M(\cdot, t) \neq \emptyset$. Now we consider the following problem:

Find $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$ such that

$$0 \in N(x, y) + M(z, u). \tag{3.1}$$

The problem (3.1) is called the general set-valued variational inclusion.

It is known that a number of problems involving the nonmonotone, nonconvex, and nonsmooth mappings arising in structural engineering, mechanics, economics, and optimization theory can be reduced to study this kind of variational inclusions (see, for example, [2], [5], [11], [17], [20]).

Some special cases:

(I) If $M(x, t) = M(x)$ for all x, t in H, then the problem (3.1) reduces to the following problem:

Find $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$ such that

$$0 \in N(x, y) + M(z), \tag{3.2}$$

which appears to be a new one. Furthermore, if G is a single-valued mapping and $\eta(x, y) = x - y$ for all x, y in H, then the problem (3.2) is equivalent to the variational inclusion considered by Huang [9].

(II) If $M(\cdot, t) = \Delta\phi(\cdot, t)$, where $\phi : H \times H \to R \cup \{+\infty\}$ is a functional such that for each fixed t in H, $\phi(\cdot, t) : H \to R \cup \{+\infty\}$ is lower semicontinuous and η-subdifferetiable on H, and $\Delta\phi(\cdot, t)$ denotes the η-subdiferentail of $\phi(\cdot, t)$, then the problem (3.1) reduces to the following problem:

Find $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$ such that

$$\langle N(x,y), \eta(v,z) \rangle \geq \phi(z,u) - \phi(v,u) \tag{3.3}$$

for all v in H, which which appears to be a new one. Furthermore, if $N(x,y) = x - y$ for all x, y in H, and S, T, G are three single-valued mappings, then the problem (3.3) reduces to the general quasi-variational-like inclusion considered by Ding and Luo [4].

(III) If $S, T : H \to H$ are single-valued mappings, G is the identity mapping, $N(x,y) = x - y$ for all x, y in H, and $M(\cdot, t) = \Delta\phi$ for all t in H, where $\Delta\phi$ denotes the η-subdifferential of a proper convex lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$, then the problem (3.1) reduces to the following problem:

Find $u \in H$ such that

$$\langle Su - Tu, \eta(v,u) \rangle \geq \phi(u) - \phi(v) \tag{3.4}$$

for all v in H, which is called the strongly nonlinear variational-like inclusion problem considered by Lee *et al.* [13].

(IV) If $\eta(x,y) = x - y$ and $M(\cdot, t) = \partial\phi$, where $\partial\phi$ denotes the subdifferential of a proper convex lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$, then the problem (3.1) reduces to finding $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$ such that

$$\langle N(x,y), v - z \rangle \geq \phi(z) - \phi(v) \tag{3.5}$$

for all v in H. Furthermore, if $N(x,y) = x - y$ for all x, y in H and G is an identity mapping, then the problem (3.5) is equivalent to the set-valued nonlinear generalized variational inclusion considered by Huang [7] and, in turn, includes the variational inclusions studied by Hassouni and Moudafi [6] and Kazmi [12] as special cases.

Summing up the above arguments, it shows that for a suitable choice of N, η, M, S, T, G, and for the space H, one can obtained a number of known and new classes of variational inclusions, variational inequalities, and corresponding optimization problems from the general set-valued variational inclusion problem (3.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in the mathematical, physical, and engineering sciences in a general and unified framework.

Lemma 3.1. *For given $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$, (u, x, y, z) is a solution of the problem (3.1) if and only if*

$$z = J_\rho^{M(\cdot,u)}(z - \rho N(x, y)), \tag{3.6}$$

where $J_\rho^{M(\cdot,u)} = (I + \rho M(\cdot, u))^{-1}$ and $\rho > 0$ is a constant.

Proof. This directly follows from the definition of $J_\rho^{M(\cdot,u)}$. □

Based on Lemma 3.1 and Nadler [14], we develop a new iterative algorithm for solving the problem (3.1) as follows:

Algorithm 3.1. For any given $u_0 \in H$, $x_0 \in Su_0$, $y_0 \in Tu_0$, and $z_0 \in Gu_0$. Define the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ as follows:

$$\begin{cases} u_{n+1} = u_n - z_n + J_\rho^{M(\cdot,u_n)}(z_n - \rho N(x_n, y_n)), \\ x_{n+1} \in Su_{n+1},\ \|x_n - x_{n+1}\| \le (1+(1+n)^{-1})H(Su_n, Su_{n+1}), \\ y_{n+1} \in Tu_{n+1},\ \|y_n - y_{n+1}\| \le (1+(1+n)^{-1})H(Tu_n, Tu_{n+1}), \\ z_{n+1} \in Gu_{n+1},\ \|z_n - z_{n+1}\| \le (1+(1+n)^{-1})H(Gu_n, Gu_{n+1}), \end{cases} \tag{3.7}$$
$$n = 0, 1, 2, 3 \ldots .$$

Theorem 3.1. *Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants δ and τ, respectively. Let $S, T, G : H \to \mathrm{CB}(H)$ be H-Lipschitz continuous with constants σ, κ, and ξ, respectively, and G is strongly monotone with constant γ. Let $N : H \times H \to H$ be Lipschitz continuous in the first and second arguments with constants α and β, respectively, and be strongly monotone with respect to S in the first argument with constant r. Let $M : H \times H \to 2^H$ be such that for each fixed $t \in H$, $M(\cdot, t)$ is maximal η-monotone. Suppose that there exist constants $\rho > 0$ and $\lambda > 0$ such that for each $x, y, z \in H$,*

$$\|J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z)\| \le \lambda\|x - y\|, \tag{3.8}$$

and

$$\begin{aligned} \theta = \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\gamma + \xi^2} + \frac{\tau}{\delta}\sqrt{1 - 2\rho r + \rho^2\alpha^2\sigma^2} \\ + \beta\kappa\rho\frac{\tau}{\delta} + \lambda < 1. \end{aligned} \tag{3.9}$$

Then the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *generated by Algorithm 3.1 converge strongly to* u^*, x^*, y^*, *and* z^*, *respectively and* (u^*, x^*, y^*, z^*) *is a solution of the problem* (3.1).

Proof. It follows from (3.7) and (3.8) that

$$
\begin{aligned}
\|u_{n+2} - u_{n+1}\| &= \|u_{n+1} - u_n - (z_{n+1} - z_n) \\
&\quad + J_\rho^{M(\cdot,u_{n+1})}(z_{n+1} - \rho N(x_{n+1}, y_{n+1})) \\
&\quad - J_\rho^{M(\cdot,u_n)}(z_n - \rho N(x_n, y_n))\| \\
&\le \|u_{n+1} - u_n - (z_{n+1} - z_n)\| \\
&\quad + \|J_\rho^{M(\cdot,u_{n+1})}(z_{n+1} - \rho N(x_{n+1}, y_{n+1})) \\
&\quad - J_\rho^{M(\cdot,u_{n+1})}(z_n - \rho N(x_n, y_n))\| \\
&\quad + \|J_\rho^{M(\cdot,u_{n+1})}(z_n - \rho N(x_n, y_n)) - J_\rho^{M(\cdot,u_n)}(z_n - \rho N(x_n, y_n))\| \\
&\le \|u_{n+1} - u_n - (z_{n+1} - z_n)\| \qquad\qquad (3.10) \\
&\quad + \frac{\tau}{\delta}\|z_{n+1} - z_n - \rho(N(x_{n+1}, y_{n+1}) \\
&\quad - N(x_n, y_n))\| + \lambda\|u_{n+1} - u_n\| \\
&\le \|u_{n+1} - u_n - (z_{n+1} - z_n)\| + \lambda\|u_{n+1} - u_n\| \\
&\quad + \frac{\tau}{\delta}\|z_{n+1} - z_n - \rho(N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1}))\| \\
&\quad + \rho\frac{\tau}{\delta}\|N(x_n, y_{n+1}) - N(x_n, y_n)\| \\
&\le \left(1 + \frac{\tau}{\delta}\right)\|u_{n+1} - u_n - (z_{n+1} - z_n)\| \\
&\quad + \lambda\|u_{n+1} - u_n\| \\
&\quad + \frac{\tau}{\delta}\|u_{n+1} - u_n - \rho(N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1}))\| \\
&\quad + \rho\frac{\tau}{\delta}\|N(x_n, y_{n+1}) - N(x_n, y_n)\|.
\end{aligned}
$$

Since G is strongly monotone and H-Lipschitz continuous, we obtain

$$
\begin{aligned}
&\|u_{n+1} - u_n - (z_{n+1} - z_n)\|^2 \\
&\quad = \|u_{n+1} - u_n\|^2 - 2\langle u_{n+1} - u_n, z_{n+1} - z_n\rangle + \|z_{n+1} - z_n\|^2
\end{aligned}
$$

$$\leq (1-2\gamma)\|u_{n+1}-u_n\|^2 + \left(1+\frac{1}{1+n}\right)^2 H^2(Gu_{n+1},Gu_n)$$

$$\leq \left(1-2\gamma+\left(1+\frac{1}{1+n}\right)^2\xi^2\right)\|u_{n+1}-u_n\|^2. \tag{3.11}$$

Further, from the assumptions, we have

$$\begin{aligned}\|N(x_n,y_{n+1})-N(x_n,y_n))\| &\leq \beta\|y_{n+1}-y_n\| \\ &\leq \beta\left(1+\frac{1}{1+n}\right)H(Tu_n,Tu_{n+1}) \\ &\leq \beta\kappa\left(1+\frac{1}{1+n}\right)\|u_n-u_{n+1}\|,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\|u_{n+1}-u_n-\rho(N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}))\|^2 \\ &\quad = \|u_{n+1}-u_n\|^2 - 2\rho\langle u_{n+1}-u_n, N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\rangle \\ &\qquad + \rho^2\|N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|^2 \\ &\quad \leq \left(1-2\rho r+\rho^2\alpha^2\sigma^2\left(1+\frac{1}{1+n}\right)^2\right)\|u_n-u_{n+1}\|^2.\end{aligned} \tag{3.13}$$

It follows from (3.10)–(3.13) that

$$\|u_{n+2}-u_{n+1}\| \leq \theta_n\|u_{n+1}-u_n\|, \tag{3.14}$$

where

$$\begin{aligned}\theta_n &= \left(1+\frac{\tau}{\delta}\right)\sqrt{1-2\gamma+\left(1+\frac{1}{1+n}\right)^2\xi^2} \\ &\quad + \frac{\tau}{\delta}\sqrt{1-2\rho r+\rho^2\alpha^2\sigma^2\left(1+\frac{1}{1+n}\right)^2} + \left(1+\frac{1}{1+n}\right)\beta\kappa\rho\frac{\tau}{\delta}+\lambda.\end{aligned}$$

Letting

$$\theta = \left(1+\frac{\tau}{\delta}\right)\sqrt{1-2\gamma+\xi^2} + \frac{\tau}{\delta}\sqrt{1-2\rho r+\rho^2\alpha^2\sigma^2} + \beta\kappa\rho\frac{\tau}{\delta}+\lambda.$$

We know that $\theta_n \searrow \theta$ as $n \to \infty$. It follows from (3.9) that $0 \le \theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (3.14) implies that $\{u_n\}$ is a Cauchy sequence in H. Let $u_n \to u^*$ as $n \to \infty$. From (3.7), we get

$$\begin{aligned} \|x_n - x_{n+1}\| &\le \left(1 + \frac{1}{1+n}\right) H(Su_n, Su_{n+1}) \\ &\le \sigma \left(1 + \frac{1}{1+n}\right) \|u_n - u_{n+1}\|, \\ \|y_n - y_{n+1}\| &\le \left(1 + \frac{1}{1+n})H(Tu_n, Tu_{n+1}\right) \\ &\le \kappa \left(1 + \frac{1}{1+n}\right) \|u_n - u_{n+1}\|, \\ \|z_n - z_{n+1}\| &\le \left(1 + \frac{1}{1+n})H(Gu_n, Gu_{n+1}\right) \\ &\le \xi \left(1 + \frac{1}{1+n}\right) \|u_n - u_{n+1}\|. \end{aligned} \tag{3.15}$$

Since $\{u_n\}$ is a Cauchy sequence, from (3.15), we know that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are also Cauchy sequences. Let $x_n \to x^*, y_n \to y^*$, and $z_n \to z^*$ as $n \to \infty$.

Furthermore, we have

$$\begin{aligned} d(x^*, Su^*) &\le \|x^* - x_n\| + d(x_n, Su^*) \\ &\le \|x^* - x_n\| + H(Su_n, Su^*) \\ &\le \|x^* - x_n\| + \sigma\|u_n - u^*\| \to 0. \end{aligned}$$

This implies that $x^* \in Su^*$. Similarly, we know that $y^* \in Tu^*$ and $z^* \in Gu^*$. Therefore, (u^*, x^*, y^*, z^*) is a solution of the problem (3.1). This completes the proof. □

Example 3.1. If there exists a constant $\rho > 0$ such that

$$\begin{cases} \left|\rho - \dfrac{\tau r - \delta(1-l)\beta}{\tau(\alpha^2\sigma^2 - \beta^2\kappa^2)}\right| < \dfrac{\sqrt{[\tau r - \delta(1-l)\beta]^2 - (\alpha^2\sigma^2 - \beta^2\kappa^2)[\tau^2 - \delta^2(1-l)^2]}}{\tau(\alpha^2\sigma^2 - \beta^2\kappa^2)}, \\ \tau r > \delta(1-l)\beta + \sqrt{(\alpha^2\sigma^2 - \beta^2\kappa^2)(\tau^2 - \delta^2(1-l)^2)}, \quad \alpha\sigma > \beta\kappa, \\ l = \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\gamma + \xi^2} + \lambda, \quad \rho\tau\beta\kappa < \delta(1-l), \ l < 1, \end{cases}$$

then the condition (3.9) is satisfied.

Remark 3.1. Theorem 3.1 extends and improves many corresponding results in [4], [7]–[9], [13], [18]–[20].

Acknowledgement. The authors wish to thank the referees for their helpful and constructive comments.

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(Received July 19, 2001; revised May 27, 2002)