Publ. Math. Debrecen
62/1-2 (2003), 165–185

Oscillation criteria of hyperbolic equations with deviating arguments

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Abstract. In this paper we shall consider a class of hyperbolic nonlinear equations with deviating arguments. Some new sufficient conditions for oscillation of all solutions with three kinds of boundary conditions are obtained.

1. Introduction

The study of oscillatory behavior of solutions of partial differential equations with deviating arguments, besides its theoretical interest, is important from the viewpoint of applications. Examples of applications can be found in [12]. But only a few results on the oscillatory behavior of hyperbolic equations with deviating arguments were recently obtained in [2], [3], [5], [6], [9]–[11] and the references cited therein. In this paper, we shall consider the nonlinear hyperbolic equation with deviating arguments

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} &= a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,\tau_i(t)) \\ &\quad - q(x,t)f(u(x,g(t))), \quad (x,t)\in\Omega\times\mathbb{R}_+ = G,\end{aligned} \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a piecewise smooth boundary $\partial\Omega$, and Δu is the Laplacian in $\mathbb{R}^n$.

Throughout, we will assume that the following conditions hold:

Mathematics Subject Classification: 34K11, 35L10.
Key words and phrases: oscillation, hyperbolic equations.

(H1) $a, a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ for $i = 1, \ldots, m$;

(H2) $q \in C(\overline{G}, \mathbb{R}_+)$ and $q(t) = \min_{x\in\overline{\Omega}} q(x,t)$ is not identically zero on $[t_0, \infty)$ for some $t_0 > 0$;

(H3) τ_i, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g'(t) > 0$, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \tau_i(t) = \infty$ for $i = 1, \ldots, m$;

(H4) $f \in C(\mathbb{R}, \mathbb{R})$ is convex in $\mathbb{R}_+$, $uf(u) > 0$ and $f(u)/u \geq k > 0$ for $u \neq 0$.

We consider three kinds of boundary conditions:

$$\frac{\partial u(x,t)}{\partial N} = 0, \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{B1}$$

$$\frac{\partial u(x,t)}{\partial N} + \gamma(x,t)u = 0, \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{B2}$$

$$u(x,t) = 0, \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{B3}$$

where N is the unit exterior normal vector to $\partial\Omega$ and γ is a nonnegative continuous function on $\partial\Omega \times \mathbb{R}_+$.

Definition 1. A function $u(x,t) \in C^2(\Omega \times [t_{-1}, \infty), \mathbb{R}) \cap C^1(\overline{\Omega} \times [t_{-1}, \infty), \mathbb{R})$ is called a solution of the problem (E), (B), if it satisfies (E) in the domain G along with the corresponding boundary condition, where

$$t_{-1} = \min\Big\{ \min_{1\leq i\leq m} \big\{ \inf_{t\geq 0} \tau_i(t) \big\}, \inf_{t\geq 0} g(t) \Big\}.$$

Definition 2. A solution $u(x,t)$ of the problem (E), (B) is said to be oscillatory in the domain G, if for each positive number μ there exists a point $(x_1, t_1) \in \Omega \times [\mu, \infty)$ where $u(x_1, t_1) = 0$.

Definition 3. A function $U(t)$ is called eventually positive (negative) if there exists a number $t_1 \geq t_0$ such that $U(t) > 0 (< 0)$ holds for all $t_1 \geq t_0$.

In [3], the author considered the equation (E) with two kinds of boundary conditions, B2 and B3, and extended the oscillation criterion of Aitkinson [1] for second order differential equation and given some sufficient conditions which guarantee that every solution of (E) and (B2), (E) and (B3), is oscillatory in G. Our aim in this paper is to give some new oscillation criteria, Kamenev-type [4] and Philos-type [8] oscillation criteria for equation (E) with the boundary conditions (B1)–(B3). Our results in this paper extend and improve the results in [3], since Kamenev-type and

Philos-type oscillation criteria improve the Atkinson criterion for oscillation of second order differential equations.

2. Main Results

In this section we will give some oscillation criteria of (E) with the boundary conditions (B1), (B2) and (B3).

First, we consider the oscillation of the problem (E), (B1).

Theorem 2.1. *Suppose that the conditions* (H_1)–(H_4) *hold. If there exists a function* $\rho \in C^1([t_0, \infty), \mathbb{R}_+)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)} \right) ds = \infty. \tag{2.1}$$

Then every solution of (E), (B1) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, $(t_0 \geq 0)$. By condition (H_3) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x,t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. Integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d^2}{dt^2} \int_\Omega u(x,t)dx = a(t) \int_\Omega \Delta u(x,t)dx + \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u(x, \tau_i(t))dx \\ - \int_\Omega q(x,t) f(u(x, g(t)))dx. \end{aligned} \tag{2.2}$$

Using Green's formula and (B_1), we have

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N} dS = 0, \quad t \geq t_1, \tag{2.3}$$

$$\int_\Omega \Delta u(x, \tau_i(t))dx = \int_{\partial\Omega} \frac{\partial u(x, \tau_i(t))}{\partial N} dS = 0, \quad i = 1, \ldots, m, \ t \geq t_1, \tag{2.4}$$

where dS is the surface element on $\partial\Omega$. Using Jensen's inequality and (H2), we have

$$\begin{aligned}\int_\Omega q(x,t)f(u(x,g(t)))dx &\geq q(t)\int_\Omega f(u(x,g(t)))dx \\ &\geq q(t)\int_\Omega dx f\left(\frac{\int_\Omega u(x,g(t))dx}{\int_\Omega dx}\right), \quad t \geq t_1.\end{aligned} \tag{2.5}$$

Therefore, from (2.2)–(2.5), we have

$$U''(t) + q(t)f(U(g(t))) \leq 0, \quad t \geq t_1, \tag{2.6}$$

where

$$U(t) = \frac{\int_\Omega u(x,t)dx}{\int_\Omega dx}, \quad t \geq t_1. \tag{2.7}$$

It is shown that $U(t) > 0$ and $U''(t) \leq 0$ for $t \geqslant t_1$. We claim that

$$U'(t) > 0, \quad \text{for} \quad t \geqslant t_1. \tag{2.8}$$

If not, there is a $t_2 > t_1$ such that $U'(t_2) < 0$. Then $U'(t) \leq U'(t_2)$ and

$$U(t) \leq U(t_2) + U'(t_2)(t - t_2). \tag{2.9}$$

Let $t \to \infty$ then $\lim_{t\to\infty} U(t) = -\infty$, which contradicts the fact that $U(t) > 0$ for $t \geq t_1$. Therefore (2.8) holds. Set

$$w(t) = \rho(t)\frac{U'(t)}{U(g(t))} \quad \text{for} \quad t \geq t_1. \tag{2.10}$$

Then $w(t) > 0$ for $t \geq t_1$. From (2.10) and (2.6), we have

$$w'(t) \leq -k\rho(t)q(t) + \rho'(t)\frac{U'(t)}{U(g(t))} - g'(t)\rho(t)\frac{U'(t)U'(g(t))}{U^2(g(t))}. \tag{2.11}$$

Using the fact that $U''(t) \leq 0$ and $g(t) \leq t$, then we obtain

$$U'(t) \leq U'(g(t)), \quad t \geq t_1. \tag{2.12}$$

Combining (2.11) and (2.12) we have

$$w'(t) \leq -k\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - g'(t)\rho(t)\frac{(U'(t))^2}{U^2(g(t))}. \tag{2.13}$$

Then, from (2.10) and (2.13), we have

$$w'(t) \leq -k\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{g'(t)}{\rho(t)}w^2(t), \quad t \geq t_1, \tag{2.14}$$

thus

$$\begin{aligned} w'(t) \leq & -k\rho(t)q(t) + \frac{(\rho'(t))^2}{4\rho(t)g'(t)} \\ & - \left[\sqrt{\frac{g'(t)}{\rho(t)}}w(t) - \frac{\rho'(t)}{2\rho(t)}\sqrt{\frac{\rho(t)}{g'(t)}}\right]^2. \end{aligned} \tag{2.15}$$

Then

$$w'(t) \leq -\left[k\rho(t)q(t) - \frac{(\rho'(t))^2}{4\rho(t)g'(t)}\right]. \tag{2.16}$$

Integrating (2.16) from t_1 to t we have

$$w(t) \leq w(t_1) - k\int_{t_1}^{t}\left[\rho(s)q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)}\right]ds. \tag{2.17}$$

Letting $t \to \infty$, we have in view of (2.1) that $w(t) \to -\infty$, a contradiction, since $w(t) > 0$. If $u(x,t) < 0$ for $\Omega \times (t_0, \infty)$ then $-u(x,t)$ is a positive solution of (E), (B1) and the proof is similar. This completes the proof. $\square$

Theorem 2.1 improve Theorems 1 and 2 in [3], since the oscillation criterion in Theorem 2.1 do not require the conditions $\rho'(t) \geq 0$ and $\left(\frac{\rho'(t)}{g'(t)}\right)' \leq 0$ for $t \geq t_0 > 0$.

Now, we present some new oscillation criterion for.(E), (B1) by using integral averages condition of Kamenev-type.

Theorem 2.2. *Assume that all the assumptions of Theorem 2.1 are satisfied, except the condition* (2.1) *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(\rho(s)q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)}\right)ds = \infty. \tag{2.18}$$

Then every solution of (E), (B1) *is oscillatory in* G.

PROOF. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, $(t_0 \geq 0)$. By condition (H_3) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x,t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. By Theorem 2.1 we have

$$w'(t) \leq -k\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{g'(t)}{\rho(t)}w^2(t), \quad t \geq t_1. \tag{2.14}$$

Multiplying both sides of (2.14) by $(t-s)^n$ and integrate from t_1 to t we have

$$\begin{aligned}\int_{t_1}^{t}(t-s)^n\left[\rho(s)q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)}\right]ds& \\ \leq -\frac{1}{k}\int_{t_1}^{t}(t-s)^n w'(s)ds, \quad t \geq t_1.&\end{aligned} \tag{2.19}$$

Since

$$\int_{t_1}^{t}(t-s)^n w'(s)ds = n\int_{t_1}^{t}(t-s)^{n-1}w(s)ds - w(t_1)(t-t_1)^n \tag{2.20}$$

we get

$$\begin{aligned}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n Q(s)ds \leq \frac{1}{k}w(t_1)\left(\frac{t-t_1}{t}\right)^n& \\ -\frac{n}{kt^n}\int_{t_1}^{t}(t-s)^{n-1}w(s)ds&\end{aligned} \tag{2.21}$$

where $Q(s) = \rho(s)q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)}$. Hence

$$\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n Q(s)ds \leq \frac{1}{k}w(t_1)\left(\frac{t-t_1}{t}\right)^n \tag{2.22}$$

where $w(t) > 0$. Then

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n Q(s)ds \to w(t_1) \equiv \text{finite number} \tag{2.23}$$

which contradicts (2.18). If $u(x,t) < 0$ for $\Omega \times (t_0, \infty)$ then $-u(x,t)$ is a positive solution of (E), (B1) and the proof is similar. This completes the proof. $\square$

Next, we present some new oscillation criteria for (E) and (B1) by using integral averages condition of Philos-type. Following PHILOS [8], we introduce a class of functions $\Re$. Let

$$D_0 = \{(t, s) : t > s \geq t_0\} \text{ and } D = \{(t, s) : t \geq s \geq t_0\}. \tag{2.24}$$

The function $H \in C(D, \mathbb{R})$ is said to belongs to the class $\Re$ if

(I) $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ on D_0;

(II) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that and there exists a continuous function $h(t, s)$ such that

$$-\frac{\partial H(t, s)}{\partial s} = h(t, s)\sqrt{H(t, s)} \quad \text{for all } (t, s) \in D_0. \tag{2.25}$$

Theorem 2.3. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[kH(t, s)\rho(s)q(s) - \frac{\rho(s)Q^2(t, s)}{4g'(s)} \right] ds = \infty, \tag{2.26}$$

where

$$Q(t, s) = h(t, s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t, s)}. \tag{2.27}$$

Then every solution of (E), (B1) *is oscillatory in* G.

PROOF. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x, t) > 0$, $u(x, t) \in \Omega \times (t_0, \infty)$, $(t_0 \geq 0)$. By condition (H_3) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x, t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. By Theorem 2.1 we obtain (2.14). In order to simplify notations we denote by

$$\gamma_1(s) = \frac{\rho'(s)}{\rho(s)}, \quad W_1(s) = \frac{g'(s)}{\rho(s)}. \tag{2.28}$$

Then from (2.14) for all $t > t_1$, we have

$$\begin{aligned} &\int_{t_1}^{t} H(t, s)w'(s)ds + \int_{t_1}^{t} kH(t, s)\rho(s)q(s)ds \\ &\quad - \int_{t_1}^{t} H(t, s)\gamma_1(s)w(s)ds + \int_{t_1}^{t} H(t, s)W_1(s)w^2(s)ds \leq 0 \end{aligned} \tag{2.29}$$

which implies that

$$\begin{aligned}\int_{t_1}^{t} kH(t,s)\rho(s)q(s)ds \leq & \int_{t_1}^{t} H(t,s)\gamma_1(s)w(s)ds \\ & - \int_{t_1}^{t} H(t,s)w'(s)ds - \int_{t_1}^{t} H(t,s)W_1(s)w^2(s)ds \\ = & -H(t,s)w(s)\Big|_{t_1}^{t} \\ & - \int_{t_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}w(s) - H(t,s)\gamma_1(s)w(s) + H(t,s)W_1(s)w^2(s)\right]ds \\ = & H(t,t_1)w(t_1) \\ & - \int_{t_1}^{t}\left[\sqrt{H(t,s)}(h(t,s) - \sqrt{H(t,s)}\gamma_1(s))w(s) + H(t,s)W_1(s)w^2(s)\right]ds \\ = & H(t,t_1)w(t_1) \\ & - \int_{t_1}^{t}\left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W_1(s)}}\right]^2 + \int_{t_1}^{t}\frac{Q^2(t,s)}{4W_1(s)}ds\end{aligned} \tag{2.30}$$

where $Q(t,s) = (h(t_1,s) - \sqrt{H(t_1,s)}\gamma_1(s))$. Thereby, we conclude that

$$\begin{aligned}&\int_{t_1}^{t}\left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)}\right]ds \\ &\quad \leq H(t,t_1)w(t_1) - \int_{t_1}^{t}\left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W_1(s)}}\right]^2 ds.\end{aligned} \tag{2.31}$$

By virtue of (2.31) and (II) for all $t > t_1$, we obtain

$$\int_{t_1}^{t}\left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t_1,s)}{4W_1(s)}\right]ds \leq H(t,t_1)|w(t_1). \tag{2.32}$$

Then by (2.32) and (II), we have

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)}\right]ds \\ &\qquad \leq \int_{t_0}^{t_1}\rho(s)q(s)ds + |w(t_1)|.\end{aligned} \tag{2.33}$$

Inequality (2.33) yields

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds < \infty, \tag{2.34}$$

and the latter inequality contradicts assumption (2.26). If $u(x,t) < 0$ for $\Omega \times (t_0, \infty)$ then $-u(x,t)$ is a positive solution of (E), (B1) and the proof is similar. This completes the proof. □

The following theorem is immediate from Theorem 2.3.

Theorem 2.4. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty, \tag{2.35}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)Q^2(t,s)}{g'(s)} ds < \infty. \tag{2.36}$$

Then every solution of (E), (B1) *is oscillatory in* G.

Corollary 2.1. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* $\rho(t) = 1$ *and let* H *belongs to the class* $\Re$ *such that*

$$\begin{gathered} \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)ds = \infty, \\ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)h^2(t,s)}{g'(s)} ds < \infty. \end{gathered} \tag{2.37}$$

Then every solution of (E), (B1) *is oscillatory in* G.

The following two oscillation criteria treat the case when it is not possible to verify easily condition (2.26).

Theorem 2.5. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ*as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that*

$$0 < \inf_{s \ge t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \le \infty. \tag{2.38}$$

Let $\psi \in C[[t_{0,}\infty), \mathbb{R}]$ *such that for* $t \geq t_1$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{Q^2(t,s)}{W_1(s)} ds < \infty, \tag{2.39}$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \psi_+^2(s) W_1(s) ds = \infty, \tag{2.40}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right) ds \geq \sup_{t\geq t_0} \psi(t), \tag{2.41}$$

where $Q(t,s)$ *as in Theorem 2.3 and* $\psi_+ = \max\{\psi(t), 0\}$, *then every solution of* (E), (B1) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, ($t_0 \geq 0$). By condition ($H_3$) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x,t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. By Theorem 2.3 we obtain (2.31). By (2.31) we have for $t > t_1$

$$\begin{aligned} &\frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds \\ &\leq w(t_1) - \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}\, w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds. \end{aligned}$$

Hence, for $t \geq t_1$

$$\begin{aligned} &\limsup_{t_1\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds \\ &\leq w(t_1) - \liminf_{t_1\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}\, w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds. \end{aligned}$$

By (2.41) and the last inequality, we have for $t \geq t_1$

$$\begin{aligned} w(t_1) \geq \psi(t_1) + \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \\ \times \int_{t_1}^{t} \left[\sqrt{H(t_1,s)W_1(s)}w(s) + \frac{Q(t_1,s)}{2\sqrt{W_1(s)}} \right]^2 ds, \end{aligned} \tag{2.42}$$

and hence

$$\begin{aligned} 0 \leq \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds \\ \leq w(t_1) - \psi(t_1) < \infty. \end{aligned} \tag{2.43}$$

Define the functions $\alpha(t)$ and $\beta(t)$ as follows

$$\begin{aligned} \alpha(t) &= \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)W_1(s)w^2(s)ds \\ \beta(t) &= \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s)}Q(t,s)w(s)ds. \end{aligned} \tag{2.44}$$

The reminder of the proof is similar to that the proof of Theorem 2.6 in [7] and hence is omitted. □

Theorem 2.6. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$, *and assume that* (2.38) *holds. Suppose there exists a function* $\psi \in C[[t_{0,}\infty),\mathbb{R}]$ *such that for* $t > t_0$, $T \geq t_0$ (2.40) *holds, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds < \infty, \tag{2.45}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right) ds \geq \sup_{t\geq t_0} \psi(t), \tag{2.46}$$

where $Q(t,s)$ *as in Theorem 2.3 and* $\psi_+ = \max\{\psi(t),0\}$, *then every solution of* (E), (B1) *is oscillatory in* G.

Proof. We proceed as in Theorem 2.3, we assume that equation (E) has an eventually positive solution. Defining again $w(t)$ by (2.18), and in the same way as in Theorem 2.3, we have that the inequality (2.31) holds. By (2.31) we have for $t > t_1$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[kH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds$$
$$\leq w(t_1) - \limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds.$$

It follows from (2.46) that for $t > t_1$

$$\begin{aligned} w(t_1) \geq \psi(t_1) + \limsup_{t\to\infty} \frac{1}{H(t,t_1)} \\ \times \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds, \end{aligned} \tag{2.47}$$

and, hence

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds$$
$$\leq w(t_1) - \psi(t_1) < \infty.$$

By defining again $\alpha(t)$ and $\beta(t)$ as in Theorem 2.5, the remainder of the proof is similar to that the proof of Theorem 2.8 in [7] and hence omitted. $\square$

Now, we consider the oscillation of the problem (E), (B2).

Theorem 2.7. *Suppose that the conditions* (H_1)–(H_4) *hold. If there exists a function* $\rho \in C^1([t_0,\infty),\mathbb{R}_+)$ *such that* (2.1) *holds, then every solution of* (E), (B2) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B2). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0,\infty)$, ($t_0 \geq 0$). By condition ($H_3$) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x,\tau_i(t)) > 0$,

$u(x, g(t)) > 0$, $(x, t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. Integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d^2}{dt^2}\int_\Omega u(x,t)dx = a(t)\int_\Omega \Delta u(x,t)dx + \sum_{i=1}^{m} a_i(t)\int_\Omega \Delta u(x,\tau_i(t))dx \\ - \int_\Omega q(x,t)f(u(x,g(t)))dx.\end{aligned} \tag{2.2}$$

Using Green's formula and (B_2), we have

$$\begin{aligned}\int_\Omega \Delta u(x,t)dx &= \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}dS \\ &= -\int_{\partial\Omega} \gamma(x,t)u(x,t)dS \le 0, \quad t \ge t_1,\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}\int_\Omega \Delta u(x,\tau_i(t))dx = \int_{\partial\Omega} \frac{\partial u(x,\tau_i(t))}{\partial N}dS \\ = -\int_{\partial\Omega} \gamma(x,t)u(x,\tau_i(t))dS \le 0, \quad i = 1,\ldots,m, \quad t \ge t_1,\end{aligned} \tag{2.49}$$

where dS is the surface element on $\partial\Omega$. Using Jensen's inequality and (H_2), we have

$$\begin{aligned}\int_\Omega q(x,t)f(u(x,g(t)))dx &\ge q(t)\int_\Omega f(u(x,g(t)))dx \\ &\ge q(t)\int_\Omega dx f\left(\frac{\int_\Omega u(x,g(t))dx}{\int_\Omega dx}\right), \quad t \ge t_1.\end{aligned} \tag{2.50}$$

Therefore, from (2.2), (2.48)–(2.50), we have

$$U''(t) + q(t)f(U(g(t))) \le 0, \quad t \ge t_1, \tag{2.51}$$

where $U(t)$ is defined by (2.7). The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Theorem 2.8. *Assume that all the assumptions of Theorem 2.7 are satisfied, except the condition* (2.1) *is replaced by* (2.18). *Then every solution of* (E), (B2) *is oscillatory in* G.

Proof. The proof is similar to that of Theorem 2.2 and hence is omitted. □

Theorem 2.9. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that* (2.26) *holds. Then every solution of* (E), (B2) *is oscillatory in* G.

Proof. The proof is similar to that of Theorem 2.3 and hence is omitted. □

The following theorem is immediate from Theorem 2.9.

Theorem 2.10. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that* (2.35) *and* (2.36) *hold. Then every solution of* (E), (B2) *is oscillatory in* G.

The following two theorems follow immediately from Theorems 2.5 and 2.6 for oscillation criteria of (E), (B2).

Theorem 2.11. *Assume that all the assumptions of Theorem 2.5 hold. Then every solution of* (E), (B2) *is oscillatory in* G.

Theorem 2.12. *Assume that all the assumptions of Theorem 2.6 hold. Then every solution of* (E), (B2) *is oscillatory in* G.

Next, we consider the oscillation of the problem (E), (B3).

With each solution $u(x,t)$ of the problem (E), (B3), we associate a function $V(t)$ defined by

$$V(t) = \frac{\int_\Omega u(x,t)\Phi(x)dx}{\int_\Omega \Phi(x)dx}, \quad t \geq t_1. \tag{2.52}$$

Consider the Dirichlet problem in the domain Ω

$$\Delta u + \alpha u = 0 \quad \text{in} \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{2.53}$$

$$u = 0 \quad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2.54}$$

in which α is a constant. It is well known [13] that the smallest eigenvalue α_1 of problem (2.53)–(2.54) is positive and the corresponding eigenfunction $\Phi(x)$ is also positive for $x \in \Omega$.

Theorem 2.13. *Suppose that the conditions* (H_1)–(H_4) *hold,* $g(t) \leq \tau_i(t) \leq t$ *for* $i = 1, 2, \ldots, m$*. If there exists a function* $\rho \in C^1([t_0, \infty), \mathbb{R}_+)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)Q(s) - \frac{(\rho'(s))^2}{4k\rho(s)g'(s)} \right) ds = \infty, \tag{2.55}$$

where

$$Q(t) = \alpha_1[a(t) + \sum_{i=1}^{m} a_i(t)] + kq(t).$$

Then every solution of (E), (B3) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B3). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, ($t_0 \geq 0$). By condition ($H_3$) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x,t) \in \Omega \times [t_1, \infty$, $i = i, \ldots, m$. Multiplying both sides of equation (E) $\Phi(x)$, and integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d^2}{dt^2} \int_\Omega u(x,t)\Phi(x)dx &= a(t) \int_\Omega \Delta u(x,t)\Phi(x)dx \\ &+ \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u(x, \tau_i(t))\Phi(x)dx \\ &- \int_\Omega q(x,t) f(u(x, g(t)))\Phi(x)dx. \end{aligned} \tag{2.56}$$

Using Green's formula and (B3), we have

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Phi dx &= \int_{\partial\Omega} \left(\Phi(x)\frac{\partial u}{\partial N} - u\frac{\partial \Phi(x)}{\partial N} \right) dS + \int_\Omega u\Delta\Phi(x)dx \\ &= -\alpha_1 \int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_1, \end{aligned} \tag{2.57}$$

and

$$\begin{gathered} \int_\Omega \Delta u(x, \tau_i(t))\Phi dx = -\alpha_1 \int_\Omega u(x, \tau_i(t))\Phi(x)dx, \\ i = 1, \ldots, m \quad t \geq t_1, \end{gathered} \tag{2.58}$$

where α_1 is the smallest eigenvalue of problem (2.53)–(2.54). Using Jensen's inequality and (H_2), we have

$$\begin{aligned}&\int_\Omega q(x,t)f(u(x,g(t)))dx \geq q(t)\int_\Omega f(u(x,g(t)))dx \\ &\geq q(t)\int_\Omega \Phi(x)dx f\left(\frac{\int_\Omega u(x,g(t))\Phi(x)dx}{\int_\Omega \Phi(x)dx}\right), \quad t\geq t_1.\end{aligned} \tag{2.59}$$

Therefore, from (2.56)–(2.59), we have

$$\begin{aligned}V''(t)+\alpha_1 V(t)+\alpha_1\sum_{i=1}^{m}a_i(t)V(\tau_i(t))& \\ +q(t)f(V(g(t)))\leq 0,& \quad t\geq t_1.\end{aligned} \tag{2.60}$$

It is shown that $V(t)>0$ and $V''(t)<0$ for $t\geqslant t_1$. As in the proof of Theorem 2.1 we can prove easily that $V'(t)>0$, then since $g(t)\leq\tau_i(t)\leq t$, then $V(t)\geq V(\tau_i(t))\geq V(g(t))$, then (2.60) reduces to

$$V''(t)+Q(t)V(g(t))\leq 0, \quad t\geq t_1. \tag{2.61}$$

Set

$$w(t)=\rho(t)\frac{V'(t)}{V(g(t))} \quad \text{for } t\geq t_2. \tag{2.62}$$

Then $w(t)>0$ for $t\geq t_1$. From (2.61) and (2.62), we have

$$w'(t)\leq -\rho(t)Q(t)+\rho'(t)\frac{U'(t)}{U(g(t))}-g'(t)\rho(t)\frac{U'(t)U'(g(t))}{U^2(g(t))}. \tag{2.63}$$

Using the fact that $V''(t)<0$ and $g(t)\leq t$, then we obtain

$$V'(t)\leq V'(g(t)), \quad t\geq t_1. \tag{2.64}$$

Combining (2.63) and (2.64) we have

$$w'(t)\leq -\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{g'(t)}{\rho(t)}w^2(t). \tag{2.65}$$

The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Theorem 2.13 improve Theorems 4 and 5 in [3], since the oscillation criterion in Theorem 2.1 do not require the conditions $\rho'(t) \geq 0$ and $\left(\frac{\rho'(t)}{g'(t)}\right)' \leq 0$ for $t \geq t_0 > 0$.

Theorem 2.14. *Assume that all the assumptions of Theorem 2.13 are satisfied, except the condition* (2.55) *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \left(\rho(s)Q(s) - \frac{(\rho'(s))^2}{4\rho(s)g'(s)} \right) ds = \infty. \tag{2.66}$$

Then every solution of (E), (B3) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, $(t_0 \geq 0)$. By Theorem 2.13 we have (2.65), and the remainder of the proof is similar to that of Theorem 2.2 and hence is omitted. □

Theorem 2.15. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function ρ as in Theorem 2.1 and let H belongs to the class $\Re$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)Q(s) - \frac{\rho(s)Q^2(t,s)}{4g'(s)} \right] ds = \infty,$$

where

$$Q(t,s) = h(t,s) - \frac{\rho'(s)}{\rho(s)} \sqrt{H(t,s)}.$$

Then every solution of (E), (B3) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0, \infty)$, $(t_0 \geq 0)$. By condition (H_3) there exists a $t_1 > \mu$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x, \tau_i(t)) > 0$, $u(x, g(t)) > 0$, $(x,t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. By Theorem 2.13 we obtain (2.65). In order to simplify notations we use (2.28). Then from (2.65) for all $t > t_1$, we have

$$\begin{aligned} &\int_{t_1}^{t} H(t,s)w'(s)ds + \int_{t_1}^{t} H(t,s)\rho(s)Q(s)ds \\ &- \int_{t_1}^{t} H(t,s)\gamma_1(s)w(s)ds + \int_{t_1}^{t} H(t,s)W_1(s)w^2(s)ds \leq 0. \end{aligned} \tag{2.67}$$

The reminder of the proof is similar to that of Theorem 2.3 and hence is omitted. □

The following theorem is immediate from Theorem 2.15.

Theorem 2.16. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)Q(s)ds = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)Q^2(t,s)}{g'(s)} ds < \infty.$$

Then every solution of (E), (B3) *is oscillatory in* G.

Theorem 2.17. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function* ρ *as in Theorem 2.1 and let* H *belongs to the class* $\Re$ *such that*

$$0 < \inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty.$$

Let $\psi \in C[[t_{0,}\infty),\mathbb{R}]$ *such that for* $t \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{Q^2(t,s)}{W_1(s)} ds < \infty,$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \psi_+^2(s)W_1(s)ds = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\rho(s)Q(s) - \frac{Q^2(t,s)}{4W_1(s)}\right) ds \geq \sup_{t\geq t_0} \psi(t),$$

where $Q(t,s)$ *as in Theorem 2.15 and* $\psi_+ = \max\{\psi(t),0\}$, *then every solution of* (E), (B3) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B1). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,t) \in \Omega \times (t_0,\infty)$, $(t_0 \geq 0)$. By condition (H_3) there exists a $t_1 > t_0$ such that $\tau_i(t) \geq t_0$ and $g(t) \geq t_0$, $t \geq t_1$. Then $u(x,\tau_i(t)) > 0$,

$u(x, g(t)) > 0$ $(x, t) \in \Omega \times [t_1, \infty)$, $i = i, \ldots, m$. From Theorem 2.15 by using (2.67) as in the proof of Theorem 2.4, we obtain find for $t > t_1$, that

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\rho(s)Q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds$$
$$\leq w(t_1) - \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\rho(s)Q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right] ds$$
$$\leq w(t_1) - \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W_1(s)}} \right]^2 ds.$$

The reminder of the proof is similar to that the proof of Theorem 2.4 and hence is omitted. □

Theorem 2.18. *Assume that* (H_1)–(H_4) *hold. Let the differentiable function ρ as in Theorem 2.1 and let H belongs to the class $\Re$, and assume that* (2.38) *holds. Suppose there exists a function $\psi \in C[[t_{0,}\infty), \mathbb{R}]$ such that for $t > t_0$, $T \geq t_0$* (2.40) *holds, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)Q(s)ds < \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\rho(s)Q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right) ds \geq \sup_{t\geq t_0} \psi(t),$$

where $Q(t,s)$ as in Theorem 2.15 and $\psi_+ = \max\{\psi(t), 0\}$, then every solution of (E), (B3) *is oscillatory in G.*

Proof. The proof is similar to that of Theorem 2.6 by using (2.65) and the details are left to the reader.

With the appropriate choice of functions H and h, it is possible to derive from Theorems 2.3–2.6 a number of oscillation criteria for equation (1.1). Defining, for example, for some integer $n > 1$, the function

$H(t, s)$ by

$$H(t, s) = (t - s)^n, \quad (t, s) \in D,$$

we can easily check that $H \in \Re$. Furthermore the function

$$h(t, s) = n(t - s)^{(n-2)/2}, \quad (t, s) \in D.$$

Another possibility is to choose the functions H and h as follows:

$$H(t, s) = \left(\ln \frac{t}{s}\right)^n, \quad h(t, s) = \frac{n}{s}\left(\ln \frac{t}{s}\right)^{n/2-1}, \qquad t \geq s \geq t_0.$$

One may also choose the more general forms for the functions H and h:

$$H(t, s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \quad h(t, s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{n/2-1} \quad t \geq s \geq t_0,$$

where $n > 1$ is an integer, and $\theta : [t_0, \infty) \to \mathbb{R}^+$ is a continuous function satisfying condition

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{du}{\theta(u)} = \infty,$$

and

$$H(t, s) = (e^t - e^s)^n, \quad h(t, s) = ne^s\left(e^t - e^s\right)^{(n-2)/2}, \qquad t \geq s \geq t_0.$$

It is a simple matter to check that in all these cases assumption (I) and (II) are verified. The details are left to the reader. □

Acknowledgement. The author thanks Prof. I. D. Iliev for his suggestions to improve the statement of the results in this paper.

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(Received November 19, 2001, revised May 21, 2002)