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Noncommutative line spaces derived from certain ovals of 4-dimensional translation planes

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Abstract. We start with a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that derivation defines a spread (partition) of $\mathbb{R}^4$. Using chords of the graph Γ of f we construct a system of curves of $\mathbb{R}^2$ having a base point. If Γ is an oval in the associated translation plane, then this system of curves can be endowed with a join operation such that we get a noncommutative line space in the sense of J. André.

1. Introduction

1.1. We construct examples of (in general) noncommutative geometries in the sense of J. André with point set $\mathbb{R}^2$. Such a geometry will be derived from any 4-dimensional compact projective translation plane containing a closed oval tangent to the translation axis. More precisely, the point set of the geometry will be the oval minus its point of tangency with the translation axis, and the blocks will be the socket curves on the oval that we shall introduce in Section 1.3. A socket curve contains a distinguished point, and this fact allows us to define an *a priori* noncommutative operation of joining ordered pairs of points.

We shall give an explicit description of four examples of such geometries. The ovals used for this are lines Γ_f of well known 4-dimensional shift planes, compare [9, Section 74]. It turns out that in these examples, each socket curve is a closed subset of Γ_f homeomorphic to $\mathbb{R}$. It remains an

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open question whether or not this is true in general. Exactly one of the four examples (namely that which is derived from a conic in the complex plane) turns out to be commutative (in fact, it is the real affine plane). Here the question arises whether or not there are other commutative (or other affine) examples that can be obtained in the same way. The non-commutative topological geometries we present are explicit examples for the theory developed in [7].

The shift planes generated by the four examples of ovals considered here are the parabola model of the complex plane and the complex skew parabola plane, see [9, 74.2], and the two single shift planes generated by the first and second Knarr surface, see [9, 74.24]. The generating ovals are either algebraic $\mathbb{R}$-varieties, or they are composed of two such varieties, a fact that makes computation easy.

It is unknown whether or not each oval in a translation plane tangent to the axis generates a shift plane, compare [9, 74.17]. The construction of the socket curves reflects some of the difficulties arising in this context. Hence, our paper can be seen as a contribution to the problems arising in connection with differentiating of shift planes and integration of translation planes, compare [9, Section 74].

1.2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \mapsto \Big(w(s,t), z(s,t)\Big)$ be an arbitrary differentiable function and $\Gamma_f := \Big\{\Big(s,t,w(s,t),z(s,t)\Big) =: p_{s,t} \;\Big|\; (s,t) \in \mathbb{R}^2\Big\}$ its graph. By virtue of $(x_1,x_2,x_3,x_4) \mapsto (1,x_1,x_2,x_3,x_4)\mathbb{R}$ we embed the affine space $\mathbb{R}^4$ into the projective space $\mathrm{PG}(4,\mathbb{R})$; by Ω we denote the 3-space at infinity ($x_0 = 0$) and by ℓ_{34} the line at infinity with $x_0 = x_1 = x_2 = 0$. Let $\tau_{s,t}$ be the tangent plane of Γ_f at the point $p_{s,t}$. We call f a *partition function*, if the line set

$$\{\tau_{s,t} \cap \Omega \mid (s,t) \in \mathbb{R}^2\} \cup \{\ell_{34}\} =: \mathcal{S}_f \tag{1}$$

is a spread of Ω. We speak of the *derived spread* $\mathcal{S}_f$ of the *partition surface* Γ_f. The spread $\mathcal{S}_f$ generates an affine translation plane $\mathcal{A}_{\mathcal{S}_f}$ whose point set is $\mathbb{R}^4$ and whose lines are the translates of the tangent planes $\tau_{s,t}$. Usually $\mathcal{A}_{\mathcal{S}_f}$ is called the *associated affine translation plane of* f and its projective closure $\mathcal{P}_{\mathcal{S}_f}$ the *associated projective translation plane of* f. The lines of the spread $\mathcal{S}_f$ can be interpreted as points at infinity of $\mathcal{P}_{\mathcal{S}_f}$, hence

$\mathcal{S}_f$ can be seen as translation line of $\mathcal{P}_{\mathcal{S}_f}$. In Section 2 we will prove that $\mathcal{P}_{\mathcal{S}_f}$ is a compact connected translation plane.

We speak of an *oval* partition surface Γ_f, if $\Gamma_h \cup \{\ell_{34}\}$ is[1] a compact oval in the associated projective translation plane $\mathcal{P}_{\mathcal{S}_f}$.

The logical transition from 1.1. to 1.2 is described in Section 5.

1.3. For the following we assume that Γ_f is a partition surface of $\mathbb{R}^4$. By a *chord of* Γ_f we mean a line of $\mathbb{R}^4$ which either is tangent to Γ_f or has at least two common points with Γ_f; by $\mathcal{C}_{\Gamma_f}$ we denote the set of all chords of Γ_f. Let L be an arbitrary line of $\mathbb{R}^4$ not meeting ℓ_{34}, then we call

$$\{X \in \mathcal{C}_{\Gamma_f} \mid X \parallel L\} =: \mathcal{Z}_L \tag{2}$$

the *chord cylinder*[2] *of* Γ_f *parallel to* L and

$$\bigcup_{X \in \mathcal{Z}_L} (X \cap \Gamma_f) =: a_L \tag{3}$$

socket curve of the chord cylinder $\mathcal{Z}_L$ or *socket curve corresponding to* L. Moreover, we consider the incidence structure $\mathfrak{G}_f = (\Gamma_f, \mathbf{A}_{\Gamma_f}, \in)$ where Γ_f is the set of points and the line set $\mathbf{A}_{\Gamma_f}$ consists of all socket curves on Γ_f; we speak of the *geometry* $\mathfrak{G}_f$ *of socket curves on* Γ_f. We can visualize $\mathfrak{G}_f$ in $\mathbb{R}^2$ by applying the *ground projection*

$$\gamma : \mathbb{R}^4 \to \mathbb{R}^4 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, 0, 0). \tag{4}$$

A point $p_{s,t} \in a_L \subseteq \Gamma_f$ is called a *base point of* a_L, if the tangent plane $\tau_{s,t}$ of Γ_f at $p_{s,t}$ is parallel to L. By a *star* $\mathbf{A}_{a,b}$*,* $(a,b) \in \mathbb{R}^2$*, with base point* $p_{a,b}$ we mean the set of all socket curves having the base point $p_{a,b}$. We will prove

Theorem 1. *Let* Γ_f *be an oval partition surface. Then each star* $\mathbf{A}_{a,b}$*,* $(a,b) \in \mathbb{R}^2$*, of socket curves on* Γ_f *with vertex* $p_{a,b}$ *is a simple (schlicht) covering of* $\Gamma_f \setminus \{p_{a,b}\}$.

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[1]Here we have to interpret ℓ_{34} as point at infinity of the associated translation plane $\mathcal{P}_{\mathcal{S}_f}$.

[2]We use the terms "chord cylinder" and, later, "socket curve" only as names.

Theorem 1 enables us to define an, in general, noncommutative geometry on Γ_f in the sense of J. André.

2. Constructing noncommutative line spaces from oval partition surfaces

Lemma 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a partition function, then the associated projective translation plane $\mathcal{P}_{\mathcal{S}_f}$ of f is compact and connected.*

Proof. The map $\sigma_f : \mathbb{R}^2 \to \mathcal{S}_f \setminus \{\ell_{34}\} : (s,t) \mapsto \tau_{s,t} \cap \Omega$ is a homeomorphism because f is of class C^1. By [9, 64.8(a), p. 355], $\mathcal{S}_f$ is compact and $\mathcal{P}_{\mathcal{S}_f}$ is a topological translation plane. Moreover, $\mathcal{S}_f$ is homeomorphic to the sphere $\mathbb{S}_2$. As $\mathbb{S}_2$ is compact and connected, so the same holds for the translation line $\mathcal{S}_f$ of $\mathcal{P}_{\mathcal{S}_f}$. Now [9, 41.7(a)] and [9, 42.1] show that $\mathcal{P}_{\mathcal{S}_f}$ is compact and connected. $\square$

By $\mathcal{L}[p_{s,t}, \tau_{s,t}]$ we denote the pencil of lines with vertex $p_{s,t}$ in the tangent plane $\tau_{s,t}$. Then for the star[3] $\mathbf{A}_{s,t}$ we have

$$\mathbf{A}_{s,t} = \{a_X \mid X \in \mathcal{L}[p_{s,t}, \tau_{s,t}]\}.$$

Proof of Theorem 1. Let $p_{s,t} \in \Gamma_f \setminus \{p_{a,b}\}$ be arbitrary and let $\tau_{a,b}$ be the tangent plane of Γ_f at $p_{a,b}$. There is exactly one plane $\tau^{\|}_{a,b}$ which is incident with $p_{s,t}$ and parallel to $\tau_{a,b}$. Now $\tau^{\|}_{a,b}$ must be different from the tangent plane of Γ_f at $p_{s,t}$ because otherwise the translation line $\mathcal{S}_f$, the line $\tau_{a,b}$, and the line $\tau^{\|}_{a,b}$ of $\mathcal{P}_{\mathcal{S}_f}$ would be three different concurrent tangents of the topological oval $\Gamma_f \cup \{\ell_{34}\}$ of $\mathcal{P}_{\mathcal{S}_f}$, a situation that contradicts [6, (3.7),p.412]. Hence the topological oval $\Gamma_f \cup \{\ell_{34}\}$ of $\mathcal{P}_{\mathcal{S}_f}$ and the line $\tau^{\|}_{a,b}$ of $\mathcal{P}_{\mathcal{S}_f}$ have exactly two points in common, namely $p_{s,t}$ and a point q with $q \in \Big(\Gamma_f \cup \{\ell_{34}\}\Big) \setminus \{p_{s,t}\}$. As $\tau_{a,b} \cap \Omega (\in \mathcal{S}_f \setminus \{\ell_{34}\})$ and ℓ_{34} are skew lines of Ω, so q and ℓ_{34} are different points of $\mathcal{P}_{\mathcal{S}_f}$, i.e., $q \in \Gamma_f$. Thus the line $p_{s,t} \vee q =: C$ is the only chord of Γ_f which is incident with $p_{s,t}$ and parallel to $\tau_{a,b}$. Therefore a_C is the only socket curve of $\mathbf{A}_{a,b}$ which contains $p_{s,t}$. $\square$

[3] We follow the notation of K. Niemann [7, p. iii].

We take the concept "line space" from [1, Chpt. 1]:

Definition 1. A structure $\mathfrak{R} = (X, \sqcup)$ with a set $X \neq \emptyset$ and a map

$$\sqcup : X^2 \to \mathfrak{P}X, \quad (x, y) \mapsto x \sqcup y \subseteq X$$

is called *line space (L-space)*, if for all $x, y \in X$ the following properties hold:

(L0) $x \sqcup x = \{x\}$

(L1) $x, y \in x \sqcup y$

(L2) $z \in (x \sqcup y) \setminus \{x\} \Rightarrow x \sqcup y = x \sqcup z$.

Subsets of the form $x \sqcup y$ are called *proper lines* for $x \neq y$ and *improper lines* for $x = y$.

Let $p_{a,b}, p_{s,t}$ be arbitrary points of the oval partition surface Γ_f; for $p_{a,b} \neq p_{s,t}$ we define $p_{a,b} \sqcup_f p_{s,t}$ to[4] be the unique socket curve of the star $\mathbf{A}_{a,b}$ which contains $p_{s,t}$; for $p_{a,b} = p_{s,t}$ we put $p_{a,b} \sqcup_f p_{s,t} := \{p_{a,b}\}$. Obviously, the structure $\mathfrak{R}_f := (\Gamma_f, \sqcup_f)$ is a line space. We call $\mathfrak{R}_f$ the *line space of socket curves on the oval partition surface* Γ_f. We can visualize $\mathfrak{R}_f$ in $\mathbb{R}^2$ by the γ-images of the socket curves.

Remark 1. Our construction can be extended to ovals Γ of any affine translation plane T, if Γ satisfies the following two conditions:

1. Γ plus one ideal point I is an oval of the projective completion of T.
2. $\Gamma \cup \{I\}$ does not have three confluent tangents.

The second condition is always true for closed ovals in compact connected planes of finite topological dimension and also for ovals in finite planes of odd order.

3. Examples of oval partition surfaces

A generating line of a 4-dimensional shift plane is also called *shift surface.*

Example 0. $f_0 : \mathbb{R}^2 \to \mathbb{R}^2 : (s, t) \mapsto (s^2 - t^2, 2st)$; by [9, 74.2], Γ_{f_0} is a shift surface and, by [9, p.430], Γ_{f_0} is a partition surface. We call Γ_{f_0} the *classical partition surface.*

[4] When it is clear from the situation we write the join symbol without subscript f.

Example 1. Assume that w is a fixed real number with $w > 1$ and

$$f_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \mapsto \begin{cases} (s^2 - t^2, 2st) & \text{for } t \geq 0, \\ (s^2 - wt^2, 2st) & \text{for } t < 0; \end{cases}$$

by [9, 74.2], Γ_{f_1} is a shift surface and, by [3, Proof of Satz 3], Γ_{f_1} is a partition surface. We call Γ_{f_1} a *skew classical partition surface*[5].

Example 2 and 3. $f_{[k]} : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \mapsto (st - \frac{1}{3}s^3 + ks, \frac{1}{2}(t^2 + ks^2) - \frac{1}{12}s^4)$; we put $f_2 =: f_{[0]}$ and $f_3 =: f_{[-1]}$ and call Γ_{f_2} and Γ_{f_3} the *first* resp. *second Knarr surface*. By [9, 74.24], Γ_{f_2} and Γ_{f_3} are shift surfaces and partition surfaces.

We show next that all four examples are oval partition surfaces.

Given two planar functions f and g on $\mathbb{R}$ which are both convex, Polster [8] constructs a planar function $f * g$ on $\mathbb{R}^2$, called the *product* of f and g, as follows

$$(f * g)(x_1, x_2) = (f(x_1) - g(x_2), x_1 x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

In particular, for $q : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ we get the shift surface Γ_{q*q} which is the image of the classical shift surface Γ_{f_0} under the affinity $\alpha : \mathbb{R}^4 \to \mathbb{R}^4 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, \frac{1}{2}x_4)$. Suppose $w > 1$ and choose $f = q$ and $g = q_w$ with

$$q_w : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x^2 & \text{for } x \leq 0 \\ wx^2 & \text{for } x > 0, \end{cases}$$

then $\Gamma_{q*q_w} = \alpha(\Gamma_{f_1})$. By [8, Prop.3.4.1 and 3.5.2], Γ_{q*q} and Γ_{q*q_w}, are oval partition surfaces, hence the same is valid for Γ_{f_0} and Γ_{f_1}.

The Knarr surfaces Γ_{f_2} and Γ_{f_3} are oval partition surfaces because of [4, Prop. 3].

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[5]This example belongs to a larger class of shift planes, see [9, 74.30 and 74.31].

Only now we are allowed to speak of the line spaces $\mathfrak{R}_{f_j}$ $(j = 0, \dots, 3)$ of socket curves on the classical, the skew classical partition surface, the first and second Knarr surface, respectively.

4. Testing the examples for commutativity

A line space $\mathfrak{R} = (X, \sqcup)$ with $x \sqcup y = y \sqcup x$ for all $x, y \in X$ is called *commutative*, otherwise *noncommutative*.

Theorem 2. (a) *The line space* $(\Gamma_{f_0}, \sqcup_{f_0})$ *of socket curves on the classical partition surface is commutative.*

(b) *The line space* $(\Gamma_{f_1}, \sqcup_{f_1})$ *of socket curves on a skew classical partition surface is noncommutative.*

(c) *The line spaces* $(\Gamma_{f_2}, \sqcup_{f_2})$ *and* $(\Gamma_{f_3}, \sqcup_{f_3})$ *of socket curves on the first resp. second Knarr surface are noncommutative.*

Proof. (a) The classical partition surface $\Gamma_{f_0} = \{(s, t, s^2 - t^2, 2st) =: p_{s,t} \mid s, t \in \mathbb{R}\}$ is the intersection of the two quadratic coordinate hypercylinders:

$$C_{\text{di}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 - x_2^2 - x_3 = 0\} \tag{5}$$

and

$$C_{\text{mu}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1x_2 - x_4 = 0\}. \tag{6}$$

Let $L = \{(d_1, d_2, d_3, d_4)\xi \mid \xi \in \mathbb{R}\}$ with $(d_1, d_2, d_3, d_4) \in \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\}$ be an arbitrary line not meeting ℓ_{34}, i.e., $(d_1, d_2) \neq (0, 0)$. In order to describe the socket curve a_L of the chord cylinder of Γ_{f_0} parallel to L, we take the line $D := \{p_{s,t} + (d_1, d_2, d_3, d_4) \cdot \xi \mid \xi \in \mathbb{R}\}$ and determine $D \cap C_{\text{di}}$ and $D \cap C_{\text{mu}}$. This is equivalent to the determination of the zeros of the two polynomials

$$\xi \cdot p_1(\xi) \in \mathbb{R}[\xi] \text{ with } p_1(\xi) := \xi({d_1}^2 - {d_2}^2) + 2sd_1 - 2td_2 - d_3 \text{ resp.}$$

$$\xi \cdot p_2(\xi) \in \mathbb{R}[\xi] \text{ with } p_2(\xi) := 2\xi d_1 d_2 + 2sd_2 + 2td_1 - d_4$$

in the unknown ξ. The line D is a chord of Γ_{f_0} if, and only if, the polynomials $p_1(\xi)$ and $p_2(\xi)$ have a common zero. Firstly, we discuss the case with $d_1 d_2 \neq 0$ and $d_1^2 - d_2^2 \neq 0$.

Case $d_1d_2 \neq 0$ and $d_1^2 - d_2^2 \neq 0$. By [10, p. 55 Elimination] the two (linear) polynomials $p_1(\xi)$ and $p_2(\xi)$ have a common zero if, and only if, their resultant vanishes. For the resultant R of $p_1(\xi)$ and $p_2(\xi)$ we compute

$$R = -2d_2(d_1^2 + d_2^2)s + 2d_1(d_1^2 + d_2^2)t + 2d_1d_2d_3 - d_4(d_1^2 - d_2^2). \quad (7)$$

For fixed (d_1, d_2, d_3, d_4) and variable $(s,t) = (x_1, x_2) \in \mathbb{R}^2$ the condition $R = 0$ describes the γ-image of the socket curve a_L; obviously, $\gamma(a_L)$ is a (straight) line[6].

Also for the remaining cases the socket curve images $\gamma(a_L)$ are (straight) lines.

This shows together with Theorem 1 that the γ-image of $\mathfrak{R}_{f_0}$ is the classical model of the real affine plane.

(b) Now we put

$$p_{s,t} := \begin{cases} (s, t, s^2 - t^2, 2st) & \text{for } t \geq 0 \\ (s, t, s^2 - wt^2, 2st) & \text{for } t < 0 \end{cases}$$

with $w > 1$, and $\Gamma_{f_1}^{\geq}$ resp. $\Gamma_{f_1}^{\leq} := \{p_{s,t} \mid s, t \in \mathbb{R} \text{ and } t \geq 0 \text{ resp. } t \leq 0\}$; we denote the tangent plane of Γ_{f_1} at $p_{s,t}$ by $\tau_{s,t}$. Moreover,

$$C_w := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 - wx_2^2 - x_3 = 0\}. \quad (8)$$

Now $\Gamma_{f_1}^{\geq}$ and $\Gamma_{f_1}^{\leq}$ are proper subsets of the two different classical partition surfaces $\Phi_1 := C_{\text{di}} \cap C_{\text{mu}}$ and $\Phi_w := C_w \cap C_{\text{mu}}$, respectively. We get Γ_{f_1} by tacking together $\Gamma_{f_1}^{\geq}$ and $\Gamma_{f_1}^{\leq}$ along their common parabola $\Gamma_{f_1}^{\geq} \cap \Gamma_{f_1}^{\leq} = \{(s, 0, s^2, 0) \mid s \in \mathbb{R}\} := p_{\text{com}}$: at each point $x \in p_{\text{com}}$ the surfaces Φ_1 and Φ_w have the same tangent plane. We prove the assertion (b) by showing:

$$p_{0,0} \sqcup p_{1,1} \neq p_{1,1} \sqcup p_{0,0}. \quad (9)$$

Let $\tau_{0,0}^{\parallel}$ be the plane which is parallel to $\tau_{0,0}$ and incident with $p_{1,1}$; we compute:

$$\tau_{0,0} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 = x_4 = 0\}$$

[6]This way of finding the description of socket curves via a resultant has the advantage that it works, mutatis mutandis, for all partition surfaces that are algebraic varieties. The first and second Knarr surface Γ_{f_2} and Γ_{f_3} are of this kind.

and

$$\tau_{0,0}^{\parallel} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 = 0,\ x_4 = 2\}.$$

By Theorem 1, $(\tau_{0,0}^{\parallel} \setminus \{p_{1,1}\}) \cap \Gamma_{f_1}$ contains exactly one point, say p_{s_0,t_0}; we get:

$$p_{s_0,t_0} = \left(-w^{1/4}, -\frac{1}{w^{1/4}}, 0, 2\right).$$

According to the definition, $p_{0,0} \sqcup p_{1,1}$ is the socket curve corresponding to $G := p_{1,1} \vee p_{s_0,t_0}$ (where $\vee$ denotes the join in the affine space $\mathbb{R}^4$). Clearly,

$$p_{s_0,t_0} \in p_{0,0} \sqcup p_{1,1}. \tag{10}$$

Let $\tau_{1,1}^{\parallel}$ be the plane which is parallel to $\tau_{1,1}$ and incident with $p_{0,0}$, i.e.,

$$\tau_{1,1}^{\parallel} = \{(\xi, \eta, 2\xi - 2\eta, 2\xi + 2\eta) \mid (\xi, \eta) \in \mathbb{R}^2\}.$$

By Theorem 1, $(\tau_{1,1}^{\parallel} \setminus \{p_{0,0}\}) \cap \Gamma_{f_1}$ contains exactly one point, namely $p_{s_1,t_1} := (2, 2, 0, 8)$. Thus we have that $p_{1,1} \sqcup p_{0,0}$ is the socket curve corresponding to

$$H := p_{0,0} \vee p_{s_1,t_1} = \{(2, 2, 0, 8)\xi \mid \xi \in \mathbb{R}\}.$$

By $H^{\parallel}$ we denote the line which is parallel to H and incident with p_{s_0,t_0}. It suffices to show

$$\big(H^{\parallel} \setminus \{p_{s_0,t_0}\}\big) \cap \Gamma_{f_1} = \emptyset, \tag{11}$$

because from (11) follows that $H^{\parallel}$ is no chord of Γ_{f_1} and therefore

$$p_{s_0,t_0} \notin p_{1,1} \sqcup p_{0,0} \tag{12}$$

which together with (10) implies (9).

We are now going to prove (11). We compute

$$H^{\parallel} = \left\{\Big(-w^{1/4} + 2\xi, -\frac{1}{w^{1/4}} + 2\xi, 0, 2 + 8\xi\Big) =: b_\xi \;\Big|\; \xi \in \mathbb{R}\right\}.$$

It turns out to be convenient to begin with the determination of the intersection of $H^{\parallel} \setminus \{p_{s_0,t_0}\}$ and the quadratic hypercylinder C_{mu} described by (6):

$$(H^{\parallel} \setminus \{p_{s_0,t_0}\}) \cap C_{\mathrm{mu}} = \{b_{\xi_0}\} \quad \text{with} \quad \xi_0 = \frac{\sqrt{w} + 1 + 2\sqrt[4]{w}}{2\sqrt[4]{w}}.$$

We will show that the point b_{ξ_0} is not on the quadratic hypercylinder C_{di} described by (5). For $b_{\xi_0} \in C_{\mathrm{di}}$ we compute the following condition:

$$\begin{aligned} 0 &= w + 4w^{3/4} - 4w^{1/4} - 1 \\ &= (z-1)(z+1)(z+2-\sqrt{3})(z+2+\sqrt{3}) \text{ with } z = w^{1/4}. \end{aligned}$$

The zeros $z = 1$ and $z = -1$ are impossible because of $w > 1$ and $w \in \mathbb{R}$, respectively; as $-2+\sqrt{3} < 0$ and $-2-\sqrt{3} < 0$, so the two other zeros are also impossible. Consequently,

$$b_{\xi_0} \notin C_{\mathrm{di}}. \tag{13}$$

Secondly, we show that the point b_{ξ_0} is not on the quadratic hypercylinder C_w described by (8). For $b_{\xi_0} \in C_w$ we compute the following condition:

$$\begin{aligned} 0 &= -4w^{1/2} - 1 - 4w^{1/4} + w^2 + 4w^{7/4} + 4w^{3/2} \\ &= (z-1)\,(z+1)^3\, A(z)B(z)C(z) \end{aligned}$$

with

$$A(z) := \left(z^2 + \left(1-\sqrt{3}\right)z + 1\right), \quad B(z) := \left(z + \frac{1}{2} + \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2}\sqrt[4]{3}\right),$$

$$\text{and} \quad C(z) := \left(z + \frac{1}{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2}\sqrt[4]{3}\right).$$

As above $z = 1$ and $z = -1$ are impossible. The polynomial $A(z)$ has no real zero. Because of $-\frac{1}{2}-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{2}\sqrt[4]{3} < 0$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{2}\sqrt[4]{3} < 0$, the zeros of $B(z)$ and $C(z)$ are also impossible. Thus we have

$$b_{\xi_0} \notin C_w. \tag{14}$$

From (13) and (14) we deduce $b_{\xi_0} \notin \Phi_1 \cup \Phi_w$ which implies $b_{\xi_0} \notin \Gamma_{f_1}$. Hence we have $(H^{\|} \setminus \{p_{s_0,t_0}\}) \cap \Gamma_{f_1} = \emptyset$.

(c) We put

$$p_{s,t}^{[k]} := \left(s, t, st - \frac{1}{3}s^3 + ks, \frac{1}{2}(t^2 + ks^2) - \frac{1}{12}s^4\right),$$

$$C^{[k]}_{\text{cub}} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; x_1 x_2 - \frac{1}{3} x_1^3 + k x_1 - x_3 = 0 \right\},$$

and

$$C^{[k]}_{\text{biq}} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \frac{1}{2}(x_2^2 + k x_1^2) - \frac{1}{12} x_1^4 - x_4 = 0 \right\}$$

$$\text{for } k \in \{0, -1\}.$$

Knarr's surfaces are the intersections of the two coordinate cylinders $C^{[k]}_{\text{cub}}$ and $C^{[k]}_{\text{biq}}$, in symbols

$$\Gamma_{f_{[k]}} = C^{[k]}_{\text{cub}} \cap C^{[k]}_{\text{biq}} \quad \text{for } k \in \{0, -1\}.$$

The line $T_{[k]} := \{(1, 0, k, 0)\xi \mid \xi \in \mathbb{R}\}$ belongs to the tangent plane of $\Gamma_{f_{[k]}}$ at the point $p^{[k]}_{0,0}$. In order to determine the socket curve $a_{T_{[k]}}$ on $\Gamma_{f_{[k]}}$ we assume that $(s, t) \in \mathbb{R}^2$ is fixed and intersect the line $T^{\|}_{[k]} := \{p^{[k]}_{s,t} + (1, 0, k, 0)\xi \mid \xi \in \mathbb{R}\}$ with $C^{[k]}_{\text{cub}}$ and $C^{[k]}_{\text{biq}}$. This is equivalent to the determination of the zeros of the two polynomials $\xi \cdot p_1(\xi) \in \mathbb{R}[\xi]$ and $\xi \cdot p_2(\xi, k) \in \mathbb{R}[\xi]$ with

$$p_1(\xi) := \xi^2 + 3s\xi - 3t + 3s^2 \quad \text{and}$$

$$p_2(\xi, k) := (\xi + 2s)\left(-2s^2 - 2s\xi - \xi^2 + 6k\right)$$

in the unknown ξ. The line $T^{\|}_{[k]}$ is a chord of $\Gamma_{f_{[k]}}$ if, and only if, the polynomials $p_1(\xi)$ and $p_2(\xi, k)$ have a common zero. Firstly, $p_1(\xi)$ and $(\xi + 2s)$ have a common zero if, and only if, their resultant $R_1 := -3t + s^2$ vanishes. Secondly, $p_1(\xi)$ and $(-2s^2 - 2s\xi - \xi^2 + 6k)$ have a common zero if, and only if, their resultant $R_2(k) := s^4 - 6ks^2 + 36k^2 - 36kt + 9t^2$ vanishes.

For variable $(s, t) \in \mathbb{R}^2$ the condition $R_1 = 0$ yields the rational curve

$$\left\{ \left(s, \frac{1}{3}s^2, ks, -\frac{1}{36}s^4 + \frac{1}{2}ks^2 \right) \;\middle|\; s \in \mathbb{R} \right\} =: c_{[k]} \quad \text{for } k \in \{0, -1\}. \tag{15}$$

We will show now: $c_{[k]}$ *is the socket curve*[7] *corresponding to* $T_{[k]}$ *and* $p_{0,0}^{[k]}$ *is a base point on* $c_{[k]}$ *for* $k \in \{0, -1\}$.

Case $k=0$. The condition $R_2(0)=0$ implies $s^4+9t^2=0$; as $(s,t)\in\mathbb{R}^2$, so $s=t=0$. The point $p_{0,0}^{[0]}$ belongs to the curve $c_{[0]}$, hence $c_{[0]}$ is the complete socket curve on $\Gamma_{f_{[0]}}$ to the direction $T_{[0]}$, in symbols $c_{[0]} = a_{T_{[0]}}$. Obviously, $p_{0,0}^{[0]}$ is a base point of $a_{T_{[0]}}$.

Case $k=-1$. The condition $R_2(-1)=0$ implies $(3t+6)^2+s^4+6s^2=0$, and consequently, $s=0$ and $t=-2$. The line $K := \{p_{0,-2}^{[-1]} + (1,0,-1,0)\xi \mid \xi \in \mathbb{R}\}$ meets $\Gamma_{f_{[-1]}}$ exactly in $p_{0,-2}^{[-1]}$ and two complex conjugate points, therefore K is no real chord of $\Gamma_{f_{[-1]}}$ and $p_{0,-2}^{[-1]} \notin a_{T_{[-1]}}$. Thus $c_{[-1]}$ is the complete socket curve on $\Gamma_{f_{[-1]}}$ to the direction $T_{[-1]}$, in symbols $c_{[-1]} = a_{T_{[-1]}}$. Obviously, $p_{0,0}^{[-1]}$ is a base point of $a_{T_{[-1]}}$.

Because of $p_{3,3}^{[k]} \in c_{[k]} = a_{T_{[k]}}$, we have $p_{3,3}^{[k]} \in p_{0,0}^{[k]} \sqcup p_{3,3}^{[k]} = a_{T_{[k]}} = c_{[k]}$ for $k \in \{0,-1\}$. By $\tau_{3,3}^{[k]}$ we denote the tangent plane to the Knarr surface $\Gamma_{f_{[k]}}$ at $p_{3,3}^{[k]}$, and by $\tau_{3,3}^{[k]\|}$ the plane which is parallel to $\tau_{3,3}^{[k]}$ and incident with $p_{0,0}^{[k]}$ ($k \in \{0,-1\}$). We get:

$$\tau_{3,3}^{[k]\|} = \Big\{ x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\Big|\; (6-k)x_1 - 3x_2 + x_3 = (9-3k)x_1 - 3x_2 + x_4 = 0 \Big\}$$

for $k \in \{0,-1\}$. It is easy to verify

$$\Big(\tau_{3,3}^{[k]\|} \setminus \{p_{0,0}^{[k]}\}\Big) \cap c_{[k]} = \emptyset \quad \text{for } k \in \{0,-1\}. \tag{16}$$

By Theorem 1 $(\tau_{3,3}^{[k]\|} \setminus \{p_{0,0}^{[k]}\}) \cap \Gamma_{f_{[k]}}$ contains a single point, say $r^{[k]}$, and $r^{[k]} \in p_{3,3}^{[k]} \sqcup p_{0,0}^{[k]}$ ($k \in \{0,-1\}$). From (16) follows $r^{[k]} \notin c_{[k]} = a_{T_{[k]}} =$

[7] We point out that for the Knarr surfaces socket curves admitting a rational parametric description are the exception to the rule. In general, socket curves on the Knarr surfaces can be described as algebraic varieties.

$p_{0,0}^{[k]} \sqcup p_{3,3}^{[k]}$, therefore

$$p_{0,0}^{[k]} \sqcup p_{3,3}^{[k]} \neq p_{3,3}^{[k]} \sqcup p_{0,0}^{[k]} \quad \text{for } k \in \{0, -1\}. \qquad \square$$

5. Remark on ovals and partition surfaces

In Subsection 1.2 we took the approach using a partition surface, and then made the additional assumption that the surface is an oval. For the purposes of the present paper the point of view taken in 1.1 is more appropriate, since the differentiability of f will not be used. If we start from any 4-dimensional compact translation plane $\mathcal{P}$ containing a compact oval O tangent to the translation axis, then after passage to an affine representation with the translation axis being the line at infinity and the point of tangency of O being the infinite point of the y-axis, the affine part of the oval becomes an oval partition surface as defined in 1.2, and the corresponding spread is the spread defining the affine part of $\mathcal{P}$. This follows from the results of [6] as we show now.

Indeed, by the definition of an oval, O is the graph Γ_f of a function f. The main result of [6] asserts that O is a topological oval. This means that given any pair of sequences x_n, y_n in O such that both converge to the same point p, the secants $x_n \vee y_n$ (or, in case $x_n = y_n$, the tangents in these points) converge to the tangent at p. This implies that the function f is continuously differentiable and that the tangent planes of Γ_f in the analytic sense are just the geometric tangents of the oval. Moreover, it is shown in [6] that the tangents of O form an oval in the dual projective plane, and this implies that each point at infinity is incident with precisely one tangent. It follows that Γ_f is a partition surface defining the spread which generates $\mathcal{P}$.

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References

[1] J. André, Endliche nichtkommutative Geometrie, *Ann. Univ. Saraviensis Ser. Math.* **2**, no. 1 (1988).

[2] J. André, On non-commutative geometry, *Ann. Univ. Saraviensis Ser. Math.* **4**, no. 2 (1993).

[3] D. Betten, Komplexe Schiefparabel-Ebenen, *Abh. Math. Sem. Univ. Hamburg* **48** (1979), 76–88.

[4] D. Betten and B. Polster, Topological ovals in 4-dimensional translation planes, *Arch. Math.* **63** (1994), 284–288.

[5] D. Betten and R. Riesinger, Discussing Knarr's 2-surface of $\mathbb{R}^4$ which generates the first single shift plane, *Geom. Dedicata* **83** (2000), 329–342.

[6] T. Buchanan, H. Hähl and R. Löwen, Topologische Ovale, *Geom. Dedicata* **9** (1980), 401–424.

[7] K. Niemann, Topologische nichtkommutative Inzidenzgeometrien, *Dissertation*, *Kiel*, 1996.

[8] B. Polster, Continuous planar functions, *Dissertation*, *Erlangen*, 1993.

[9] H. R. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel, Compact Projective Planes, *de Gruyter*, *Berlin*, 1995.

[10] B. L. van der Waerden, Einführung in die algebraische Geometrie, 2. Auflage, *Springer*, *Berlin, Heidelberg, New York*, 1973.

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