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On the Todorov's conjecture for Nevanlinna classes

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Abstract. Let $f \in \mathcal{N}_2$ and $f^{-1}(w) = w + n_2(f)w^2 + n_3(f)w^3 + \dots$ where $\mathcal{N}_2$ is the well known Nevanlinna's class of the second type. The problem of finding the sharp lower and upper bounds of $n_k(f)$ over $\mathcal{N}_2$ for $n = 5, 6, \dots$ is open. We solve this problem for $n = 5, 6$.

1. Introduction

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the classes of Nevanlinna functions of the first and second type, respectively. $\mathcal{N}_1$ consists of all functions $g(z)$ of the form

$$g(z) = \int_{-1}^{1} \frac{d\mu(t)}{z - t}, \quad z \notin \{z - 1 \leq z \leq 1\} \tag{1}$$

where $\mu(t)$ is a probability measure in $[-1\,1]$, and $\mathcal{N}_2$ consists of all functions $f(z)$ of the form

$$f(z) \equiv g\left(\frac{1}{z}\right) \equiv \int_{-1}^{1} \frac{z\,d\mu(t)}{1 - tz} \tag{2}$$

in the appropriate cut of z-plane. In [1] it was noted that the functions (1) and (2) are univalent for $|z| > 1$ and $|z| < 1$, respectively. Now let

$$f^{-1}(w) = w + \sum_{n=1}^{\infty} n_k(f) w^k \tag{3}$$

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denote the inverse for any function defined by (2). The largest common region of series (3) is the disc $|w| < \frac{1}{2}$, (see [3], p. 345).

P. G. Todorov in [2] posed the problem of calculating the values M_k and m_k, $k = 2, 3, \dots$ where

$$M_k = \max_{f \in \mathcal{N}_2} n_k(f) \quad \text{and} \quad m_k = \min_{f \in \mathcal{N}_2} n_k(f).$$

He also proved that

$$M_2 = M_3 = 1, \quad m_2 = m_3 = -1.$$

Let now

$$f_\lambda(z) = \frac{\lambda+1}{2}\,\frac{z}{1+z} + \frac{\lambda-1}{2}\,\frac{z}{1-z}, \quad -1 \le \lambda \le 1,$$

$$f_\lambda^{-1}(w) = \sum_{n=1}^{\infty} b_n(\lambda) w^n,\ M_n^* = \max_{-1 \le \lambda \le 1} b_n(\lambda),\ m_n^* = \min_{-1 \le \lambda \le 1} b_n(\lambda).$$

Todorov in [4] and [5] shows that $m_{2n} = -M_{2n}$, $n = 1, 2, \dots$, $(M_k^*, m_k^*) = (M_k, m_k)$, $k = 2, 3, 4$ where

$$M_4 = -m_4 = \frac{16\sqrt{15}}{45} = 1.3770607\dots,$$

calculates (M_k^*, m_k^*) for all $k \le 7$ and conjectures that $(M_k^*, m_k^*) = (M_k, m_k)$ for $k = 2, 3, \dots$.

In the next theorem we find M_k and m_k for $k = 5, 6$. The results are in accordance with Todorov's conjecture.

We think that it will be helpful to describe in brief below the basic ideas and technics.

If $\mathcal{F}$ is a class of holomorphic functions in the unit disk it is known that the n-th coefficient region $C_n(\mathcal{F})$ consists of the points $(w_0, w1, \dots, w_{n-1})$ such that $w_k = f^{(k)}(0)/k!$ $(k = 0, 1, \dots, n-1)$ for some f in $\mathcal{F}$. By $\mathcal{P}_\mathbb{R}$ we denote the class of holomorphic functions in the unit disk with real Taylor coefficients, $f(0) = 1$ and $\Re f(z) > 0$, $(|z| < 1)$. Since we deal with a problem of estimation of quantities on which Taylor coefficients are involved, it is natural to search for the stronger conditions holding between them. For the class $\mathcal{P}_\mathbb{R}$ such conditions are given by the Caratheodory–Toeplitz (C–T) Theorem. Using an one-to-one correspondence between $C_n(\mathcal{P}_\mathbb{R})$ and $C_n(\mathcal{N}_2)$ we get analogous conditions for the class $\mathcal{N}_2$.

In this way the initial problem for nth coefficient is converted to a problem of finding the maximum and the minimum of a polynomial of n variables over the compact set $C_{n+1}(\mathcal{P}_{\mathbb{R}})$.

In order to find all necessary critical points, we had to calculate all the solutions of some polynomial systems

$$D_i(x_1, x_2, \ldots, x_k) = 0, \quad i = 1, 2, \ldots, k \ (k \leq 5).$$

In order to solve this systems we follow the following procedure: We consider D_i as polynomials with respect to x_k and we find the division's remainder between D_1 and D_2. Next we find the division's remainder between D_2 and the previous remainder and we continue until the elimination of x_k. Next we eliminate x_k between D_i, $i > 2$, and the first degree polynomial that was obtained in the previous part of the procedure. After elimination of x_k, repeating the procedure we eliminate $x_{k-1}, x_{k-2}, \ldots$ and we finally get a polynomial equation $p(x_1) = 0$.

It is a fact that the procedure described, leads us to hard calculations containing operations of symbolic algebra, which were completed with computer algebra system Mathematica 4. Also, for each final polynomial, it was possible to obtain all its roots (complex and real) with Mathematica 4, even with a 200-decimal points precision. Notice that although the roots of the polynomials are given below with precision of less than 10-decimal points, our computations were performed with precision of 200-decimal points. Because the solutions calculated through the procedure of successive polynomial divisions are a superset of the critical points demanded, they are checked and verified over the initial system.

We now state our main results:

2. Main results

Theorem.

$$\text{(i)} \qquad M_5 = 2, \ m_5 = -\frac{113}{56};$$

$$\text{(ii)} \qquad M_6 = \frac{2(19\sqrt{14} + 28\sqrt{31})}{175\sqrt{5}} \sqrt{28 - \sqrt{434}} = 3.10592\ldots, \ m_6 = -M_6.$$

For the proof of the Theorem we will need the following lemmas.

Lemma 1. (i) $C_{n+1}(\mathcal{P}_{\mathbb{R}}) = \{1\} \times \overline{A}_n$, $n = 1, 2, \ldots$, *where* A_n *is the set of points* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *such that* $D_k(x) > 0$, $k = 1, 2, \ldots, n$ *where*

$$D_k(x) = \begin{vmatrix} 2 & x_1 & x_2 & \ldots & x_k \\ x_1 & 2 & x_1 & \ldots & x_{k-1} \\ x_2 & x_1 & 2 & \ldots & x_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k-1} & x_{k-2} & \ldots & 2 \end{vmatrix}.$$

(ii) *If* $(1, x_1, x_2, \ldots, x_n) \in C_{n+1}(\mathcal{P}_{\mathbb{R}})$ *such that* $D_k(x) = 0$ *for some* $k < n$ *then* $D_k(x) = D_{k+1}(x) = \ldots,\ D_n(x) = 0$.

Lemma 1 is a part of Caratheodory's Toeplitz's theorem (see [6]).

Lemma 2. *For* $n \leq 6$ *the following propositions are equivalent:*

(i) $(0, 1, q_2, \ldots, q_n) \in C_{n+1}(\mathcal{N}_2)$.

(ii) *There is a point* $(1, p_1, \ldots, p_n) \in C_{n+1}(\mathcal{P}_{\mathbb{R}})$ *such that* $q_k = Q_{k-1}(p_1, p_2, \ldots, p_{k-1})$ *where:*

$$Q_1(p_1) = \frac{-p_1}{2},$$

$$Q_2(p_1, p_2) = \frac{2p_1^2 - p_2 - 2}{4},$$

$$Q_3(p_1, p_2, p_3) = \frac{7p_1 - 5p_1^3 + 5p_1p_2 - p_3}{8},$$

$$Q_4(p_1, p_2, ptp_3, p_4) = \frac{6 - 24p_1^2 + 14p_1^4 + 8p_2 - 21p_1^2p_2 + 3p_2^2 + 6p_1p_3 - p_4}{16}$$

and

$$\begin{aligned} Q_5(p_1, p_2, p_3, p_4, p_5) = \frac{1}{32}(&-38p_1 + 84p_1^3 - 42p_1^5 - 63p_1p_2 + 84p_1^3p_2 \\ &- 28p_1p_2^2 + 9p_3 - 28p_1^2p_3 + 7p_2p_3 + 7p_1p_4 - p_5). \end{aligned}$$

Proof. To every

$$f(z) = \int_{-1}^{1} \frac{z}{1 - tz} d\mu(t) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{N}_2,$$

we correspond the function

$$L(f)(z) = \int_{-1}^{1} \frac{1-z^2}{1-2tz+z^2} d\mu(t) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Since the operators

$$\mu \to \int_{-1}^{1} \frac{z}{1-tz} d\mu(t) \quad \text{and} \quad \mu \to \int_{-1}^{1} \frac{1-z^2}{1-2tz+z^2} d\mu(t)$$

from the class of probability measures in $[-1,1]$ to the classes $\mathcal{N}_2$ and $\mathcal{P}_{\mathbb{R}}$ respectively are one-to-one and onto then the operator L is one-to-one and in addition $L(\mathcal{N}_2) = \mathcal{P}_{\mathbb{R}}$. Using now Taylor's expansion for f and $L(f)$ we get

$$p_1 = 2b_2, \quad p_2 = -2 + 4b_3, \quad p_3 = 2(-3b_2 + 4b_4),$$

$$p_4 = 2(1 - 8b_3 + 8b_5), \quad p_5 = 2(5b_2 - 20b_4 + 16b_6)$$

or

$$b_2 = \frac{p_1}{2}, \quad b_3 = \frac{(p_2+2)}{4}, \quad b_4 = \frac{(p_3+3p_1)}{8},$$

$$b_5 = \frac{p_4 + 4p_2 + 6}{16}, \quad b_6 = \frac{10p_1 + 5p_3 + p_5}{32}.$$

Let now

$$f(z) = z + b_2 z^2 + b_3 z^3 + \dots, \quad f \in \mathcal{N}_2 \text{ and } f^{-1}(w) = w + q_2 z^2 + \dots .$$

Since

$$(f \circ f^{-1})'(0) = 1 \quad \text{and} \quad (f \circ f^{-1})^{(n)}(0) = 0, \; n = 2, 3 \dots$$

after the calculations we get

$$q_k = Q_{k-1}(p_1, p_2, \dots, p_{k-1}), \quad k = 2, 3, \dots, 6. \qquad \square$$

3. Proof of the theorem

We will prove part (ii) of the theorem. At first we make the following remarks.

(a) By Lemma 1 and 2 we obtain that M_6 and m_6 coincides with maximum and minimum respectively, of the polynomial Q_5 over $\overline{A}_5$. Since Q_5 is of the first degree with respect to p_5 the previous maximum and minimum are not given over the open set A_5.

(b) Since $D_k(p_1, p_2, \ldots, p_k)$ are polynomials of the second degree with respect to p_k, the equations $D_k(p_1, p_2, \ldots, p_k) = 0$ give as roots the continuous functions $P_{k,i}(p_1, p_2, \ldots, p_{k-1})$, $i = 1, 2$, $k \leq 5$. (The functions $P_{k,i}$ are given in the appendix). In the case that $D_k(p_1, p_2, \ldots, p_k) = 0$ ($k \leq 5$) after the calculations we get:

$$P_{\rho,1}(p_1, p_2, \ldots, p_\rho) = P_{\rho,2}(p_1, p_2, \ldots, p_\rho), \quad \rho = k + 1, \ldots, 5.$$

We now define the restrictions $H_{k,i}^{(5)}$ over A_{k-1} of the functions Q_5 as follows: $H_{k,i}^{(5)}(p_1, p_2, \ldots, p_{k-1}) = Q_5(p_1, p_2, \ldots, p_5)$ with

$$p_\rho = P_{\rho,i}(p_1, p_2, \ldots, p_{\rho-1}), \quad \rho = k, k + 1, \ldots, 5.$$

We consider the set $C_{k,i}^{(5)}$ of all $(p_1, p_2 \ldots, p_{k-1})$ which are the critical points in the functions $H_{k,i}^{(5)}$ over A_k and the set $V_{k,i}^{(5)}$ which includes the maximum and minimum of the respective values.

By the two previous remarks it is easy to see that M_6 and m_6 are the maximum and minimum of all the values $V_{k,i}^{(5)}$.

We will now find the set $C_{5,1}^{(5)}$. We consider the polynomial system that is obtained after elimination of the denominators of the equations

$$\frac{\partial H_{5,1}^{(5)}}{\partial p_k} = 0, \quad k = 1, 2, 3, 4.$$

In the procedure of polynomial remainders that we apply in order to solve the above system after factorization of every remainder expect of the denominators we omit also and the factors of the form $D_k(p_1, p_2, \ldots, p_k)$ ($k < 5$). Among the equations which are obtained in the above procedure we consider the following:

$-24p_1 - 12p_1^2 + 7p_1^3 - 2p_1p_2 + 14p_1^2p_2 - 2p_2^2$
$+7p_1p_2^2 - 16p_1p_3 - 7p_1^2p_3 - 2p_2p_3 + 4p_4 + 2p_1p_4 = 0,$

$-252p_1 - 112p_1^2 + 287p_1^3 + 168p_1^4 - 4p_2 - 196p_1p_2 - 161p_1^2p_2 + 24p_3 + 28p_1p_3 = 0$

$27584 + 114240p_1 + 223440p_1^2 + 260960p_1^3 + 161700p_1^4 + 44100p_1^5 + 3675p_1^6 = 0.$

Let $R_{5,1}^{(5)}$ be the set of all points (p_1, p_2, p_3, p_4) that are solutions of the above system. Solving this system we obtain:

$R_{5,1}^{(5)} = \{(784.11, -686.303, 64.2589, -6.99709), (-141.238, 21.7373, 0.904313, -2.32053), (8.54405, -2.65501, 0.76221, -1.34458), (0.515794, 1.12714, -1.32933, -0.656607), (4.83205 + 0.49452i, 0.64871 + 2.48806i, -2.29805+0.198561i),(-0.340597-0.638435i),4.83205+0.49452i,0.64871- 2.48806i, -2.29805 - 0.198561i, 0.340597 + 0.638435i)\}.$

Checking all the above solutions we obtain
$C_{5,1}^{(5)} = \{(0.515794, 1.12714, -1.32933, -0.656607)\}$ and $V_{5,1}^{(5)} = \{0.00140562\}$. Since $H_{5,2}^{(5)}(p_1, p_2, p_3, p_4) = -H_{5,2}^{(5)}(-p_1, p_2, -p_3, p_4)$ we obtain that $V_{5,2}^{(5)} = -V_{5,1}^{(5)} = \{-0.00140562\}$.

Following the same procedure for the function $H_{4,1}$ we get the system of equations:

$934p_1 - 1050p_1^3 + 371p_1p_2 + 560p_1^3p_2 - 455p_1p_2^2 - 66p_3 - 70p_1^2p_3 + 105p_2p_3 = 0,$

$-8087582453760 + 162830301622272p_1^2 - 889886700581760p_1^4 + 2028040709688000p_1^6 - 208912199563200p_1^8 + 1013704871970000p_1^{10} - 220626705506250p_{12}^6 + 18461401546875p_1^{14} + p_2(-556223 + 108234507709440p_1^2 - 443713854024000p_1^4 + 692136093600000p_1^6 - 455672481600000p_1^8 + 116130727650000p_1^{10} - 10723428734375p_1^{12}) = 0$

and

$(5076 - 131760p_1^2 + 51725p_1^4)(-57517056 + 60963840p_1^2 + 2070841920p_1^6 + 1129156000p_1^8 - 168070000p_1^{10} + 7503125p_1^{12})(-41929933824 - 844096619520p_1^2 + 22180104313920p_1^4 - 60190321766400p_1^6 + 39504762948000p_1^8 - 9536171750000p_1^{10} + 722288328125p_1^{12}) = 0.$

Solving the above system and checking its solutions we obtain
$C_{4,1}^{(5)} = \{(0.460717327, -0.728901779, 0.413929249104), (-0.460717327, -0.728901779, -0.413929249104)\}$ and $V_{4,1}^{(5)} = \{\pm 0.1011430991\}$.

The same procedure applied to the function $H_{4,2}$ gives that $C_{4,2}^{(5)} = \emptyset$.

Continuing with the remaining functions we obtain
$C_{3,1}^{(5)} = \{(-1.57229, 0.573931)\}$, $V_{3,1}^{(5)} = \{-0.00374433\}$,
$V_{3,2}^{(5)} = \{0.00374433\}$, $C_{2,1}^{(5)} = \{\emptyset\}, C_{2,2}^{(5)} = \{\pm 0.63997, \pm 1.67046\}$,
$V_{2,2}^{(5)} = \{\pm 3.10592\}$
and
$V_{1,1}^{(5)} = -V_{1,2}^{(5)} = \{-1\}$.

Comparing the values of all the sets $V_{i,k}^{(5)}$ we get that

$$\{M_6, m_6\} = V_{2,2}^{(5)}.$$

Notice that the procedure of polynomial remainders we obtain that the set $C_{2,2}^{(5)}$ coincides with the set of the roots of the equation

$$40 - 112p_1^2 + +35p_1^4 = 0,$$

therefore the sets $C_{2,2}^{(5)}$ and $V_{2,2}^{(5)}$ get the exact form

$$C_{2,2}^{(5)} = \left\{ \pm \sqrt{\frac{8}{5} - \frac{2\sqrt{\frac{62}{7}}}{5}} \right\}$$

and

$$V_{2,2}^{5} = \left\{ \pm \frac{2(19\sqrt{14} + 28\sqrt{31}\,)}{175\sqrt{5}} \sqrt{28 - \sqrt{434}} \right\}.$$

The proof of the part (iii) is now complete.

Because the procedure of the proof of part (i) is the same as for part (ii), we omit to give all the intermediate expressions explicitly, and we only give the final sets $C_{k,i}^{(4)}$ and $V_{k,i}^{(4)}$. Actually the intermediate expressions that we omit are simpler than the corresponding ones in part (ii).

Part (ii).
$C_{4,1}^{(4)} = \{(0, -\frac{1}{3}, 0)\}$, $V_{4,1}^{(4)} = \{-\frac{4}{12}\}$ and $C_{4,2}^{(4)} = \emptyset$.
$C_{3,1}^{(4)} = \{(-0.573434, 0.92694)\}$, $V_{3,1}^{(4)} = \{0.0202514\}$,
$C_{3,2}^{(4)} = \{(0.573434, 0.92694)\}$ and $V_{3,2}^{(4)} = \{0.0202514\}$.
$C_{2,1}^{(4)} = \{0\}$, $V_{2,1}^{(4)} = \{0\}$, $C_{2,2}^{(4)} = \{(0, \pm\sqrt{\frac{15}{7}})\}$ and $V_{2,2}^{(4)} = \{2, -\frac{113}{56}\}$.

$V_{1,1}^{(4)} = \{1\}$ and $V_{1,2}^{(4)} = \{1\}$.

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4. Appendix

$$P_{1,1} = -2, \qquad P_{1,2} = 2,$$
$$P_{2,1} = 2, \qquad P_{2,2} = -2 + p_1^2,$$
$$P_{3,1} = (-4 + 2p_1 + p_1^2 - 2p_1p_2 + p_2^2)\,(-2 + p_1)^{-1},$$
$$P_{3,2} = (-4 - 2p_1 + p_1^2 + 2p_1p_2 + p_2^2)\,(2 + p_1)^{-1},$$
$$P_{4,1} = (-4 + p_1^2 + 2p_2 - 2p_1p_3 + p_3^2)\,(-2 + p_2)^{-1},$$
$$\begin{aligned} P_{4,2} = {} & (-4 + 3p_1^2 - 2p_2 - 2p_1^2p_2 + 2p_2^2 + p_2^3 + 2p_1p_3 - 2p_1p_2p_3 + p_3^2) \\ & \times (2 - p_1^2 + p_2)^{-1}, \end{aligned}$$
$$\begin{aligned} P_{5,1} = {} & (8 + 4p_1 - 4p_1^2 - p_1^3 - 4p_1p_2 + 2p_1^2p_2 - 4p_2^2 + 2p_1p_2^2 + 4p_3 + 2p_1p_3 \\ & \quad + 2p_1^2p_3 - 4p_2p_3 + 2p_1p_2p_3 - p_2^2p_3 \\ & \quad - 2p_3^2 - 2p_2p_3^2 - p_3^3 - 4p_1p_4 - 2p_1^2p_4 + 2p_1p_2p_4 \\ & \quad + 2p_2^2p_4 + 2p_1p_3p_4 + 2p_2p_3p_4 - 2p_4^2 - p_1p_4^2) \\ & \times (-4 - 2p_1 + p_1^2 + 2p_1p_2 + p_2^2 - 2p_3 - p_1p_3)^{-1}, \end{aligned}$$
$$\begin{aligned} P_{5,2} = {} & (-8 + 4p_1 + 4p_1^2 - p_1^3 - 4p_1p_2 - 2p_1^2p_2 + 4p_2^2 + 2p_1p_2^2 + 4p_3 - 2p_1p_3 \\ & \quad + 2p_1^2p_3 - 4p_2p_3 - 2p_1p_2p_3 - p_2^2p_3 + 2p_3^2 + 2p_2p_3^2 - p_3^3 - 4p_1p_4 \\ & \quad + 2p_1^2p_4 + 2p_1p_2p_4 - 2p_2^2p_4 - 2p_1p_3p_4 + 2p_2p_3p_4 + 2p_4^2 - p_1p_4^2) \\ & \times (-4 + 2p_1 + p_1^2 - 2p_1p_2 + p_2^2 + 2p_3 - p_1p_3)^{-1}. \end{aligned}$$

References

[1] P. G. Todorov, On the coefficients of certain class of analytic functions, *Ann. Univ, Mariae Curie-Sklodowska Sect.* A (17) **42** (1988), 151–157.

[2] P. G. Todorov, On the coefficients of the univalent functions of the Nevanlinna classes N_1 and N_2, Current Topics in Analytic Function Theory, *World Scientific, Singapore – New Jersey – London – Hong Kong.*

[3] P. G. Todorov and M. O. Reade, The Koebe domain of the classes $N_1(a)$ and $N_2(a)$ of Nevanlinna analytic functions, *Complex Variables Theory Appl.* **7** (1987), 343–348.

[4] P. G. Todorov, A conjecture for the coefficients of the inverse functions of the Nevanlinna univalent functions of the classes N_1 and N_2, *J. Tech. Univ. Plovdiv Fundam. Sci. Appl. Ser. A Pure Appl. Math.* **5** (1997), 113–120.

[5] P. G. Todorov, Sharp estimates for the coefficients of the inverse functions of the Nevanlinna univalent functions of the classes N_1 and N_2, *Abh. Math. Sem. Univ. Hamburg* **68** (1998), 91–102.

[6] U. Grenander and G. Szego, Toeplitz forms and their applications, *Univ. of California Press*, *Berkeley and Los Angeles*, 1958.

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