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Hereditary properties of special Finsler manifolds

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Abstract. A necessary and sufficient condition for the horizontal second fundamental form to be symmetric is derived. Necessary and sufficient conditions on the imbedded Finsler manifold are established so that it inherits from the ambient Finsler manifold the properties of being C-reducible, Landsberg, general Landsberg, $S3$-like, Berwald–Cartan and p-reducible. Some special classes of Finsler submanifolds are found.

Introduction

The theory of Finsler submanifolds is one of the most interesting theories in Finsler geometry. However, in considering submanifolds of a Finsler manifold the immensity of difficulties is great. One of these difficulties is the lack of symmetry of the horizontal second fundamental form; consequently the horizontal Weingarten operator is not self-adjoint in general. A second difficulty is that the induced and intrinsic connections of a Finsler submanifold do not coincide in general ([1], [2], [9], [18], [22] etc.). Even if we particularize our study to hypersurfaces of a Finsler manifold, the unavoidable obstacles are still great ([6], [13], [15], [16], [19] etc.).

Various interesting results of Finsler submanifolds have been obtained by a new method [3], [4] and [5]. By this method the induced and the intrinsic connections are the same and all the fundamental tensor fields are symmetric. Even by this method the theory of Finsler submanifolds remains one of the most difficult theories in Finsler geometry. Some global

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results for submanifolds in a Minkowskian manifold have been found by Z. Shen [17] by introducing the notions of mean and normal curvatures which do not depend on any connection. However, our approach to the study of the theory of submanifolds is based on the induced Finsler metric in a natural way following the path of Riemannian geometry, so that if the ambient manifold is Riemannian our formulas will reduce naturally to the classical Riemannian formulas.

Because the geometry of submanifolds is described in terms of the second fundamental form, it is natural to study isometric immersions whose second fundamental forms are simple in some sense. We consider totally umbilic Finsler immersions, i.e. immersions whose horizontal (respectively vertical) second fundamental form is proportional to the metric (respectively to the angular metric) tensor.

The main aim of this paper is twofold. On the one hand, we develop the author's studies of Finsler submanifolds ([18], [22]) to find new results so as to enable us to solve some of the above mentioned problems; for example we derive a necessary and sufficient condition for the horizontal second fundamental form to be symmetric. On the other hand, we find a special class of Finsler submanifolds for which the induced and intrinsic connections coincide. Among the various items investigated in this paper we find out another special class of submanifold for which the normal connection is either v-flat or h-flat. Moreover, we show that a Finsler submanifold of a C-reducible Finsler manifold is C-reducible. Thus the C-reducibility property of a Finsler manifold is a hereditary property. The totally geodesic property is also hereditary under vanishing of the normal curvature [18]. The locally Minkowskian property is hereditary under the totally geodesic condition [22]. The Landsberg and general Landsberg properties are hereditary under the totally geodesic condition. The infinitesimal isometry property is hereditary under the auto-parallel condition [18]. Lastly, the $S3$-likeness property is hereditary under the v-umbilical condition. The p-reducibility property is hereditary under some conditions. Therefore, we establish necessary and sufficient conditions on the imbedded Finsler manifold so that it inherits from the ambient Finsler manifold the properties of being C-reducible, Landsberg, general Landsberg, $S3$-like, Berwald–Cartan and p-reducible.

It should be noticed that the present work is formulated in a prospective modern coordinate-free form.

1. Notation and preliminaries

In this section, we give a brief account of the basic concepts necessary for this work. For more details, we refer to [23] or [20]. We make the general assumption that all geometric objects we consider are of class C^∞, except for the fundamental function on the zero section etc. The following notations will be used throughout the paper:

V: a differentiable manifold of finite dimension n and of class C^∞.

$\pi_V : TV \longrightarrow V$: the tangent bundle of V.

$\pi_V^* : T^*V \longrightarrow V$: the cotangent bundle of V.

$\pi : \mathcal{T}V \longrightarrow V$: the subbundle of nonzero vectors tangent to V.

$P : \pi^{-1}(TV) \longrightarrow \mathcal{T}V$: the bundle, with base space $\mathcal{T}V$, induced by π and TV.

$P^* : \pi^{-1}(T^*V) \longrightarrow \mathcal{T}V$: the bundle, with base space $\mathcal{T}V$, induced by π and T^*V.

$\mathfrak{F}(V)$: the $\mathbb{R}$-algebra of differentiable functions on V.

$\mathfrak{X}(V)$: the $\mathfrak{F}(V)$-module of vector fields on V.

$\mathfrak{X}(\pi(V))$: the $\mathfrak{F}(\mathcal{T}V)$-module of differentiable sections of $\pi^{-1}(TV)$.

$\mathfrak{I}_s^r(\pi)$: the module of π-tensor fields of type (r, s)).

Elements of $\mathfrak{X}(\pi(V))$ will be called π-vector fields. Tensor fields on $\pi^{-1}(TV)$ will be called π-tensor fields. The canonical vector field is the π-vector field ϑ defined by $\vartheta(u) = (u, u)$ for all $u \in \mathcal{T}V$. A tangent vector $X \in T_u(\mathcal{T}V)$ is said to be vertical if $\pi_*(X) = 0$. The linear space of all such vectors is called the vertical space at u and is denoted by $V_u(\mathcal{T}V)$.

The tangent bundle $T(\mathcal{T}V)$ is related to the vector bundle $\pi^{-1}(TV)$ by the exact sequence $0 \longrightarrow \pi^{-1}(TV) \xrightarrow{\gamma} T(\mathcal{T}V) \xrightarrow{\rho} \pi^{-1}(TV) \longrightarrow 0$, where the vector bundle morphisms are defined by $\rho = (\pi_{\mathcal{T}V}, d\pi)$ and $\gamma(u, v) = j_u(v)$ respectively, where j_u is the natural isomorphism $j_u : T_{\pi_V(v)}V \longrightarrow T_u(T_{\pi_V(v)}V)$.

Let ∇ be a linear connection (or simply a connection) in the vector bundle $\pi^{-1}(TV)$. We associate to ∇ the map K defined by $K = \nabla\vartheta$, called the deflection map of ∇. A tangent vector $X \in T_u(\mathcal{T}V)$ is said to be horizontal if $K(X) = 0$. The linear space of all such vectors is called the horizontal space at u and is denoted by $H_u(\mathcal{T}V)$. The connection ∇ is said to be regular [8] if $T_u(\mathcal{T}V) = V_u(\mathcal{T}V) \oplus H_u(\mathcal{T}V)$ for all $u \in \mathcal{T}V$. If

V is endowed with a regular connection, then we define a section β of the morphism ρ by $\beta = (\rho|_{H(\mathcal{T}V)})^{-1}$ and call it the horizontal map associated with ∇.

The torsion $\mathbf{T}$ and the curvature $\mathbf{R}$ of the connection ∇ are defined by

$$\mathbf{T} = d^{\nabla}\rho, \quad \mathbf{R}(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \qquad \forall\ X, Y \in \mathfrak{X}(\mathcal{T}V).$$

The horizontal and mixed torsion tensors, denoted respectively by A and T, are defined, for all $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(V))$, by

$$A(\bar{X},\bar{Y}) = \mathbf{T}(\beta\bar{X}, \beta\bar{Y}), \quad T(\bar{X},\bar{Y}) = \mathbf{T}(\gamma\bar{X}, \beta\bar{Y}).$$

The horizontal, mixed and vertical curvature tensors, denoted respectively by R, P and S, are defined, for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(V))$, by

$$R(\bar{X},\bar{Y})\bar{Z} = \mathbf{R}(\beta\bar{X}, \beta\bar{Y})\bar{Z}, \quad P(\bar{X},\bar{Y})\bar{Z} = \mathbf{R}(\gamma\bar{X}, \beta\bar{Y})\bar{Z},$$

$$S(\bar{X},\bar{Y})\bar{Z} = \mathbf{R}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}.$$

2. Some specifications

Throughout this work, $(V, \bar{L})$ will denote an n-dimensional Finsler manifold and (M, L) an m-dimensional Finsler submanifold of $(V, \bar{L})$, where $n = m + p$, $p \geq 1$. Entities of $(V, \bar{L})$ will be marked by a bar "–". Let $\bar{g}$ be the Finsler metric on $\mathcal{T}V$ and g the Finsler metric on $\mathcal{T}M$ induced by $\bar{g}$. The two metrics g and $\bar{g}$ are related by

$$g = \bar{g}|_{\pi^{-1}(TM)}. \tag{1}$$

For every $u \in \mathcal{T}M$, let $\mathcal{N}_u$ be the orthogonal complement of $\pi_u^{-1}(TM)$ in $\pi_u^{-1}(TV)$ with respect to $\bar{g}_u$. Then $\mathcal{N} = \bigcup_{u \in \mathcal{T}M} \mathcal{N}_u \longrightarrow \mathcal{T}M$ is a vector bundle, called the induced normal bundle for the given immersion. Let $\Gamma(\mathcal{N})$ be the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\mathcal{N}$. The elements of $\Gamma(\mathcal{N})$ will be called π-normal vector fields.

The Cartan connection $\bar{\nabla}$ in $\pi^{-1}(TV)$ and the induced connection ∇ in $\pi^{-1}(TM)$ are related by the Gauss formula [9]

$$\bar{\nabla}_X \bar{Y} = \nabla_X \bar{Y} + \tilde{H}(X, \bar{Y}), \tag{2}$$

for all $X \in \mathfrak{X}(\mathcal{T}M)$ and all $\bar{Y} \in \mathfrak{X}(\pi(M))$, where $\tilde{H}$ is the second fundamental form for the given immersion. For every $\xi \in \Gamma(\mathcal{N})$, the induced normal connection (or simply the normal connection) $\nabla^{\perp}$ in the normal vector bundle $\mathcal{N}$ is related to $\bar{\nabla}$ by the Weingarten formula

$$\bar{\nabla}_X \xi = -\tilde{B}_\xi X + \nabla^{\perp}{}_X \xi, \tag{3}$$

where $\tilde{B}_\xi$ is the Weingarten operator associated to ξ. We have

$$\bar{g}(\tilde{H}(X,\bar{Y}),\xi) = g(\tilde{B}_\xi X, \bar{Y}). \tag{4}$$

It will be noticed that the normal connection $\nabla^{\perp}$ is $g^{\perp}$-metric, where

$$g^{\perp} = \bar{g}|_{\mathcal{N}}. \tag{5}$$

For every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, the horizontal and vertical second fundamental forms, denoted respectively by H and Q, are defined by

$$H(\bar{X},\bar{Y}) = \tilde{H}(\beta\bar{X},\bar{Y}), \quad Q(\bar{X},\bar{Y}) = \tilde{H}(\gamma\bar{X},\bar{Y}). \tag{6}$$

As the induced connection is regular and $\bar{S}(\bar{X},\bar{Y})\vartheta = 0$, we have

$$Q(\bar{X},\vartheta) = 0, \quad S(\bar{X},\bar{Y})\vartheta = 0, \quad Q(\bar{X},\bar{Y}) = Q(\bar{Y},\bar{X}). \tag{7}$$

The normal curvature vector N and the normal curvature N_o are given by

$$N(\bar{X}) = H(\bar{X},\vartheta) \quad \text{and} \quad N_o = N(\vartheta). \tag{8}$$

The horizontal maps β and $\bar{\beta}$ are related, for every $\bar{X} \in \mathfrak{X}(\pi(M))$, by

$$\beta(\bar{X}) = \bar{\beta}(\bar{X}) + \gamma N(\bar{X}). \tag{9}$$

As a vector field X on $\mathcal{T}M$ can be represented in the form

$$X = \gamma K(X) + \beta\rho X, \tag{10}$$

it follows from (6) and (10) that

$$\tilde{H}(X,\bar{Y}) = H(\rho X,\bar{Y}) + Q(K(X),\bar{Y}), \quad \forall\, \bar{Y} \in \mathfrak{X}(\pi(M)). \tag{11}$$

Using equations (2), (8) and (9), the π-torsion tensors T and $\bar{T}$ are related, for every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, by

$$\left.\begin{aligned} \bar{T}(\bar{X},\bar{Y}) &= T(\bar{X},\bar{Y}) + Q(\bar{X},\bar{Y}),\\ \bar{T}(N(\bar{X}),\bar{Y}) - \bar{T}(N(\bar{Y}),\bar{X}) &= A(\bar{X},\bar{Y}) + H(\bar{X},\bar{Y}) - H(\bar{Y},\bar{X}). \end{aligned}\right\} \tag{12}$$

Theorem 1. *The horizontal second fundamental form H is symmetric if and only if $N_o = 0$ or $\bar{T}(\bar{X}, \xi) = 0$ for every $\bar{X} \in \mathfrak{X}(\pi(M))$, $\xi \in \Gamma(\mathcal{N})$.*

Proof. It should firstly be noticed that [18] $N_o = 0$ if and only if $N = 0$ and that $N = 0$ implies $A = 0$ and H is symmetric. By equation (12), it is clear that H is symmetric if $\bar{T}(\bar{X}, \xi) = 0 \ \ \forall \xi \in \Gamma(\mathcal{N})$.

Conversely, if H is symmetric, then it follows from (12) that

$$\bar{g}(\bar{T}(N(\bar{X}), \bar{Y}), \xi) = \bar{g}(\bar{T}(N(\bar{Y}), \bar{X}), \xi) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)), \ \xi \in \Gamma(\mathcal{N}).$$

By the symmetry of the torsion tensor $\bar{T}$, the above equation takes the form

$$\bar{g}(\bar{T}(\bar{X}, \xi), N(\bar{Y})) = \bar{g}(\bar{T}(\bar{Y}, \xi), N(\bar{X})).$$

Setting $\bar{Y} = \vartheta$ in this equation, it follows that $\bar{g}(\bar{T}(\bar{X}, \xi), N_o) = 0$. But $\bar{g}(\bar{T}(\bar{X}, \xi), N_o) = 0$ if and only if $N_o = 0$ or $\bar{T}(\bar{X}, \xi) = 0$. This completes the proof. □

For $\xi \in \Gamma(\mathcal{N})$, the horizontal and vertical Weingarten operators, denoted respectively by B_ξ and W_ξ, are defined by

$$B_\xi = \tilde{B}_\xi \circ \beta, \qquad W_\xi = \tilde{B}_\xi \circ \gamma. \tag{13}$$

It follows from (10) and (13) that

$$\tilde{B}_\xi = B_\xi \circ \rho + W_\xi \circ K. \tag{14}$$

Equations (4), (6), (11) and (14) imply that

$$\bar{g}(Q(\bar{X}, \bar{Y}), \xi) = g(W_\xi \bar{X}, \bar{Y}), \quad \bar{g}(H(\bar{X}, \bar{Y}), \xi) = g(B_\xi \bar{X}, \bar{Y}). \tag{15}$$

We recall the following lemma (from [18]) which will be used in the sequel:

Lemma 1. *For every $X, Y \in \mathfrak{X}(\mathcal{T}M)$ and every $\bar{Z} \in \mathfrak{X}(\pi(M))$, we have*

$$\begin{aligned}\bar{\mathbf{R}}(X, Y)\bar{Z} = \mathbf{R}(X, Y)\bar{Z} &+ \tilde{B}_{\tilde{H}(Y, \bar{Z})}X - \tilde{B}_{\tilde{H}(X, \bar{Z})}Y + (\nabla_Y H)(\rho X, \bar{Z}) \\ &- (\nabla_X H)(\rho Y, \bar{Z}) + (\nabla_Y Q)(K(X), \bar{Z}) - (\nabla_X Q)(K(Y), \bar{Z}) \\ &- H(\mathbf{T}(X, Y), \bar{Z}) + Q(\mathbf{R}(X, Y)\vartheta, \bar{Z}),\end{aligned}$$

where ∇Q *is given by* $(\nabla_X Q)(\bar{Y}, \bar{Z}) = \nabla^{\perp}{}_X Q(\bar{Y}, \bar{Z}) - Q(\nabla_X \bar{Y}, \bar{Z}) - Q(\bar{Y}, \nabla_X \bar{Z})$.

Applying Lemma 1 for $X = \gamma\bar{X}$, $Y = \gamma\bar{Y}$ and taking equations (6), (7) and (13) into account, we have

Proposition 1. *For every* $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, *we have*

$$\begin{aligned}\bar{S}(\bar{X}, \bar{Y})\bar{Z} = S(\bar{X}, \bar{Y})\bar{Z} + W_{Q(\bar{Y},\bar{Z})}\bar{X} - W_{Q(\bar{X},\bar{Z})}\bar{Y} \\ + (\nabla_{\gamma\bar{Y}} Q)(\bar{X}, \bar{Z}) - (\nabla_{\gamma\bar{X}} Q)(\bar{Y}, \bar{Z}).\end{aligned}$$

Definition 1. A Finsler manifold $(V, \bar{L})$ is said to be a Berwald–Cartan manifold [7] if its third curvature tensor $\bar{S}$ vanishes.

Proposition 1 and Definition 1 imply

Corollary 1. *For a Finsler submanifold* M *of a Berwald–Cartan manifold* V, *we have*

(a) $S(\bar{X}, \bar{Y})\bar{Z} = W_{Q(\bar{X},\bar{Z})}\bar{Y} - W_{Q(\bar{Y},\bar{Z})}\bar{X}$,

(b) $(\nabla_{\gamma\bar{Y}} Q)(\bar{X}, \bar{Z}) = (\nabla_{\gamma\bar{X}} Q)(\bar{Y}, \bar{Z})$.

Let $\mathbf{R}^N$ be the curvature transformation of the normal connection $\nabla^{\perp}$. Using equations (2) and (3), we get

Lemma 2. *For every* $X, Y \in \mathfrak{X}(\mathcal{T}M)$ *and for every* $\xi \in \Gamma(\mathcal{N})$, *we have*

$$\begin{aligned}\bar{\mathbf{R}}(X, Y)\xi = \mathbf{R}^{\mathbf{N}}(X, Y)\xi + \tilde{H}(X, \tilde{B}_\xi Y) - \tilde{H}(Y, \tilde{B}_\xi X) \\ + \tilde{B}_{\nabla^{\perp}_Y \xi} X - \tilde{B}_{\nabla^{\perp}_X \xi} Y + \mathbf{T}_{\tilde{B}_\xi}(X, Y),\end{aligned}$$

where by $\mathbf{T}_{\tilde{B}_\xi}(X, Y)$ *we mean the operator defined by*

$$\mathbf{T}_{\tilde{B}_\xi}(X, Y) = \nabla_X \tilde{B}_\xi Y - \nabla_Y \tilde{B}_\xi Y - \tilde{B}_\xi [X, Y].$$

For every $X, Y \in \mathfrak{X}(\mathcal{T}V)$ and every $\bar{Z}, \bar{W} \in \mathfrak{X}(\pi(V))$, we will use the notation

$$\bar{\mathbf{R}}(X, Y, \bar{Z}, \bar{W}) = \bar{g}(\bar{\mathbf{R}}(X, Y)\bar{Z}, \bar{W}).$$

The vertical Ricci tensor is the covariant π-tensor field $\overline{\text{Ric}}$ of degree 2, defined for every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(V))$ by $\overline{\text{Ric}}(\bar{X}, \bar{Y})$ = trace of the map $\bar{Z} \longrightarrow$

$\bar{S}(\bar{X},\bar{Z})\bar{Y}$. The vertical Ricci map is the linear transformation $\overline{\text{Ric}}_o$ defined by $\overline{\text{Ric}}(\bar{X},\bar{Y}) = \bar{g}(\overline{\text{Ric}}_o(\bar{X}),\bar{Y})$. The vertical scalar curvature is the map $\overline{\text{Sc}} : \mathcal{T}V \to \mathbb{R}$ defined by $\overline{\text{Sc}}$ = trace of the map $\bar{X} \longrightarrow \overline{\text{Ric}}_o(\bar{X})$.

Using Proposition 1 and Lemma 2, and taking into account equations (1), (5)–(7), (13), (15) and the fact that the vertical Weingarten operator is self-adjoint, one can prove

Proposition 2. *For every* $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4 \in \mathfrak{X}(\pi(M))$ *and for every* $\xi_1, \xi_2 \in \Gamma(\mathcal{N})$, *we have*

(1)
$$\begin{aligned}&\bar{R}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) + \bar{P}(N(\bar{X}_1),\bar{X}_2,\xi_1,\xi_2) - \bar{P}(N(\bar{X}_2),\bar{X}_1,\xi_1,\xi_2)\\ &\quad + \bar{S}(N(\bar{X}_1),N(\bar{X}_2),\xi_1,\xi_2)\\ &= R^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) + g(B_{\xi_2}\bar{X}_1, B_{\xi_1}\bar{X}_2) - g(B_{\xi_1}\bar{X}_1, B_{\xi_2}\bar{X}_2),\end{aligned}$$

(2)
$$\begin{aligned}&\bar{S}(\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4) = S(\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4)\\ &\quad + \bar{g}(Q(\bar{X}_1,\bar{X}_4),Q(\bar{X}_2,\bar{X}_3)) - \bar{g}(Q(\bar{X}_1,\bar{X}_3),Q(\bar{X}_2,\bar{X}_4)),\end{aligned}$$

(3)
$$\bar{S}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = S^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) + g([W_{\xi_1},W_{\xi_2}]\bar{X}_1,\bar{X}_2).$$

Let $\bar{\ell}$ be the π-form [21] defined by $\bar{\ell}(\bar{X}) = \bar{L}^{-1}\bar{g}(\bar{X},\vartheta), \forall\, \bar{X} \in \mathfrak{X}(\pi(V))$. The angular metric tensor $\bar{h}$ is defined by $\bar{h} = \bar{g} - \bar{\ell} \otimes \bar{\ell}$. Let $\bar{\varphi}$ be the π-tensor field defined by $\bar{\varphi} = I - \bar{L}^{-1}\bar{\ell} \otimes \vartheta$, where I is the identity transformation.

Definition 2. An n-dimensional Finsler manifold $(V,\bar{L})$, $n \geq 3$, is an $S3$-like manifold [23] (or in terms of local coordinates [12]) if the π-tensor field $\bar{S}$ has the form

$$\bar{S}(\bar{X},\bar{Y},\bar{Z},\bar{W}) = \bar{\lambda}\bar{L}^{-2}\big[\bar{h}(\bar{X},\bar{Z})\bar{h}(\bar{Y},\bar{W}) - \bar{h}(\bar{X},\bar{W})\bar{h}(\bar{Y},\bar{Z})\big], \qquad (16)$$

for every $\bar{X},\bar{Y},\bar{Z},\bar{W} \in \mathfrak{X}(\pi(V))$, where $\bar{\lambda} \in \mathfrak{F}(V)$; that is, $\bar{\lambda}$ depends on position only.

In [23] we have shown

Proposition 3. *For an $S3$-like manifold, we have:*

(a) $\bar{S}(\bar{X},\bar{Y})\bar{Z} = \bar{\lambda}\bar{L}^{-2}[\bar{h}(\bar{X},\bar{Z})\bar{\varphi}(\bar{Y}) - \bar{h}(\bar{Y},\bar{Z})\bar{\varphi}(\bar{X})] \quad \forall \bar{X},\bar{Y},\bar{Z} \in \mathfrak{X}(\pi(V))$,

(b) $\overline{\text{Ric}} = (n-2)\bar{\lambda}\bar{L}^{-2}\bar{h}$,

(c) $\overline{\text{Ric}}_o = (n-2)\bar{\lambda}\bar{L}^{-2}\bar{\varphi}$,

(d) $\overline{\text{Sc}} = (n-1)(n-2)\bar{\lambda}\bar{L}^{-2}$.

Corollary 2. *If the ambient Finsler manifold V is $S3$-like, then the π-tensor field S of the imbedded manifold M is given by*

$$\begin{aligned} S(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = & \frac{\overline{\text{Sc}}}{(n-1)(n-2)} \\ & \times [h(\bar{X}_1, \bar{X}_3)h(\bar{X}_2, \bar{X}_4) - h(\bar{X}_1, \bar{X}_4)h(\bar{X}_2, \bar{X}_3)] \\ & + \bar{g}(Q(\bar{X}_1, \bar{X}_3), Q(\bar{X}_2, \bar{X}_4)) - \bar{g}(Q(\bar{X}_1, \bar{X}_4), Q(\bar{X}_2, \bar{X}_3)). \end{aligned}$$

In fact, this follows from Proposition 2 by taking into account Proposition 3 and the fact that $h = \bar{h}|_{\pi^{-1}(TM)}$.

A result of [23] states that:

Lemma 3. *For every $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(V))$, we have*

(a) $\bar{\nabla}_{\gamma\bar{X}}\bar{L} = \bar{\ell}(\bar{X})$,

(b) $(\bar{\nabla}_{\gamma\bar{X}}\bar{\ell})(\bar{Y}) = \bar{L}^{-1}\bar{h}(\bar{X}, \bar{Y})$,

(c) $(\bar{\nabla}_{\gamma\bar{X}}\bar{\varphi})(\bar{Y}) = -\bar{L}^{-1}\{\bar{\varphi}(\bar{X})\bar{\ell}(\bar{Y}) + \bar{L}^{-1}\bar{h}(\bar{X}, \bar{Y})\vartheta\}$.

(d) $(\bar{\nabla}_{\gamma\bar{X}}\bar{h})(\bar{Y}, \bar{Z}) = -\bar{L}^{-1}[\bar{h}(\bar{X}, \bar{Y})\bar{\ell}(\bar{Z}) + \bar{h}(\bar{X}, \bar{Z})\bar{\ell}(\bar{Y})]$,

(e) $\bar{\nabla}_{\bar{\beta}\bar{X}}\bar{L} = \bar{\nabla}_{\bar{\beta}\bar{X}}\bar{\ell} = \bar{\nabla}_{\bar{\beta}\bar{X}}\bar{h} = \bar{\nabla}_{\bar{\beta}\bar{X}}\bar{\varphi} = 0$.

The mixed π-torsion tensor $\bar{T}$ induces a π-tensor field of type $(0,3)$, denoted again by $\bar{T}$, defined by, $\bar{T}(\bar{X}, \bar{Y}, \bar{Z}) = \bar{g}(\bar{T}(\bar{X}, \bar{Y}), \bar{Z})$ $\forall \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(V))$; it also induces a π-form $\bar{C}$ defined by $\bar{C}(\bar{X})$ = trace of the map $\bar{Y} \longrightarrow \bar{T}(\bar{X}, \bar{Y})$.

Definition 3. An n-dimensional Finsler manifold $(V, \bar{L})$, where $n \geq 3$, is a C-reducible [23] (or in terms of local coordinates [10]) manifold if the π-tensor field $\bar{T}$ can be expressed in the form

$$\bar{T}(\bar{X}, \bar{Y}, \bar{Z}) = \bar{h}(\bar{X}, \bar{Y})\bar{\alpha}(\bar{Z}) + \bar{h}(\bar{Y}, \bar{Z})\bar{\alpha}(\bar{X}) + \bar{h}(\bar{X}, \bar{Z})\bar{\alpha}(\bar{Y})$$

for any $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(\pi(V))$, where $\bar{\alpha}$ is the π-form given by $(n+1)\bar{\alpha} = \bar{C}$.

If $\bar{\mathbf{c}}$ is the π-vector field associated with $\bar{\alpha}$ under the duality defined by the metric $\bar{g}$, then we have [23]

Proposition 4. *For a C-reducible manifold* $(V, \bar{L})$, *the* π*-tensor field* $\bar{S}$ *takes the form*

$$\bar{S}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \bar{h}(\bar{X}, \bar{W})\bar{\sigma}(\bar{Y}, \bar{Z}) - \bar{h}(\bar{X}, \bar{Z})\bar{\sigma}(\bar{Y}, \bar{W}) + \bar{h}(\bar{Y}, \bar{Z})\bar{\sigma}(\bar{X}, \bar{W}) - \bar{h}(\bar{Y}, \bar{W})\bar{\sigma}(\bar{X}, \bar{Z}),$$

where $\bar{\sigma}(\bar{X}, \bar{Z}) = \frac{1}{2}\boldsymbol{c}^2\bar{h}(\bar{X}, \bar{Z}) + \bar{\alpha}(\bar{X})\bar{\alpha}(\bar{Z})$ *and* $\mathbf{c}^2 = \bar{g}(\bar{\mathbf{c}}, \bar{\mathbf{c}})$.

The mixed curvature tensor $\bar{P}$ induces a π-tensor field of type $(0, 3)$, denoted by $\widehat{\bar{P}}$, defined by $\widehat{\bar{P}}(\bar{X}, \bar{Y}, \bar{Z}) = \bar{g}(\bar{P}(\bar{X}, \bar{Y})\vartheta, \bar{Z})$ for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(M))$.

Proposition 5. *For every* $\bar{X}_1, \bar{X}_2, \bar{X}_3 \in \mathfrak{X}(\pi(M))$, *we have*

$$\widehat{\bar{P}}(\bar{X}_1, \bar{X}_2, \bar{X}_3) = \widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) + \bar{g}(Q(\bar{X}_1, \bar{X}_3), N(\bar{X}_2)).$$

Proof. Applying Lemma 1 for $X = \gamma\bar{X}_1$, $Y = \beta X_2$, $\bar{Z} = \vartheta$ and using (6)–(8) and (9), we get

$$\bar{P}(\bar{X}_1, \bar{X}_2)\vartheta = P(\bar{X}_1, \bar{X}_2)\vartheta + W_{N(\bar{X}_2)}\bar{X}_1 - (\nabla_{\gamma\bar{X}_1}N)(\bar{X}_2) + H(\bar{X}_2, \bar{X}_1) - N(T(\bar{X}_1, \bar{X}_2)).$$

Consequently, by equation (15), we deduce that

$$\widehat{\bar{P}}(\bar{X}_1, \bar{X}_2, \bar{X}_3) = \widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) + \bar{g}(Q(\bar{X}_1, \bar{X}_3), N(\bar{X}_2)).$$

This completes the proof. □

Definition 4. An n-dimensional Finsler manifold $(V, \bar{L})$, where $n \geq 3$, is called a p-reducible manifold [23] (or in terms of local coordinates [11]) if the π-tensor field $\widehat{\bar{P}}$ can be expressed in the form

$$\widehat{\bar{P}}(\bar{X}, \bar{Y}, \bar{Z}) = \bar{\delta}(\bar{Z})\bar{h}(\bar{X}, \bar{Y}) + \bar{\delta}(\bar{X})\bar{h}(\bar{Y}, \bar{Z}) + \bar{\delta}(\bar{Y})\bar{h}(\bar{X}, \bar{Z})$$

for every $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(V))$, where $\bar{\delta}$ is given by $\bar{\delta} = -\bar{\nabla}_{\bar{\beta}\vartheta}\bar{\alpha}$.

Corollary 3. *If the ambient Finsler manifold is* p*-reducible, then the* π*-tensor field* $\widehat{P}$ *of the imbedded manifold* M *is given, for every* $\bar{X}_1, \bar{X}_2, \bar{X}_3 \in \mathfrak{X}(\pi(M))$, *by*

$$\widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) = \bar{\delta}(\bar{X}_3)h(\bar{X}_1, \bar{X}_2) + \bar{\delta}(\bar{X}_1)h(\bar{X}_2, \bar{X}_3) + \bar{\delta}(\bar{X}_2)h(\bar{X}_1, \bar{X}_3) - \bar{g}(Q(\bar{X}_1, \bar{X}_3), N(\bar{X}_2)).$$

Indeed, this follows from Proposition 5 and Definition 4 by taking into account the fact that $h = \bar{h}|_{\pi^{-1}(TM)}$.

3. Totally v-umbilic submanifolds

As in Section 2, (M, L) will denote an m-dimensional Finsler submanifold of an $(m+p)$-dimensional Finsler manifold $(V, \bar{L})$, $p \geq 1$.

Let $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_m$ denote an orthonormal π-frame in the vector bundle $\pi^{-1}(TM)$ at a point $u \in \mathcal{T}M$. The vertical mean curvature normal vector ν at u is defined by

$$\nu = \frac{1}{m} \sum_{\alpha=1}^{m} Q(e_\alpha, e_\alpha). \tag{17}$$

Definition 5. A submanifold M of a Finsler manifold V is said to be totally v-umbilic if $Q = L^{-1}\ h \otimes \nu$.

Lemma 4. *For a totally v-umbilic submanifold M of a Finsler manifold V, we have $W_\xi = L^{-1}\ \bar{g}(\nu, \xi)\varphi$ for every $\xi \in \Gamma(\mathcal{N})$.*

Proof. Since M is totally v-umbilic, it follows from (15) that $L^{-1} h(\bar{X}, \bar{Y})\bar{g}(\nu, \xi) = g(W_\xi \bar{X}, \bar{Y})$ for every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, $\xi \in \Gamma(\mathcal{N})$. Consequently the result follows from the nondegeneracy of the metric g. □

Theorem 2. *Suppose that M is a totally v-umbilic submanifold of a Finsler manifold V. Then either ${\nabla^\perp}_{\gamma\vartheta}\ \nu = 0$ or M is a regular curve in V.*

Proof. Applying Lemma 1 for $X = \gamma\bar{X}$, $Y = \gamma\vartheta$ and using (7), we get $(\nabla_{\gamma\vartheta} Q)(\bar{X}, \bar{Z}) = (\nabla_{\gamma\bar{X}} Q)(\vartheta, \bar{Z})$. As a matter of fact $(\nabla_{\gamma\bar{X}} Q)(\vartheta, \bar{Z}) = -Q(\bar{X}, \bar{Z})$, we have

$$\nabla_{\gamma\vartheta} Q = -Q. \tag{18}$$

As the imbedded manifold M is v-umbilic, it follows from Lemma 3 that

$$\nabla_{\gamma\vartheta} Q = L^{-1} h \otimes ({\nabla^\perp}_{\gamma\vartheta}\nu - \nu). \tag{19}$$

Substituting (19) into (18) we get $h \otimes {\nabla^\perp}_{\gamma\vartheta}\ \nu = 0$. But $h \otimes {\nabla^\perp}_{\gamma\vartheta}\ \nu = 0$ if, and only if, $h = 0$ or ${\nabla^\perp}_{\gamma\vartheta}\ \nu = 0$. As $h(\bar{X}, \bar{Y}) = g(\varphi(\bar{X}), \bar{Y})$ for every

$\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, then $h = 0$ implies $\varphi = 0$. Consequently $I = L^{-1}\ell \otimes \vartheta$ which means that M is a submanifold of dimension one. This completes the proof. □

Combining Proposition 2 and Lemma 4, we have

Proposition 6. *For a totally v-umbilic submanifold M of a Berwald–Cartan manifold V, S^N vanishes.*

Definition 6. A Finsler submanifold M of a Finsler manifold V is said to be a v-minimal submanifold if its vertical mean curvature vector ν vanishes identically.

Theorem 3. *If M is a totally v-umbilic and minimal submanifold of a Finsler manifold V, then Q vanishes.*

Proof. If M is totally v-umbilic, then $Q = L^{-1}\ h \otimes \nu$. It follows from Lemma 4 that trace $W_\xi = (m-1)L^{-1}\ \bar{g}(\nu, \xi)$ for all $\xi \in \Gamma(\mathcal{N})$. If M is minimal, then trace $W_\xi = 0$ for every ξ. Hence W vanishes identically. Consequently, we deduce from (15) that Q vanishes identically. □

Proposition 2 and Definitions 1 and 5 imply

Lemma 5. *Let M be a totally v-umbilic submanifold of a Berwald–Cartan manifold V. We have, for every $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4 \in \mathfrak{X}(\pi(M))$,*

$$S(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = \|\nu\|^2 L^{-2} \times \big[h(\bar{X}_1, \bar{X}_3)h(\bar{X}_2, \bar{X}_4) - h(\bar{X}_1, \bar{X}_4)h(\bar{X}_2, \bar{X}_3)\big],$$

where $\|\nu\|^2 = \bar{g}(\nu, \nu)$.

Theorem 4. *Let M be a submanifold of a Berwald–Cartan manifold V. Then*

(a) *If M is totally v-umbilic and minimal, then M is Berwald–Cartan.*

(b) *If M is totally v-umbilic, then M is $S3$-like.*

Proof. (a) follows from Lemma 5.

(b) follows from Lemma 5 by using the method illustrated in the course of the proof of Proposition 16 of [23]. □

The Cartan connection in $\pi^{-1}(TM)$ associated with the metric g is called the intrinsic connection of the submanifold M. It should be emphasized that the induced and intrinsic connections do not coincide in general ([2], [9], [18], [22] etc.), contrary to the Riemannian case. Nevertheless, we have

Theorem 5. *For a totally v-umbilic submanifold M of a Finsler manifold V, the induced and intrinsic connections of M coincide if and only if M is v-minimal, totally geodesic or is a regular curve in V.*

Proof. It should firstly be noticed that [18] M is totally geodesic if and only if $N_o = 0$ and that $N_o = 0$ if and only if $N = 0$. Secondly, the induced and intrinsic connections coincide if and only if the horizontal torsion tensor A vanishes [2].

Now, by equation (12), we have, for every $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(M))$,

$$g(A(\bar{X}, \bar{Y}), \bar{Z}) = \bar{g}(\bar{T}(N(\bar{X}), \bar{Y}), \bar{Z}) - \bar{g}(\bar{T}(N(\bar{Y}), \bar{X}), \bar{Z}).$$

By the symmetry of the torsion tensor $\bar{T}$ it follows, using equation (12) again, that

$$g(A(\bar{X}, \bar{Y}), \bar{Z}) = \bar{g}(Q(\bar{Y}, \bar{Z}), N(\bar{X})) - \bar{g}(Q(\bar{X}, \bar{Z}), N(\bar{Y})). \tag{20}$$

As M is totally v-umbilic, equation (20) implies that

$$g(A(\bar{X}, \bar{Y}), \bar{Z}) = L^{-1}[h(\bar{Y}, \bar{Z})\bar{g}(\nu, N(\bar{X})) - h(\bar{X}, \bar{Z})\bar{g}(\nu, N(\bar{Y}))]. \tag{21}$$

If $h = 0$, $\nu = 0$ or $N = 0$, equation (21) shows that the horizontal torsion tensor A vanishes. (Note that $h = 0$ implies that M is a regular curve in V, as shown in the proof of Theorem 3.) Therefore, the induced and intrinsic connections coincide.

Conversely, assume that the horizontal torsion tensor A vanishes. Then equation (21) reduces to

$$h(\bar{Y}, \bar{Z})\bar{g}(\nu, N(\bar{X})) = h(\bar{X}, \bar{Z})\bar{g}(\nu, N(\bar{Y})).$$

Setting $\bar{Y} = \vartheta$ in the last equality, we have $h(\bar{X}, \bar{Z})\bar{g}(\nu, N_o) = 0$. But $h(\bar{X}, \bar{Z})\bar{g}(\nu, N_o) = 0$ if and only if $h = 0$, $\nu = 0$ or $N_o = 0$. Hence the result. □

From Theorem 3 and from Theorem 1 of [18], we have

Corollary 4. *The induced and intrinsic connections coincide on a totally v-umbilic minimal submanifold M of a Finsler manifold V.*

4. Totally umbilic Finsler submanifolds

As before, let (M, L) be an m-dimensional Finsler submanifold of an $(m+p)$-dimensional Finsler manifold $(V, \bar{L})$, $p \geq 1$.

In [22] we have shown that a submanifold M of a Finsler manifold V is totally h-umbilic if and only if $H = g \otimes \mu$, where $\mu = \frac{1}{m} \sum_{i=1}^{p} (\text{trace } B_{\xi_i}) \xi_i$, and $\xi_1, \xi_2, \ldots, \xi_p$ is an orthonormal frame in the induced normal bundle at a point $u \in \mathcal{T}M$. Therefore, for a totally h-umbilic submanifold M, we have $H = L^{-2} g \otimes N_o$, $N = L\ell \otimes \mu$ and $N_o = L^2 \mu$.

Definition 7. A Finsler submanifold M of a Finsler manifold V is called an h-minimal submanifold [9] if the horizontal mean curvature vector μ vanishes identically.

Definition 8. A Finsler manifold $(V, \bar{L})$ is a Landsberg manifold [18] if it satisfies the condition that $\bar{P}(\bar{X}, \bar{Y})\vartheta = 0$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(V))$.

Proposition 7. *For a totally h-umbilic and totally v-umbilic submanifold M of a Landsberg manifold V, the π-tensor field $P \otimes \vartheta$ takes the form $P \otimes \vartheta = -\bar{g}(\mu, \nu)\varphi \otimes \ell$.*

Proof. Applying Lemma 1 for $X = \gamma\bar{X}$, $Y = \beta\bar{Y}$, $\bar{Z} = \vartheta$ and using equations(6)–(9) and (13), we get

$$\bar{P}(\bar{X}, \bar{Y})\vartheta = P(\bar{X}, \bar{Y})\vartheta + W_{N(\bar{Y})}\bar{X} - (\nabla_{\gamma\bar{X}} H)(\bar{Y}, \vartheta) - N(T(\bar{X}, \bar{Y})).$$

As the ambient manifold V is Landsberg, $\bar{P} \otimes \vartheta = 0$. Consequently, we have $P(\bar{X}, \bar{Y})\vartheta = -W_{N(\bar{Y})}\bar{X}$. Using equation (15) we deduce, for every $\bar{Z} \in \mathfrak{X}(\pi(M))$, that

$$\widehat{P}(\bar{X}, \bar{Y}, \bar{Z}) = -\bar{g}(Q(\bar{X}, \bar{Z}), N(\bar{Y})). \tag{22}$$

Assume that M is totally h-umbilic and totally v-umbilic. Then $N(\bar{Y}) = L\ell(\bar{Y})\mu$ and $Q(\bar{X}, \bar{Z}) = L^{-1}h(\bar{X}, \bar{Z})\nu$ respectively. Substituting into (22), we get $\widehat{P}(\bar{X}, \bar{Y}, \bar{Z}) = -\bar{g}(\mu, \nu)h(\bar{X}, \bar{Z})\ell(\bar{Y})$. The result then follows from this relation and the nondegeneracy of the metric g. □

Corollary 5. *A totally h-umbilic and totally v-umbilic submanifold M of a Landsberg manifold V is Landsberg if and only if $\mu = 0$, $\nu = 0$ or M is a regular curve in V.*

Lemma 6. *For a totally h-umbilic and totally v-umbilic submanifold M of a Finsler manifold V, the horizontal torsion tensor A takes the form $A = \bar{g}(\nu, \mu)\ \ell \wedge \varphi$.*

Proof. If M is totally h-umbilic and totally v-umbilic, it follows from (20) that $g(A(\bar{X}, \bar{Y}), \bar{Z}) = \bar{g}(\nu, \mu)[h(\bar{Y}, \bar{Z})\ell(\bar{X}) - h(\bar{X}, \bar{Z})\ell(\bar{Y})]$. By this equality and the nondegeneracy of g the result follows. □

From Lemma 6, we have

Theorem 6. *A necessary and sufficient condition for the induced and intrinsic connections of a totally h-umbilic and totally v-umbilic submanifold M of a Finsler manifold V to coincide is that M is h-minimal, v-minimal or a regular curve in V.*

5. Submanifolds of C-reducible manifolds

Throughout this section, unless otherwise stated, the ambient Finsler manifold $(V, \bar{L})$ will be a C-reducible manifold.

Theorem 7. *A submanifold of a C-reducible manifold is C- reducible.*

Proof. Taking into account equation (1), it follows that

$$h = \bar{h}|_{\pi^{-1}(TM)}, \quad C = \left(\frac{n+1}{m+1}\right) \bar{C}|_{\pi^{-1}(TM)}.$$

Consequently, equation (12) and Definition 3 imply that $\bar{T}(\bar{X}, \bar{Y}, \bar{Z}) = T(\bar{X}, \bar{Y}, \bar{Z})$ for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(M))$. Hence the result. □

Remark 1. A special case of the above result was obtained (in terms of local coordinates) by Matsumoto [13] and Singh [16], where hypersurfaces had been considered.

Proposition 8. *For a submanifold M of a C-reducible manifold V, the vertical second fundamental form Q takes the form $Q = h \otimes \bar{\mathbf{c}}$, where $\bar{\mathbf{c}}$ is as defined in Proposition* 4.

Proof. Using equation (12), it follows that $\bar{g}(Q(\bar{X}, \bar{Y}), \xi) = \bar{g}(\bar{T}(\bar{X}, \bar{Y}), \xi)$ for all $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, $\xi \in \Gamma(\mathcal{N})$. Using Definition 3 the above equality reduces to $\bar{g}(Q(\bar{X}, \bar{Y}), \xi) = h(\bar{X}, \bar{Y})\bar{\alpha}(\xi)$, since $\bar{h}(\bar{X}, \xi) = 0$ and $h = \bar{h}|_{\pi^{-1}(TM)}$. Thus, we have $Q = h \otimes \bar{\mathbf{c}}$. □

Corollary 6. *If $\bar{\mathbf{c}}$ is tangential to a submanifold M of a C-reducible manifold V, then the vertical second fundamental form Q vanishes.*

Proposition 9. *The horizontal torsion tensor A of a submanifold M of a C-reducible manifold V is given, for every $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$, by*

$$A(\bar{X}, \bar{Y}) = \bar{g}(\bar{\mathbf{c}}, N(\bar{X}))\varphi(\bar{Y}) - \bar{g}(\bar{\mathbf{c}}, N(\bar{Y}))\varphi(\bar{X}).$$

Proof. Using Proposition 8, equation (20) takes the form

$$g(A(\bar{X}, \bar{Y}), \bar{Z}) = h(\bar{Y}, \bar{Z})\bar{g}(\bar{\mathbf{c}}, N(\bar{X})) - h(\bar{X}, \bar{Z})\bar{g}(\bar{\mathbf{c}}, N(\bar{Y})).$$

The result follows from the nondegeneracy of g and the above equation. □

Corollary 7. *The horizontal torsion tensor A of a totally h-umbilic submanifold M of a C-reducible manifold V is given by $A = L\bar{g}(\bar{\mathbf{c}}, \mu)\, \ell \wedge \varphi$.*

The following result is a generalization of Theorem 2.2 of [15].

Theorem 8. *For a submanifold M of a C-reducible manifold V, the induced and intrinsic connections of M coincide if and only if M is totally geodesic, $\bar{\mathbf{c}} = 0$ or $\bar{\mathbf{c}}$ is tangential to M.*

Proof. Notice first that the induced and intrinsic connections of M coincide if and only if the horizontal torsion tensor A vanishes [2]. If M is totally geodesic, $\bar{\mathbf{c}} = 0$ or $\bar{\mathbf{c}}$ is tangential to M, Proposition 9 implies that A vanishes.

Conversely, assume that the horizontal torsion tensor A vanishes. It follows, from Proposition 9 again, that

$$\bar{g}(\bar{\mathbf{c}}, N(\bar{X}))\varphi(\bar{Y}) = \bar{g}(\bar{\mathbf{c}}, N(\bar{Y}))\varphi(\bar{X}) \quad \text{for every } \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)).$$

Setting $\bar{Y} = \vartheta$ in the above equality, the left hand side vanishes identically. Hence $\bar{g}(\bar{\mathbf{c}}, N_o)\varphi = 0$. But $\bar{g}(\bar{\mathbf{c}}, N_o)\varphi = 0$ if and only if either $\varphi = 0$, or $\bar{g}(\bar{\mathbf{c}}, N_o) = 0$. If $\varphi = 0$, then M is a submanifold of dimension one, which contradicts Theorem 7. Therefore $\bar{g}(\bar{\mathbf{c}}, N_o)\varphi = 0$ if and only if M is totally geodesic, $\bar{\mathbf{c}} = 0$ or $\bar{\mathbf{c}}$ is tangential to M. This completes the proof. □

The proof of the following result is not difficult.

Lemma 7. *For all $\bar{X} \in \mathfrak{X}(\pi(M))$, we have*

$$(\bar{\nabla}_{\bar{\beta}\vartheta}\bar{\alpha})(\bar{X}) = (\nabla_{\beta\vartheta}\alpha)(\bar{X}) - (\bar{\nabla}_{\gamma N_o}\bar{\alpha})(\bar{X}) - \bar{\alpha}(H(\vartheta, \bar{X})).$$

Corollary 8. *If $N_o = 0$, then we have, for every $\bar{X} \in \mathfrak{X}(\pi(M))$,*

$$(\bar{\nabla}_{\bar{\beta}\vartheta}\bar{\alpha})(\bar{X}) = (\nabla_{\beta\vartheta}\alpha)(\bar{X}).$$

Definition 9. A Finsler manifold $(V, \bar{L})$ is a Berwald manifold [21] if the torsion tensor $\bar{T}$ has the property that $\bar{\nabla}_{\bar{\beta}\bar{X}}\bar{T} = 0$ for every $\bar{X} \in \mathfrak{X}(\pi(V))$.

Definition 10. A general Landsberg manifold [21] is a Finsler manifold $(V, \bar{L})$ such that the trace of the linear map $\bar{Y} \to \bar{P}(\bar{X}, \bar{Y})\vartheta$ is zero for all $\bar{X} \in \mathfrak{X}(\pi(V))$. It is characterized by the condition that $\bar{\nabla}_{\bar{\beta}\vartheta}\bar{C} = 0$.

Definition 10 together with Corollary 8 imply

Proposition 10. *A totally geodesic submanifold of a general Landsberg manifold is a general Landsberg manifold.*

In [23] we have shown the following results:

Proposition 11. *For a Finsler manifold $(V, \bar{L})$, we have:*

(a) *A C-reducible manifold is a Landsberg manifold if and only if it is a general Landsberg manifold.*

(b) *A C-reducible manifold is a Berwald manifold if and only if it is a general Landsberg manifold.*

Propositions 11(a), 10 and Theorem 7 imply

Proposition 12. *A totally geodesic submanifold of a C-reducible general Landsberg manifold is a Landsberg manifold.*

From Propositions 11(b), 10 and 12 together with Theorem 7, we have

Theorem 9. *A totally geodesic submanifold of a C-reducible general Landsberg manifold is a Berwald manifold.*

Definition 11. The normal connection $\nabla^{\perp}$ is v-flat [1], if S^N vanishes identically.

Proposition 6 and Definition 11 imply

Theorem 10. *If M is a totally v-umbilic submanifold of a Berwald–Cartan manifold V, then $\nabla^{\perp}$ is v-flat.*

Theorem 11. *If M is a totally v-umbilic submanifold of a C-reducible manifold V, then $\nabla^{\perp}$ is v-flat.*

Proof. Since M is totally v-umbilic, we have, by Lemma 4,$W_\xi = L^{-1}\bar{g}(\nu,\xi)\varphi$ for every $\xi \in \Gamma(\mathcal{N})$. Consequently, it follows from Proposition 2 that

$$\bar{S}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = S^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2),$$

for all $\bar{X}_1,\bar{X}_2 \in \mathfrak{X}(\pi(M))$, $\xi_1,\xi_2 \in \Gamma(\mathcal{N})$. On the other hand, since the ambient Finsler manifold V is C-reducible, Proposition 4 implies

$$\begin{aligned}\bar{S}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = \bar{h}(\bar{X}_1,\xi_2)\bar{\sigma}(\bar{X}_2,\xi_1) - \bar{h}(\bar{X}_1,\xi_1)\ \bar{\sigma}(\bar{X}_2,\xi_2)\\ + \bar{h}(\bar{X}_2,\xi_1)\ \bar{\sigma}(\bar{X}_1,\xi_2) - \bar{h}(\bar{X}_2,\xi_2)\ \bar{\sigma}(\bar{X}_1,\xi_1).\end{aligned}$$

But since $\bar{h}(\bar{X}_i,\xi_i) = 0$; $i = 1,2$, then $\bar{S}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) = 0$ and consequently $S^N = 0$. Hence $\nabla^\perp$ is v-flat. □

Theorem 12. *A totally v-umbilic submanifold M of a Landsberg manifold V is a Landsberg manifold if, and only if, M is v-minimal, totally geodesic or is a regular curve in V.*

Proof. In the course of the proof of Proposition 7 we have shown that the mixed curvature tensor P of a submanifold M of a Landsberg manifold V can be represented in the form $P(\bar{X},\bar{Y})\vartheta = -W_{N(\bar{Y})}\bar{X}$ for every $\bar{X},\bar{Y} \in \mathfrak{X}(\pi(M))$. As M is totally v-umbilic, the last equation takes the form

$$P(\bar{X},\bar{Y})\vartheta = -L^{-1}\ g(\nu, N(\bar{Y}))\varphi(\bar{X}).$$

The result then follows from the above equation and Definition 8. □

6. Submanifolds of $S3$-like manifolds

In this section, unless otherwise stated, the ambient Finsler manifold $(V,\bar{L})$ will be an $S3$-like manifold.

Using Proposition 2 and taking Definition 2 into account, the following result can be proved in an analogous way as Theorem 11.

Theorem 13. *Suppose that M is a totally v-umbilic submanifold of an $S3$-like manifold V. Then $\nabla^\perp$ is v-flat.*

From Definition 5 and Corollary 2, we have

Lemma 8. *The vertical scalar curvature* Sc *of a totally v-umbilic submanifold M of an S3-like manifold V is given by*

$$\mathrm{Sc} = (m-1)(m-2)\left[\frac{\overline{\mathrm{Sc}}}{(n-1)(n-2)} + L^{-2}\|\nu\|^2\right].$$

Theorem 14. *A totally v-umbilic submanifold M of an S3-like manifold V is S3-like.*

Proof. Using Definition 5, Corollary 2 implies, for all $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4 \in \mathfrak{X}(\pi(M))$, that

$$S(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = \left(\frac{\overline{\mathrm{Sc}}}{(n-1)(n-2)} + L^{-2}\|\nu\|^2\right) \times \{h(\bar{X}_1, \bar{X}_3)h(\bar{X}_2, \bar{X}_4) - h(\bar{X}_1, \bar{X}_4)h(\bar{X}_2, \bar{X}_3)\}.$$

Using Lemma 8, we conclude from the above equality that

$$S(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = \frac{\mathrm{Sc}}{(m-1)(m-2)} \times \big(h(\bar{X}_1, \bar{X}_3)h(\bar{X}_2, \bar{X}_4) - h(\bar{X}_1, \bar{X}_4)h(\bar{X}_2, \bar{X}_3)\big),$$

which means that M is an $S3$-like manifold. This completes the proof. □

Remark 2. Theorem 14 generalizes Theorem 6.1 of Abatangelo, Dragomir and Hojo [1], where the authors obtained the same conclusion but under much more restrictive conditions.

Definition 12. A Finsler manifold $(V, \bar{L})$ is locally Minkowskian [23] if and only if $\bar{R} = 0$ and $\bar{\nabla}_{\bar{\beta}\bar{X}}\bar{T} = 0$ for every $\bar{X} \in \mathfrak{X}(\pi(V))$.

Definition 13. The normal connection $\nabla^{\perp}$ is h-flat [1], if R^N vanishes identically.

Proposition 13. *For a submanifold M of an S3-like locally Minkowskian manifold V, the π-tensor field R takes the form*

$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \bar{g}(H(\bar{X}, \bar{Z}), H(\bar{Y}, \bar{W})) - \bar{g}(H(\bar{X}, \bar{W}), H(\bar{Y}, \bar{Z})),$$

for every $\bar{X}, \bar{Y}, \bar{Z}, \bar{W} \in \mathfrak{X}(\pi(M))$.

Proof. Applying Lemma 1 for $X = \beta\bar{X}, Y = \beta\bar{Y}$ and taking equations (6), (8), (9) and (13) into account, we deduce, for any $\bar{W} \in \mathfrak{X}(\pi(M))$, that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y},\bar{Z},\bar{W}) &+ \bar{P}(N(\bar{X}),\bar{Y},\bar{Z},\bar{W}) - \bar{P}(N(\bar{Y}),\bar{X},\bar{Z},\bar{W}) \\ &+ \bar{S}(N(\bar{X}),N(\bar{Y}),\bar{Z},\bar{W}) \\ &= R(\bar{X},\bar{Y},\bar{Z},\bar{W}) + \bar{g}(H(\bar{X},\bar{W}),H(\bar{Y},\bar{Z})) - \bar{g}(H(\bar{X},\bar{Z}),H(\bar{Y},\bar{W})).\end{aligned}$$

If the ambient manifold V is locally Minkowskian, then $\bar{R} = \bar{P} = 0$. Consequently the above equation takes the form

$$\begin{aligned}R(\bar{X},\bar{Y},\bar{Z},\bar{W}) = \bar{S}(N(\bar{X}),N(\bar{Y}),\bar{Z},\bar{W}) &+ \bar{g}(H(\bar{X},\bar{Z}),H(\bar{Y},\bar{W})) \\ &- \bar{g}(H(\bar{X},\bar{W}),H(\bar{Y},\bar{Z})).\end{aligned} \tag{23}$$

As V is an $S3$-like manifold, we conclude that $\bar{S}(N(\bar{X}),N(\bar{Y}),\bar{Z},\bar{W}) = 0$. Hence the desired result follows from (23). □

Theorem 15. *Suppose that M is a totally h-umbilic submanifold of a locally Minkowskian manifold V. Then $\nabla^{\perp}$ is h-flat.*

Proof. Since M is totally h-umbilic, we have $N = L\ell \otimes \mu$, $B_\xi = \bar{g}(\mu,\xi)I$, and $\bar{S}(N(\bar{X}_1),N(\bar{X}_2))\xi = 0$ for every $\xi \in \Gamma(\mathcal{N})$. Consequently, for every $\bar{X}_1,\bar{X}_2 \in \mathfrak{X}(\pi(M))$ and every $\xi_1,\xi_2 \in \Gamma(\mathcal{N})$, it follows from Proposition 2 that

$$\begin{aligned}\bar{R}(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2) + \bar{P}(N(\bar{X}_1),\bar{X}_2,\xi_1,\xi_2) - \bar{P}(N(\bar{X}_2),\bar{X}_1,\xi_1,\xi_2)& \\ = R^N(\bar{X}_1,\bar{X}_2,\xi_1,\xi_2).&\end{aligned}$$

As the ambient manifold V is locally Minkowskian, we deduce that $\bar{R} = \bar{P} = 0$. Hence the result follows from the above equation. □

7. Submanifolds of p-reducible manifolds

Throughout this section the ambient Finsler manifold $(V,\bar{L})$ will be a p-reducible manifold.

Proposition 14. *For a totally v-umbilic submanifold M of a p-reducible manifold V, the π-tensor field $\widehat{P}$ is given, for every $\bar{X}_1, \bar{X}_2, \bar{X}_3 \in \mathfrak{X}(\pi(M))$, by*

$$\begin{aligned}\widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) &= \delta(\bar{X}_3)h(\bar{X}_1, \bar{X}_2) + \delta(\bar{X}_1)h(\bar{X}_2, \bar{X}_3) + \delta(\bar{X}_2)h(\bar{X}_1, \bar{X}_3)\\ &+ \frac{L^{-1}}{m+1}\{\bar{g}(\nu, N(\varphi(\bar{X}_3)))h(\bar{X}_1, \bar{X}_2) + \bar{g}(\nu, N(\varphi(\bar{X}_1)))h(\bar{X}_2, \bar{X}_3)\\ &+ \bar{g}(\nu, N(\varphi(\bar{X}_2)))\ h(\bar{X}_1, \bar{X}_3)\} - L^{-1}h(\bar{X}_1, \bar{X}_3)\ \bar{g}(\nu, N(\bar{X}_2)).\end{aligned}$$

Proof. Since M is totally v-umbilic, Corollary 3 implies that, for every $\bar{X}_1, \bar{X}_2, \bar{X}_3 \in \mathfrak{X}(\pi(M))$,

$$\begin{aligned}\widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) &= \bar{\delta}(\bar{X}_3)h(\bar{X}_1, \bar{X}_2) + \bar{\delta}(\bar{X}_1)h(\bar{X}_2, \bar{X}_3)\\ &+ \bar{\delta}(\bar{X}_2)\ h(\bar{X}_1, \bar{X}_3) - L^{-1}h(\bar{X}_1, \bar{X}_3)\ \bar{g}(\nu, N(\bar{X}_2)).\end{aligned} \tag{24}$$

By the nondegeneracy of the metric $\bar{g}$, equation (24) gives rise to

$$\begin{aligned}P(\bar{X}_1, \bar{X}_2)\vartheta &= h(\bar{X}_1, \bar{X}_2)\bar{m} + \bar{\delta}(\bar{X}_1)\varphi(\bar{X}_2) + \bar{\delta}(\bar{X}_2)\varphi(\bar{X}_1)\\ &- L^{-1}\ \bar{g}(\nu, N(\bar{X}_2))\varphi(\bar{X}_1),\end{aligned}$$

where $\bar{m}$ is the π-vector field associated with $\bar{\delta}$ under the duality defined by the metric $\bar{g}$. Taking the trace of the above equation, we have

$$\bar{\delta}(\bar{X}_1) = \delta(\bar{X}_1) + \frac{L^{-1}}{m+1}\bar{g}(\nu, N(\varphi(\bar{X}_1))). \tag{25}$$

Substituting (25) into (24), the result follows. □

Theorem 16. *A totally v-umbilic submanifold M of a p-reducible manifold is p-reducible if and only if M is v-minimal or totally geodesic.*

Proof. If M is either totally geodesic or v-minimal, then Proposition 14 implies, for every $\bar{X}_1, \bar{X}_2, \bar{X}_3 \in \mathfrak{X}(\pi(M))$, that

$$\begin{aligned}\widehat{P}(\bar{X}_1, \bar{X}_2, \bar{X}_3) &= \delta(\bar{X}_3)h(\bar{X}_1, \bar{X}_2)\\ &+ \delta(\bar{X}_1)h(\bar{X}_2, \bar{X}_3) + \delta(\bar{X}_2)h(\bar{X}_1, \bar{X}_3),\end{aligned}$$

which means that M is a p-reducible manifold.

Conversely, assume that the imbedded manifold M is p-reducible, then it follows from Proposition 14, that

$$\bar{g}(\nu, N(\bar{X}_2))h(\bar{X}_1, \bar{X}_3) = \frac{1}{m+1}\{\bar{g}(\nu, N(\varphi(\bar{X}_3)))h(\bar{X}_1, \bar{X}_2) + \bar{g}(\nu, N(\varphi(\bar{X}_1)))h(\bar{X}_2, \bar{X}_3) + \bar{g}(\nu, N(\varphi(\bar{X}_2)))h(\bar{X}_1, \bar{X}_3)\}.$$

Setting $\bar{X}_2 = \vartheta$ in the above equation, the right hand side vanishes identically. Hence $\bar{g}(\nu, N_o)h = 0$. But $\bar{g}(\nu, N_o)h = 0$ if and only if either $h = 0$ or $\bar{g}(\nu, N_o) = 0$. If $h = 0$, then $\varphi = 0$. But $\varphi = 0$ implies that M is a submanifold of dimension one (as shown in the proof of Theorem 2) which contradicts Definition 4. Therefore $\bar{g}(\nu, N_o)h = 0$ if and only if $\nu = 0$ or $N_o = 0$. This completes the proof. □

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