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On the projectivized r-th order cotangent bundle

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Abstract. Let $P^{r*} = P(T^{r*}) : \mathcal{M}f_n \to \mathcal{FM}$ be the projectivized r-th order cotangent bundle functor. That for $n \geq r+1$ every natural operator $T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ is a constant multiple of the complete lifting is deduced. That for $n \geq r+1$ every natural affinor on P^{r*} over n-manifolds is a constant multiple of the identity affinor is obtained. That for $n \geq 2$ every natural operator $T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$ is a constant multiple of the vertical lifting is verified.

0. Introduction

Let M be an n-dimensional manifold. In [16], we considered the naturality problem how a vector field X on M induces a vector field $A(X)$ on the projectivized cotangent bundle $P(T^*M)$ and proved that for $n \geq 2$ every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T(P(T^*))$ is a constant multiple of the complete lifting. We also studied the naturality problem with affinors $C : T(P(T^*M)) \to T(P(T^*M))$ on $P(T^*M)$ and derived that for $n \geq 2$ every natural affinor $C : T(P(T^*)) \to T(P(T^*))$ on $P(T^*)$ over n-manifolds is a constant multiple of the identity one. Moreover, we considered the naturality problem how a 1-form ω on M can induce a 1-form $D(\omega)$ on $P(T^*M)$ and proved that for $n \geq 2$ every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*(P(T^*))$ is a constant multiple of the vertical lifting.

We inform the reader that the results presented above are particular cases (for $r = k = 1$) of the respective (proved in [16]) facts for the bundle $K^{r*}_k M = regT^{r*}_k M/L^r_k$ of the so called contact (k, r)-coelements. The

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mentioned results for $K_k^{r*}M$ are "dualizations" of respective facts from [5] and [6] (and generalized in [10]) for the bundle $K_k^r M = regT_k^r M/L_k^r$ of contact (k,r)-elements in the sense of C. Ehresmann, [2].

In the present paper we generalize the cited above results concerning $P(T^*)$ as follows. Let $P^{r*}M = P(T^{r*}M)$ denote the projectivized r-th order cotangent bundle $T^{r*}M = J^r(M,\mathbb{R})_0$. We consider the naturality problem how a vector field X on M induces a vector field $A(X)$ on $P^{r*}M$ and prove that for $n \geq r+1$ every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ is a constant multiple of the complete lifting $\mathcal{P}^{r*}$. We also study the naturality problem with affinors $C : TP^{r*}M \to TP^{r*}M$ on $P^{r*}M$ and obtain that for $n \geq r+1$ every natural affinor $C : TP^{r*} \to TP^{r*}$ on P^{r*} is a constant multiple of the identity one. Moreover, we consider the naturality problem how a 1-form ω on M induces a 1-form $D(\omega)$ on $P^{r*}M$ and prove that for $n \geq 2$ every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$ is a constant multiple of the vertical lifting. If $r = 1$ we have $T^* \tilde{=} T^{*1}$ and we reobtain the results for $P(T^*)$.

Natural operators lifting vector fields, functions and 1-forms to some natural bundles were used practically in all papers in which problem of prolongations of geometric structures was studied, see [17], [18], etc. That is why such natural operators are studied, see e.g. [3], [5], [11], [13]–[15], [19], etc.

The respective results of the present paper shows that if $\dim(M) \geq r+1$ then $P^{r*}M$ is poor with respect to liftings of vector fields and 1-forms. This indicate that there are small possibilities to prolonge classical geometric structures from M to $P^{r*}M$. However, it seems to be interesting that the complete lifting $\mathcal{P}^{r*}$ can be characterized as the unique natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ such that $A(X)$ is a projectable vector field on $P^{r*}M$ covering X for any vector field X on M.

Natural affinors on some natural bundle FM play importrant roles in the differential geometry. We present the following reason.

A generalized connection on FM is an affinor $\Gamma : TFM \to VFM \subset TFM$ on FM (horizontal projector) such that $\Gamma \circ \Gamma = \Gamma$ and $\dim(\Gamma) = VFM$, [5]. Given a natural affinor $C : TFM \to TFM$ on FM the Frolicher–Nijenhuis bracket $[C,\Gamma]$ is the so called generalized torsion of Γ with respect to C. Such generalized torsions were studied in [7], [1], etc. (The classical torsion of a linear connection Γ on TM is proportional to $[J,\Gamma]$, where J is the canonical tangent structure affinor on TM.)

That is why natural affinors are studied, see [4]–[6], [9], [10], [12] etc.

The result of the present paper concerning natural affinors on $P^{r*}M$ brings the following two negative answers. The first one is that if $\dim(M) \geq r+1$ then there is no canonical generalized connection on $P^{r*}M$. The second one is that if $\dim(M) \geq r+1$ then the notion of generalized torsions of a generalized connection Γ on $P^{r*}M$ makes no sense because $[id, \Gamma] = 0$.

From now on $x^1, \dots, x^n$ denote the usual coordinates on $\mathbb{R}^n$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the vector fields on $\mathbb{R}^n$.

All manifolds are assumed to be without boundary, finite dimensional, Hausdorff and smooth, i.e. of class $\mathcal{C}^\infty$. All maps between manifolds are assumed to be smooth. Natural operators and natural transformations are in the sense of [5].

1. On the projectivized cotangent bundle functor $P(T^*) : \mathcal{M}f_n \to \mathcal{FM}$

For a comfort we cite below some results about the projectivized cotangent bundle functor $P(T^*) : \mathcal{M}f_n \to \mathcal{FM}$.

For every n-manifold M we have the cotangent bundle T^*M and its projectivization $P(T^*M) = \bigcup_{x \in M} P(T^*_x M)$ over M, $P(T^*_x M) =$ the projective space corresponding to $T^*_x M$. Every embedding $\varphi : M \to N$ of two n-manifolds induces a bundle map $P(T^*\varphi) = \bigcup_{x \in M} P(T^*_x \varphi) : P(T^*M) \to P(T^*N)$. The correspondence $P(T^*) : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor.

In [16], we proved the following results.

Theorem 1. *Every natural transformation $B : P(T^*) \to P(T^*)$ over n-manifolds is the identity one.*

Theorem 2. *Let $n \geq 2$. Every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T(P(T^*))$ is a constant multiple of the complete lifting.*

Theorem 3. *Let $n \geq 2$. Every natural affinor $C : T(P(T^*)) \to T(P(T^*))$ on $P(T^*)$ over n-manifolds is a constant multiple of the identity affinor.*

Theorem 4. *Let $n \geq 2$. Every natural operator $A : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*(P(T^*))$ is a constant multiple of the vertical lifting.*

2. The projectivized r-th order cotangent bundle functor $P^{r*} = P(T^{r*}) : \mathcal{M}f_n \to \mathcal{FM}$

For every n-manifold M we have the r-cotangent vector bundle $T^{r*}M = J^r(M,\mathbb{R}^k)_0$ over M. Every embedding $\varphi : M \to N$ of two n-manifolds induces a vector bundle map $T^{r*}\varphi : T^{r*}M \to T^{r*}N$, $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma \circ \varphi^{-1})$, $\gamma : M \to \mathbb{R}$, $x \in M$, $\gamma(x) = 0$.

It is well-known that the correspondence $T^{r*} : \mathcal{M}f_n \to \mathcal{VB}$ is a vector bundle functor.

For every n-manifold M we have the bundle $P^{r*}M = P(T^{r*}M) = \bigcup_{x\in M} P(T^{r*}_x M)$ over M, $P(T^{r*}_x M)$= the projective space corresponding to the fibre $T^{r*}_x M$. Every embedding $\varphi : M \to N$ of two n-manifolds induces a bundle map $P^{r*}\varphi = P(T^{r*}\varphi) = \bigcup_{x\in M} P(T^{r*}_x\varphi) : P^{r*}M \to P^{r*}N$. The correspondence $P^{r*} = P(T^{r*}) : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor. It is called the projectivized r-th order cotangent bundle functor.

3. On natural endomorphisms of $P^{r*} = P(T^{r*})$

Theorem 5. *Let $n \geq 2$. Every natural transformation $B : P^{r*} \to P^{r*}$ over n-manifolds is the identity one.*

Proof. Consider a natural transformation $B : P^{r*} \to P^{r*}$ over n-manifolds, $n \geq 2$. Since $\sigma_o = [j^r_0(x^1)] \in P^{r*}_0\mathbb{R}^n$ has dense orbit in $P^{r*}\mathbb{R}^n$ with respect to $\mathrm{Diff}(\mathbb{R}^n,\mathbb{R}^n)$, it is sufficient to verify that $B(\sigma_o) = \sigma_o$.

We can write $B(\sigma_o) = [j^r_0(\sum_{\alpha\in G} a_\alpha x^\alpha)]$ for some $(a_\alpha)_{\alpha\in G} \in \mathbb{R}^G \setminus \{0\}$, where G is the set of all $\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{N}\cup\{0\})^n$ with $1 \leq |\alpha| \leq r$.

Using the invariance of $B(\sigma_o)$ with respect to $(\frac{1}{\tau_1}x^1,\dots,\frac{1}{\tau_n}x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau = (\tau_1,\dots,\tau_n) \in (\mathbb{R}\setminus\{0\})^n$ we get $[j^r_0(\sum\limits_{\alpha\in G} a_\alpha x^\alpha)] = [j^r_0(\sum\limits_{\alpha\in G} a_\alpha \tau^\alpha x^\alpha)]$. So, only one of the a_α's is not equal to 0. Hence $B(\sigma_o) = [j^r_0(x^\alpha)]$ for some $\alpha \in G$.

If $\alpha_i \neq 0$ for some $i \geq 2$, then by the invariance of $B(\sigma_o)$ with respect to the isomorphism $(x^1,\dots,x^{i-1},x^i - x^1,x^{i+1},\dots,x^n) : \mathbb{R}^n \to \mathbb{R}^n$ we get $[j^r_0((x^1)^{\alpha_1}\dots\ \dots(x^n)^{\alpha_n})] = [j^r_0((x^1)^{\alpha_1}\dots(x^i + x^1)^{\alpha_i}\dots(x^n)^{\alpha_n})]$, i.e. we obtain the contradiction. Hence $B(\sigma_o) = [j^r_0((x^1)^q)]$ for some $q = 1,\dots,r$.

If $q \geq 2$ then the local diffeomorphisms $(\tau x^1 + (x^1)^r + (x^2)^r, x^2,\dots,x^n)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau \neq 0$ preserve $[j^r_0((x^1)^q)]$.

Using the invariance of B with respect to these diffeomorphisms we obtain that $B([j^r_0(\tau x^1 + (x^1)^r + (x^2)^r)]) = [j^r_0((x^1)^q)]$ for $\tau \neq 0$. Putting $\tau \to 0$ we get $B([j^r_0((x^1)^r + (x^2)^r)]) = [j^r_0((x^1)^q)]$. Now, changing x^1 by x^2 and vice-versa we get $[j^r_0((x^1)^q)] = [j^r_0((x^2)^q)]$. Contradiction.

Hence $q = 1$, i.e. $B(\sigma_o) = [j^r_0(x^1)] = \sigma_o$. This ends the proof. □

Remark 1. If $n = 1$ and $r \geq 2$, then $P^{r*} : \mathcal{M}f_1 \to \mathcal{FM}$ is not rigid. For, $\sigma^o = [j^r_0((x^1)^r)] \in P^{r*}_0\mathbb{R}$ is L^r_1-invariant. Hence there exists the natural transformation $B : P^{r*} \to P^{r*}$ over 1-manifolds corresponding to the constant L^r_1-equivariant map $P^{r*}_0\mathbb{R} \to \{\sigma^o\} \subset P^{r*}_0\mathbb{R}$.

Corollary 1. *If $n \geq 2$ then every absolute natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ is 0.*

Proof. Every such A is a canonical vector field on $P^{r*}M$ for any $M \in obj(\mathcal{M}f_n)$. On the other hand $F_0\mathbb{R}^n$ is compact, then the flow of a canonical vector field on FM is formed by authomorphisms $FM \to FM$. Then the flow of A is trivial because of Theorem 5. So, $A = 0$. □

4. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$

In general, if $F : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor then given a vector field X on $M \in obj(\mathcal{M}f_n)$ we have the vector field $\mathcal{F}X$ on FM via prolongation of flows. It is called the complete lifting of X to FM. If $\{\varphi_t\}$ is the flow of X then $\{F\varphi_t\}$ is the flow of $\mathcal{F}X$, see [5].

In the case $F = P^{r*}$ we have the following theorem.

Theorem 6. *If $n \geq r + 1$ then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ is a constant multiple of the complete lifting $\mathcal{P}^{r*}$.*

The proof of Theorem 6 will occupy the rest of this section and Section 5.

Given $b = (b_0, \dots, b_{r-1}) \in \mathbb{R}^r$ and $d = (d_1, \dots, d_{r-1}) \in \mathbb{R}^{r-1}$ let

$$\sigma_{d,b} = [j^r_0(\eta_{d,b})] \in P^{r*}_0\mathbb{R}^n, \tag{1}$$

where $\eta_{d,b} := (x^1)^r + \sum_{l=1}^{r-1} d_l(x^1)^l + \sum_{q=0}^{r-1} b_q x^{q+2}(x^1)^q : \mathbb{R}^n \to \mathbb{R}$.

We have the following reducibility lemma.

Lemma 1 (First Reducibility Lemma)**.** *Let $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ be a natural operator, $n \geq r+1$. If $A(\partial_1)_{\sigma_{d,b}} = 0$ for any $b \in \mathbb{R}^k$ and $d \in \mathbb{R}^{k-1}$, then $A = 0$. If $A(\partial_1)_{\sigma_{d,b}}$ is vertical for any $d \in \mathbb{R}^{r-1}$ and $b \in \mathbb{R}^n$, then A is of vertical type.*

Proof. It is sufficient to show that $A(\partial_1)_\sigma$ is equal to 0 (vertical) for any $\sigma \in P^{r*}_0\mathbb{R}^n$.

By the density argument we can assume that

$$\sigma = \Big[j^r_0\Big((x^1)^r + \sum_{l=1}^{r-1} d_l(x^1)^l + \sum_{q=0}^{r-1}\gamma_q(x^2,\dots,x^n)(x^1)^q\Big)\Big]$$

for some smooth maps $\gamma_q : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\gamma_q(0) = 0$.

By the density argument we can assume that the system $(\gamma_q(x^2,\dots,x^n))_{q=0}^{r-1} : \mathbb{R}^n \to \mathbb{R}^r$ is of rank r at $0 \in \mathbb{R}^n$. Then there exists an embedding $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ preserving 0, ∂_1 and x^1 near 0 and sending $(\gamma_q(x^2,\dots,x^n))_{q=0}^{r-1}$ into $(x^2,\dots,x^{r+1})$. Now using the invariance of A with respect to φ we can assume that $\sigma = \sigma_{d,b}$. □

Now, we prove the following decomposition lemma.

Lemma 2 (Decomposition Lemma)**.** *Let $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ be a natural operator, $n \geq r+1$. Then there exists $\alpha \in \mathbb{R}$ such that $A - \alpha\mathcal{P}^{r*}$ is a vertical operator.*

Proof. For every $a \in \mathbb{R}$, $d \in \mathbb{R}^{r-1}$ and $b \in \mathbb{R}^r$ we can write

$$T\pi(A(a\partial_1)_{\sigma_{d,b}}) = \sum_{i=1}^n \alpha_i(a,d,b)\partial_{i|0}$$

for some smooth maps $\alpha_i : \mathbb{R} \times \mathbb{R}^{r-1} \times \mathbb{R}^r \to \mathbb{R}$.

Using the invariance of A with respect to the homotheties $(\tau x^1, x^2, \dots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau \neq 0$ we get

$$\begin{aligned}&\tau\alpha_1(a,d,b)\partial_{1|0} + \sum_{i=2}^n \alpha_i(a,d,b)\partial_{i|0}\\ &\quad = T\pi\Big(A(\tau a\partial_1)_{[j^r_0(\frac{1}{\tau^r}(x^1)^r + \sum_{l=1}^{r-1}\frac{1}{\tau^l}d_l(x^1)^l + \sum_{q=0}^{r-1}\frac{1}{\tau^q}b_q x^{q+2}(x^1)^q)]}\Big)\end{aligned}$$

$$= T\pi\Big(A(\tau a\partial_1)_{[(x^1)^r+\sum_{l=1}^{r-1}\tau^{r-l}d_l(x^1)^l+\sum_{q=0}^{r-1}\tau^{r-q}b_q x^{q+2}(x^1)^q)]}\Big).$$

Then we get homogeneity conditions
$\tau\alpha_1(a,d,b) = \alpha_1(\tau a, (\tau^{r-l}d_l), (\tau^{r-q}b_q))$ and
$\alpha_i(a,d,b) = \alpha_i(\tau a, (\tau^{r-l}d_l), (\tau^{r-q}b_q))$ for $i = 2,\dots,n$ and $\tau \neq 0$.

Now, by the homogeneous function theorem, [5], $\alpha_1(a,d,b)$ is the linear combination of a, d_{r-1} and b_{r-1} with real coefficients and $\alpha_i(a,d,b) = const$ for $i = 2,\dots,n$.

Since $A(0)$ corresponds to the absolute operator, $A(0) = 0$ because of Corollary 1. Then $\alpha_i = 0$ for $i = 2,\dots,n$, and $\alpha_1(a,d,b) = \alpha_1 a$ for some $\alpha_1 \in \mathbb{R}$. Then $T\pi(A(\partial_1)_{\sigma_{d,b}}) = \alpha_1\partial_{1|0} = \alpha_1 T\pi(\mathcal{P}^{r*}(\partial_1)_{\sigma_{d,b}})$. Hence $A - \alpha_1\mathcal{P}^{r*}$ is a vertical operator because of the reducibility lemma (Lemma 1). □

5. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ of vertical type

Thanks to the decomposition lemma (Lemma 2), Theorem 6 will be proved after proving the following proposition.

Proposition 1. *If $n \geq r+1$ then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ of vertical type is* 0.

Proof. From now on $A : T_{|\mathcal{M}f_n} \rightsquigarrow TP^{r*}$ is a natural operator of vertical type, where $n \geq r+1$.

We will use the notations of Section 4.

Since A is vertical, $A(X)|P^{r*}_0\mathbb{R}^n$ is a vector field on $P^{r*}_0\mathbb{R}^n$ for every $X \in \mathcal{X}(\mathbb{R}^n)$. Let $\{F^{A(X)}_t\}$ denotes the flow of $A(X)|P^{r*}_0\mathbb{R}^n$, $X \in \mathcal{X}(\mathbb{R}^n)$. Since every projective space is compact, the flow $\{F^{A(X)}_t\}$ is global.

Let $a \in \mathbb{R}$, $b = (b_q)_{q=0}^{r-1} \in \mathbb{R}^r$, $d = (d_l)_{l=1}^{r-1} \in \mathbb{R}^{r-1}$ and $t \in \mathbb{R}$ be arbitrary. Then we have $\sigma_{d,b} \in P^{r*}_0\mathbb{R}^n$, see Section 4.

Step 1. On the points $F^{A(a\partial_1)}_t(\sigma_{d,b})$.

Clearly, $F^{A(0)}_0(\sigma_{(0),(0)}) = [j^r_0((x^1)^r)]$. So, there is $\epsilon > 0$ such that

$$F^{A(a\partial_1)}_t(\sigma_{d,b}) = \Big[j^r_0\Big((x^1)^r + \sum_{\alpha\in G} B_\alpha(t,a,d,b)x^\alpha\Big)\Big] \tag{2}$$

for all $(t,a,d,b) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$, where $B_\alpha : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r \to \mathbb{R}$ are the smooth maps. Here G is the set of all $\alpha=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$ with $1\le|\alpha|\le r$ and $\alpha\ne(r,0,\dots,0)$.

Step 2. On the maps $B_{\alpha,s}(t,a,d,b)$.

We use the invariance of $A(a\partial_1)$ with respect to $(x^1,\frac{1}{\tau_2}x^2,\dots,\frac{1}{\tau_n}x^n):\mathbb{R}^n\to\mathbb{R}^n$ for all $\tau_i\ne 0$ with $|\tau_i|<1$. We obtain the homogeneity condition

$$B_\alpha(t,a,d,(\tau_{q+2}b_q)) = B_\alpha(t,a,d,b)(\tau_2)^{\alpha_2}\dots(\tau_n)^{\alpha_n}$$

for $\alpha\in G$, where $(t,a,d,b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$. Now, we apply the (obviously adapted) homogeneous function theorem, [2]. We deduce that $B_\alpha=0$ for all $\alpha\in G$ with $\alpha_{r+2}+\dots+\alpha_n\ne 0$, and

$$B_\alpha(t,a,d,b) = B_\alpha(t,a,d)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}} \tag{3}$$

for all $\alpha\in G$ with $\alpha_{r+2}+\dots+\alpha_n=0$, where $B_\alpha:(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\to\mathbb{R}$ are the smooth maps.

Hence

$$F_t^{A(a\partial_1)}(\sigma_{d,b}) = \Big[j_0^r\Big((x^1)^r + \sum_{\alpha\in H} B_\alpha(t,a,d)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}}x^\alpha\Big)\Big] \tag{4}$$

for all $(t,a,d,b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$. Here H is the set of all $\alpha=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$ with $1\le|\alpha|\le r$, $\alpha\ne(r,0,\dots,0)$ and $\alpha_{r+2}+\dots+\alpha_n=0$.

Step 3. On the maps $B_\alpha(t,a,d)$ for $\alpha\in H$.

Using the invariance of $F_t^{A(a\partial_1)}(\sigma_{d,b})$ with respect to the local diffeomorphisms $(x^1,(x^{q+2}+\mu(x^{q+2})^{r-q+1})_{q=0}^{r-1},x^{r+2},\dots,x^n)^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ for all μ we get the condition

$$\begin{aligned} &j_0^r\Big(\sum_{\alpha\in H}B_\alpha(t,a,d)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}}(x^1)^{\alpha_1}\prod_{q=0}^{r-1}(x^{q+2})^{\alpha_{q+2}}\Big)\\ &= j_0^r\Big(\sum_{\alpha\in H}B_\alpha(t,a,d)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}}(x^1)^{\alpha_1}\prod_{q=0}^{r-1}(x^{q+2}+\mu(x^{q+2})^{r-q+1})^{\alpha_{q+2}}\Big).\end{aligned}$$

Both sides of the last equality are polynomials in μ. Considering the coefficients corresponding to $\mu = \mu^1$ we get

$$j^r_0\Big(\sum_{k=0}^{r-1}\sum_{\alpha\in H}\alpha_{k+2}B_\alpha(t,a,d)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}}(x^1)^{\alpha_1} \times\prod_{q=0}^{r-1}(x^{q+2})^{\alpha_{q+2}}(x^{k+2})^{r-k}\Big) = 0.$$

Therefore we have the implication:

($*$) If $\alpha \in H$ and $k = 0, \dots, r-1$ are such that $B_\alpha \neq 0$ and $\alpha_{k+2} \neq 0$, then $\alpha_2 + \dots + \alpha_{r+1} \geq k + 1 - \alpha_1$.

Step 4. On the maps $B_\alpha(t,a,d)$ for $\alpha \in H$ anew.

Let $\alpha \in H$.

Using the invariance of $A(a\partial_1)$ with respect to $(\tau x^1, x^2, \dots, , x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau \neq 0$ with $|\tau| < 1$ we obtain $B_\alpha(t,a,d,b)\tau^{r-\alpha_1} = B_\alpha(t,\tau a, d_l\tau^{r-l}, b_q\tau^{r-q})$ for all $(t,a,d,b) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$. Applying (3) we can write this condition in the form

$$B_\alpha(t,a,d)\tau^{r-\alpha_1} = B_\alpha(t,\tau a, d_l\tau^{r-l})\prod_{q=0}^{r-1}(\tau^{r-q})^{\alpha_{q+2}}$$

for all $(t,a,d) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}$ and $0 < |\tau| < 1$.

Hence

$$B_\alpha = 0 \text{ if } \sum_{q=0}^{r-1}(r-q)\alpha_{q+2} > r - \alpha_1,$$

$$B_\alpha \text{ depends only on } t \text{ if } \sum_{q=0}^{r-1}(r-q)\alpha_{q+2} = r - \alpha_1$$

and

$$B_\alpha = 0 \text{ if } \alpha_{k+2} \neq 0 \text{ for some } k = 0,\dots,r-1 \text{ and } \sum_{q=0}^{r-1}(r-q)\alpha_{q+2} < r - \alpha_1.$$

(The last implication we can prove as follows. Suppose that $B_\alpha \neq 0$, $\alpha_{k+2} \neq 0$ and $\sum_{q=0}^{r-1}(r-q)\alpha_{q+2} < r - \alpha_1$. Then $r - k - 1 + (\alpha_2 + \dots +$

$\alpha_{r+1}) \leq \sum_{q=0}^{r-1}(r-q)\alpha_{q+2} < r-\alpha_1$ and by the implication $(*)$ we get $r-k-1+\alpha_1 < r-(\alpha_2+\cdots+\alpha_{r+1}) \leq r-k-1+\alpha_1$. Contradiction.)

Step 5. On the points $F_t^{A(a\partial_1)}(\sigma_{d,b})$ anew.

Because of the Step 4 and (4) we can write

$$F_t^{A(a\partial_1)}(\sigma_{d,b}) = \Big[j_0^r\Big((x^1)^r + \sum_{j=1}^{r-1} B_j(t,a,d)(x^1)^j + \sum_{\alpha\in J} B_\alpha(t)\prod_{q=0}^{r-1}(b_q)^{\alpha_{q+2}}x^\alpha\Big)\Big]$$

for all $(t,a,d,b) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$, where $B_\alpha : (-\epsilon,\epsilon)\to\mathbb{R}$ and $B_j : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\to\mathbb{R}$ are the smooth maps. Here J is the set of all $\alpha\in H$ such that $\sum_{q=0}^{r-1}(r-q)\alpha_{q+2} = r-\alpha_1$ (then $\alpha_{q+2}\neq 0$ for some $q=0,\dots,r-1$).

If $a=0$ we get $F_t^{A(0)}(\sigma_{d,b}) = \sigma_{d,b}$ as $A(0)=0$ because of Corollary 1. Hence

$$F_t^{A(a\partial_1)}(\sigma_{d,b}) = \Big[j_0^r\Big((x^1)^r + \sum_{j=1}^{r-1} B_j(t,a,d)(x^1)^j + \sum_{q=0}^{r-1} b_q x^{q+2}(x^1)^q\Big)\Big] \tag{5}$$

for all $(t,a,d,b) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\times(-\epsilon,\epsilon)^r$, where $B_j : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}\to\mathbb{R}$ are the smooth maps.

Step 6. On the maps $B_j(t,a,d)$.

Let B_j be as in Step 5. We have

$$\begin{aligned} &F_t^{A(a\partial_1)}\Big(\Big[j_0^r\Big((x^1)^r + \sum_{l=1}^{r-1} d_l(x^1)^l\Big)\Big]\Big) \\ &= \Big[j_0^r\Big((x^1)^r + \sum_{j=1}^{r-1} B_j(t,a,d)(x^1)^j\Big)\Big] \end{aligned} \tag{6}$$

for all $(t,a,d) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-1}$.

Using the invariance of A with respect to $(\tau x^1, x^2, \dots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau \neq 0$ with $|\tau| < 1$ we get the homogeneity conditions

$$B_j(t,a,d)\tau^{r-j} = B_j(t,\tau a, \big(\tau^{r-l}d_l)\big)$$

for all $(t,a,d) \in (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)^{r-1}$. Now, by the homogeneous function theorem we have

$$B_j(t,a,d) = C_j(t)a^{r-j} + D_j(t)d_j + \tilde{B}_j(t,a,d_{j+1},\dots,d_{r-1}) \qquad (7)$$

for $j = 1,\dots,r-1$, where $C_j : (-\epsilon,\epsilon) \to \mathbb{R}$, $D_j : (-\epsilon,\epsilon) \to \mathbb{R}$ and $\tilde{B}_j$: $(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)^{r-j-1} \to \mathbb{R}$ are smooth and $\tilde{B}_j(t,a,d_{j+1},\dots,d_{r-1})$ is the finite linear combination of monomials in a and $d_{j+1},\dots,d_{r-1}$, not equal to a^{r-j}, with coefficients being smooth maps depending on t. In particular, $\tilde{B}_{r-1}(t,a,d) = 0$.

Step 7. $A(a\partial_1)_{|[j_0^r((x^1)^r)]} = 0$

Using the invariance of $A(a\partial_1)_{|[j_0^r((x^1)^r)]}$ with respect to the diffeomorphisms $(x^1 + \mu(x^2)^2, x^2, \dots, x^n)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ for all μ we get

$$\left[j_0^r\left((x^1)^r + \sum_{j=1}^{r-1} B_j(t,a,(0))(x^1)^j\right)\right] = F_t^{A(a\partial_1)}\big([j_0^r((x^1)^r)]\big)$$
$$= \left[j_0^r\left((x^1)^r + \sum_{j=1}^{r-1} B_j(t,a,(0))\big(x^1 + \mu(x^2)^2\big)^j\right)\right]$$

for $t, a \in (-\epsilon,\epsilon)$. Hence

$$j_0^r\left(\sum_{j=1}^{r-1} B_j(t,a,(0))(x^1)^j\right) = j_0^r\left(\sum_{j=1}^{r-1} B_j(t,a,(0))(x^1 + \mu(x^2)^2)^j\right).$$

Both sides of this equality are polynomials in μ. Considering the coefficients on $\mu = \mu^1$ we get $j_0^r(\sum_{j=1}^{r-1} jB_j(t,a,(0))(x^1)^{j-1}(x^2)^2) = 0$. Hence $B_j(t,a,(0)) = 0$ for $j = 1,\dots,r-1$ and small t, a. Therefore $F_t^{A(a\partial_1)}([j_0^r((x^1)^r)]) = [j_0^r((x^1)^r)]$, i.e. $A(a\partial_1)_{|[j_0^r((x^1)^r)]} = 0$ for $a \in (-\epsilon,\epsilon)$.

Step 8. $B_{r-1}(t,a,d) = d_{r-1}$

By (7) we have $B_{r-1}(t,a,d_{r-1}) = C_{r-1}(t)a + D_{r-1}(t)d_{r-1}$.

Since $A(a\partial_1)_{[j_0^r((x^1)^r)]} = 0$, we get $[j_0^r((x^1)^r + C_{r-1}(t)a(x^1)^{r-1} + \dots)] = F_t^{A(a\partial_1)}([j_0^r((x^1)^r)]) = [j_0^r((x^1)^r)]$, i.e. $C_{r-1}(t) = 0$.

Since $A(0)_{\sigma_{d,(0)}} = 0$ (Corollary 1), $[j_0^r((x^1)^r + D_{r-1}(t)d_{r-1}(x^1)^{r-1} + \dots)] = F_t^{A(0)}(\sigma_{d,(0)}) = \sigma_{d,(0)} = [j_0^r((x^1)^r + d_{r-1}(x^1)^{r-1} + \dots)]$. Hence $D_{r-1}(t) = 1$.

Then $B_{r-1}(t, a, d) = d_{r-1}$ for small t, a, d.

Step 9. $B_j(t, a, d) = d_j$ *for* $j = 1, \dots, r-1$.

We will proced by the induction on j.

(1) If $j = r-1$, $B_{r-1}(t, a, d) = d_{r-1}$, see Step 8.

(2) Assume that $B_{j+1}(t, a, d) = d_{j+1}, \dots, B_{r-1}(t, a, d) = d_{r-1}$ for small t, a, d. We prove that $B_j(t, a, d) = d_j$ as follows.

From the inductive assumption it follows that

$$
\begin{aligned}
&F_t^{A(a\partial_1)}\left(\left[j_0^r\left((x^1)^r + \sum_{l=j+1}^{r-1} d_l(x^1)^l\right)\right]\right) \\
&\quad = \left[j_0^r\left((x^1)^r + \sum_{l=j+1}^{r-1} d_l(x^1)^l \right.\right. \\
&\qquad\qquad \left.\left. + \sum_{l=1}^{j} B_l(t, a, 0, \dots, 0, d_{j+1}, \dots, d_{r-1})(x^1)^l\right)\right].
\end{aligned}
$$

Now, by the invariance with respect to the diffeomorphisms $(x^1 + \mu(x^2)^{r-j+1}, x^2, \dots, x^n)^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ preserving $a\partial_1$ and $[j_0^r((x^1)^r + \sum_{l=j+1}^{r-1} d_l(x^1)^l)]$ for all μ we get

$$
\begin{aligned}
&\left[j_0^r\left((x^1)^r + \sum_{l=j+1}^{r-1} d_l(x^1)^l \right.\right. \\
&\qquad \left.\left. + \sum_{l=1}^{j} B_l(t, a, 0, \dots, 0, d_{j+1}, \dots, d_{r-1})(x^1)^l\right)\right] \\
&= \left[j_0^r\left((x^1)^r + \sum_{l=j+1}^{r-1} d_l(x^1)^l \right.\right.
\end{aligned}
$$

$$+\sum_{l=1}^{j} B_l(t,a,0,d_{j+1},\dots,d_{r-1})(x^1+\mu(x^2)^{r-j+1})^l\Big)\Big].$$

Hence

$$j_0^r\Big(\sum_{l=1}^{j} B_l(t,a,0,\dots,0,d_{j+1},\dots,d_{r-1})(x^1)^l\Big)$$
$$= j_0^r\Big(\sum_{l=1}^{j} B_l(t,a,0,d_{j+1},\dots,d_{r-1})(x^1+\mu(x^2)^{r-j+1})^l\Big).$$

Both sides of the last equality are the polynomials in μ. Considering the coefficients on μ we get $j_0^r\big(\sum_{l=1}^{j} lB_l(t,a,0,d_{j+1},\dots,d_{r-1})(x^1)^{l-1}(x^2)^{r-j+1}\big)=0$. Then, in particular, $B_j(t,a,0,d_{j+1},\dots,d_{r-1})=0$, i.e. $\tilde{B}_j(t,a,d_{j+1},\dots,d_{r-1})=0$ and $C_j(t)=0$. Hence $B_j(t,a,0,d_j,\dots,d_{r-1})=D_j(t)d_j$. Now, since $A(0)=0$ (see Corollary 1), $F_t^{A(0)}\sigma_{d,0}=\sigma_{d,0}$, i.e. $D(t)=1$. So, $B_j(t,a,d_j,\dots,d_{r-1})=d_j$.

Step 10. The end of the proof of Proposition 1.

Because of formula (5) and Step 9 we have $F_t^{A(a\partial_1)}(\sigma_{d,b})=\sigma_{d,b}$ for all small t,a,d,b. Hence $A(a\partial_1)_{\sigma_{d,b}}=0$ for small a, d, b.

Using the naturality of A with respect to $(\tau x^1,x^2,\dots,x^n)$ for $\tau\neq 0$ it is easy to show that $A(a\partial_1)_{\sigma_{d,b}}=0$ for any $a\in\mathbb{R}^n$, $b\in\mathbb{R}^k$ and $d\in\mathbb{R}^{k-1}$. Then the reducibility lemma (Lemma 1) ends the proof of Proposition 1.

The proof of Theorem 6 is complete. □

6. The natural affinors on $P^{r*}=P(T^{r*})$

In this section we study the natural affinors on P^{r*}. We prove the following theorem.

Theorem 7. *Let $n\geq r+1$. Every natural affinor C on P^{r*} over n-manifolds is a constant multiple of the identity one.*

At first we prove the following reducibility lemma.

Lemma 3 (Second Reducibility Lemma)**.** *Let $C : TP^{r*} \to TP^{r*}$ be a natural affinor on $P^{r*} : \mathcal{M}f_n \to \mathcal{FM}$, $n \geq 2$. Assume that $C(\mathcal{P}^{r*}(\partial_1)_\sigma) = 0$ for every $\sigma \in P^{r*}_0\mathbb{R}^n$. Then $C = 0$.*

PROOF. Since $\sigma_o = [j^r_0(x^1)] \in P^{r*}_0\mathbb{R}^n$ has dense orbit in $P^{r*}\mathbb{R}^n$ with respect to $\mathrm{Diff}(\mathbb{R}^n, \mathbb{R}^n)$, it is sufficient to verify that $C(v) = 0$ for any $v \in T_{\sigma_o}P^{r*}\mathbb{R}^n$.

Because of the linearity we can assume $v = \mathcal{P}^{r*}(\partial_i)_{\sigma_o}$ for $i = 1, \ldots, k$ or $v = \frac{d}{dt}_{t=0}[j^r_0(x^1) + tj^r_0\gamma]$, where $\gamma : \mathbb{R}^n \to \mathbb{R}$, $\gamma(0) = 0$.

Since the isomorphism $(x^1, \ldots, x^{i-1}, x^i + x^1, x^{i+1}, \ldots, x^n) : \mathbb{R}^n \to \mathbb{R}^n$ preserves σ_o and sends ∂_1 into $\partial_1 + \partial_i$ and C is natural and fibre linear we can assume $v = \mathcal{P}^{r*}(\partial_1)_{\sigma_o}$ instead of $v = \mathcal{P}^{r*}(\partial_i)_{\sigma_o}$.

By the density argument one can assume that $(x^1, \gamma) : \mathbb{R}^n \to \mathbb{R}^2$ is of rank 2 at $0 \in \mathbb{R}^n$. Then using a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ preserving x^1 and sending γ into x^2 near $0 \in \mathbb{R}^n$ we can assume that $\gamma = x^2$.

Using the flow method it is easy to verify that $\mathcal{P}^{r*}(x^2\partial_1)_{\sigma_o} = \frac{d}{dt}_{t=0}[j^r_0(x^1) + tj^r_0(x^2)]$.

So, it is sufficient to assume that $v = \mathcal{P}^{r*}(\partial_1 + x^2\partial_1)_{\sigma_o}$ or $v = \mathcal{P}^{r*}(\partial_1)_{\sigma_o}$. Since $\partial_1 + x^2\partial_1 = \varphi_*\partial_1$ near $0 \in \mathbb{R}^n$ for some diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ preserving 0, it is sufficient to assume that $v = \mathcal{P}^{r*}(\partial_1)_\sigma$, $\sigma \in P^{r*}_0\mathbb{R}^n$. □

PROOF of Theorem 7. Using C we have the natural operator $C \circ \mathcal{P}^{r*} : T_{|\mathcal{M}f_n} \to TP^{r*}$. By Theorem 6, $C \circ \mathcal{P}^{r*} = \alpha\mathcal{P}^{r*}$ for some $\alpha \in \mathbb{R}$. Then $C(\mathcal{P}^{r*}(\partial_1)_\sigma) = \alpha\mathcal{P}^{r*}(\partial_1)_\sigma$ for all $\sigma \in P^{r*}_0\mathbb{R}^n$. Hence $C = \alpha id$ because of the second reducibility lemma (Lemma 3). □

7. The natural operators $T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$

Let $\omega : TM \to \mathbb{R}$ be a 1-form on M and $q : Y \to M$ be a fibre bundle. Then we have a 1-form $\omega^V = \omega \circ Tq : TY \to \mathbb{R}$ on Y. It is called the vertical lifting of ω to Y.

Theorem 8. *Let $n \geq 2$. Every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$ is a constant multiple of the vertical lifting $D^V : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$.*

Lemma 4 (Third Reducibility Lemma). *Let $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*P^{r*}$ be a natural operator, $n \geq 2$. Suppose that $D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma) = 0$ for any $\sigma \in P^{r*}_0\mathbb{R}^n$ and any $\omega \in \Omega^1(\mathbb{R}^n)$. Then $D = 0$.*

Proof. The proof is an obvious modification of Lemma 3. □

Proof of Theorem 8. Because of Theorem 4 we can assume $r \geq 2$.

Consider arbitrary $\sum_{i=1}^n \omega_i dx^i \in \Omega^1(\mathbb{R}^n)$ and arbitrary $\sigma = [\sum_{\alpha\in G} a_\alpha x^\alpha] \in P^{r*}_0\mathbb{R}^n$, where G is the set of all $\alpha \in (\mathbb{N}\cup\{0\})^n$ with $1 \leq |\alpha| \leq r$. Because of Lemma 4 we will study $D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma) \in \mathbb{R}$.

By the density argument we can assume that $a_{(1,0,\dots,0)} \neq 0$. Then replacing a_α by $\frac{a_\alpha}{a_{(1,0,\dots,0)}}$ we can assume that $a_{(1,0,\dots,0)} = 1$.

Using the naturality of D with respect to $(x^1, \frac{1}{\tau}x^2, \dots, \frac{1}{\tau}x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $\tau \neq 0$ and next putting $\tau \to 0$ we obtain

$$D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma) = D(\omega_1(x^1, 0, \dots, 0)dx^1) \times \Big(\mathcal{P}^{r*}(\partial_1)_{[j^r_0(x^1+\sum_{j=2}^r a_{(j,0,\dots,0)}(x^1)^j)]}\Big).$$

Now, by the nonlinear Petree theorem, [5], there is $R \in \mathbb{N}$ such that

$$D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma) = D\Big(\sum_{k=1}^R \omega_{1,k}(x^1)^k dx^1\Big) \times \Big(\mathcal{P}^{r*}(\partial_1)_{[j^r_0(x^1+\sum_{j=2}^r a_{(j,0,\dots,0)}(x^1)^j)]}\Big),$$

where $\omega_{1,k} = \frac{1}{k!}\frac{\partial^k\omega_1}{\partial(x^1)^k}(0)$.

Using the naturality of D with respect to $(\tau x^1, x^2, \dots, x^n)$ for $\tau \neq 0$ we get the homogeneity condition

$$D\Big(\sum_{k=1}^R \omega_{1,k}(x^1)^k dx^1 \tau^{k+1}\Big)\Big(\mathcal{P}^{r*}(\partial_1)_{[j^r_0(x^1+\sum_{j=2}^r \tau^{j-1}a_{(j,0,\dots,0)}(x^1)^j)]}\Big) = \tau D\Big(\sum_{k=1}^R \omega_{1,k}(x^1)^k dx^1\Big)\Big(\mathcal{P}^{r*}(\partial_1)_{[j^r_0(x^1+\sum_{j=2}^r a_{(j,0,\dots,0)}(x^1)^j)]}\Big).$$

Hence by the homogeneous function theorem

$$D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma) = \alpha\omega_1(0) + \beta a_{(2,0,\dots,0)}$$

for some real numbers α and β.

Replacing D by $D-\alpha D^V$ we can assume that $\alpha=0$. Then

$$D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma)=\beta\frac{a_{(2,0,\dots,0)}}{a_{(1,0,\dots,0)}}.$$

Suppose $a_{(2,0,\dots,0)}=1$. Then $\mathcal{P}^{r*}(\partial_1)_\sigma$ has the limit in $TP_0^{r*}\mathbb{R}^n$ as $a_{(1,0,\dots,0)}$ tends to 0. Then $D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma)$ has the (finite) limit as $a_{(1,0,\dots,0)}$ tends to 0. Then $\beta=0$.

Then $D(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma)=\alpha D^V(\omega)(\mathcal{P}^{r*}(\partial_1)_\sigma)$ for any $\sigma\in P_0^{r*}\mathbb{R}^n$ and any $\omega\in\Omega^1(\mathbb{R}^n)$. Hence $D=\alpha D^V$ because of the third reducibility lemma.

This ends the proof of Theorem 8. □

8. Counterexamples

Let n, r and k be natural numbers.

Let $T_k^{r*}=J^r(.,\mathbb{R}^k)_0:\mathcal{M}f_n\to\mathcal{VB}$ be the vector bundle functor of (k,r)-covelocities and let $P(T_k^{r*}):\mathcal{M}f_n\to\mathcal{FM}$ be the projectivized (k,r)-covelocities functor. Clearly, $T_1^{r*}=T^{r*}$ and $P(T_1^{r*})=P(T^{r*})$.

Example 1. ($P(T_k^{r*})$ *is not rigid for* $k\geq 2$.) We define a natural transformation $B:P(T_k^{r*})\to P(T_k^{r*})$, $B:P(T_k^{r*}M)\to P(T_k^{r*}M)$, $B([j^r_{x_o}\gamma])=[j^r_{x_o}(1\gamma^1,2\gamma^2,\dots,k\gamma^k)]$, $[j^r_0(\gamma)]\in P_k^{r*}M$, $\gamma=(\gamma^1,\dots,\gamma^k):M\to\mathbb{R}^k$ $x_o\in M$, $\gamma(x_o)=0$, $M\in obj(\mathcal{M}f_n)$. Clearly, B is a well-defined natural transformation and if $k\geq 2$ then $B\neq id$.

Example 2. ($P(T_k^{r*})$ *is not poor for* $k\geq 2$.) We define a natural operator of vertical type $A:T_{|\mathcal{M}f_n}\rightsquigarrow T(P(T_k^{r*}))$, $A:\mathcal{X}(M)\to\mathcal{X}(P(T_k^{r*}M))$, $A([j^r_{x_o}\gamma])=\frac{d}{dt}_{t=0}[j^r_{x_o}\gamma+tj^r_{x_o}(1\gamma^1,2\gamma^2,\dots,k\gamma^k)]$, $[j^r_0(\gamma)]\in P_k^{r*}M$, $\gamma=(\gamma^1,\dots,\gamma^k):M\to\mathbb{R}^k$, $x_o\in M$, $\gamma(x_o)=0$, $M\in obj(\mathcal{M}f_n)$. Clearly, A is a well-defined natural operator of vertical type and if $k\geq 2$ then $A\neq 0$.

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