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Completely generalized multivalued strongly quasivariational inequalities

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Abstract. In this paper, we introduce and study two new classes of quasivariational inequalities and construct some iterative algorithms by using the projection technique. We establish the existence of solutions for these classes of quasivariational inequalities involving relaxed Lipschitz, relaxed monotone and generalized pseudocontractive mappings. Under suitable conditions, the convergence analyses of the iterative algorithms are also studied. Our results are the extension and improvements of the earlier and recent results in this field.

1. Introduction

Variational inequality theory is a very powerful tool of the current mathematical technology. Up to now it has been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences etc. For details we refer to [1]–[3] and [6]–[16]. Quasivariational inequalities are the extended form of variational inequalities in which the constrained set depends upon the solutions. Recently, Yao [16] and Verma [11]–[15] studied the solvability of some classes of variational inequalities involving relaxed Lipschitz, relaxed

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monotone and generalized pseudocontractive mappings. Bai–Tang–Liu [1] extended the results due to Yao [16] and Verma [11]–[15] from the variational inequalities to the completely generalized strongly nonlinear implicit quasivariational inequalities. On the other hand, Guo–Yao [2], Lee–Lee–Huang [3], Noor [6]–[8], Siddiqi–Ansari [10] and others introduced and studied some classes of variational inequalities and quasivariational inequalities dealing with strongly monotone mappings, respectively. Inspired and motivated by the results in [1]–[4] and [6]–[16], in this paper, we introduce and study two new classes of quasivariational inequalities and construct some iterative algorithms by using the projection technique. We establish the existence of solutions for these quasivariational inequalities involving relaxed Lipschitz, relaxed monotone and generalized pseudocontractive mappings. Under suitable conditions, the convergence analyses of the iterative algorithms are also studied. Our results extend, improve and unify recent results due to Bai–Tang–Liu [1], Guo–Yao [2], Lee–Lee–Huang [3], Noor [6]–[8], Siddiqi–Ansari [10], Verma [11]–[15], Yao [16] and others.

2. Preliminaries

In what follows, we assume that H is a real Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and I denotes the identity mapping on H. Let 2^H, $CB(H)$ and $CC(H)$ denote the families of all nonempty subsets, nonempty bounded closed subsets and nonempty closed convex subsets of H, respectively. Let P_K be the projection of H onto K, where K is a subset of H. Given $f \in H$, $g, h : H \to H$, $A, B, C, D : H \to 2^H$, $K : H \to CC(H)$ and $N : H \times H \times H \to H$, we consider the following problem:

Find $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle hgu, v - gu\rangle \geq \langle N(x, y, z) - f, v - gu\rangle \quad \text{for all } v \in K(w), \tag{2.1}$$

which is called the *completely generalized multivalued strongly quasivariational inequality.*

Special cases

If $h = I$, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle gu, v - gu \rangle \geq \langle N(x, y, z) - f, v - gu \rangle \quad \text{for all } v \in K(w), \tag{2.2}$$

which is known as the *completely generalized multivalued quasivariational inequality.*

If $C = I$ and $N(x, y, z) = by - \rho(ax - f)$ for all $x, y, z \in H$, where $\rho > 0$ is a constant and $a, b : H \to H$ are mappings, then problem (2.1) collapses to finding $u \in H$, $x \in Au$, $y \in Bu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle hgu, v - gu \rangle \geq \langle by - \rho(ax - f), v - gu \rangle \quad \text{for all } v \in K(w), \tag{2.3}$$

which is called the *generalized set-valued strongly nonlinear implicit quasivariational inequality* introduced by Lee–Lee–Huang [3].

If $A = B = C = D = I$, $f = 0$ and $N(x, y, z) = hx - \rho(by + cz)$ for all $x, y, z \in H$, where $\rho > 0$ is a constant and $b, c : H \to H$ are mappings, then problem (2.1) is equivalent to finding $u \in K$ such that $gu \in K(u)$ and

$$\langle hgu, v - gu \rangle \geq \langle hu, v - gu \rangle - \rho\langle (b + c)u, v - gu \rangle \\ \text{for all } v \in K(u), \tag{2.4}$$

which is called the *nonlinear variational inequality* introduced and studied by Verma [15].

If $A = I$, $f = 0$ and $N(x, y, z) = hgx - N(y, z)$ for all $x, y, z \in H$, where $N : H \times H$ is a nonlinear mapping, then problem (2.1) is equivalent to fining $u \in H$, $y \in Bu$, $z \in Cu$ such that $gu \in K(u)$ and

$$\langle N(y, z), v - gu \rangle \geq 0 \quad \text{for all } v \in K(u), \tag{2.5}$$

which is called the *generalized multivalued quasi-variational inequality* and introduced and studied by Noor [7].

If $h = C = I$, $f = 0$ and $N(x, y, z) = ax - by$ for all $x, y \in H$, where $a, b : H \to H$ are mappings, then problem (2.1) reduces to finding $u \in H$, $x \in Au$, $y \in Bu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle gu, v - gu \rangle \geq \langle ax - by, v - gu \rangle \quad \text{for all } v \in K(w), \tag{2.6}$$

which is known as the *completely generalized strongly nonlinear implicit quasivariational inequality* and introduced by BAI–TANG–LIU [1].

For suitable and appropriate choice of the mappings g, h, A, B, C, D, K, N and the element f, one can obtain various classes of variational inequalities or quasivariatioal inequalities in [2], [8]–[14] as special cases of problem (2.1).

Problem 2.1 has potential applications in mechanics, physics, differential equations, pure and applied sciences. Furthermore, there exist problems arising in structural analysis, which can be studied by the completely generalized multivalued strongly quasivariational inequality (2.1) only.

Example 2.1. For simplicity, we consider a elastoplasticity problem, which is mainly due to PANAGIOTOPOULOS–STAVROULAKIS [9]. It is assumed that a general hyperelastic material law holds for the elastic behavior of the elastoplastic material under consideration. Moreover, a nonconvex yield function $\sigma \to F(\sigma)$ is introduced for the plasticity. For the basic definitions and concepts, see [9]. Let us assume the decomposition

$$E = E^e + E^p, \tag{2.7}$$

where E^e denotes the elastic, and E^p the plastic deformation of the three-dimensional elastoplastic body. We write the complementary virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma\rangle + \langle E^p, \tau - \sigma\rangle = \langle f, \tau - \sigma\rangle \quad \text{for all } \tau \in Z. \tag{2.8}$$

Here we have assumed that the body on a part Γ_U of its boundary has given displacements, that is, $\mu_i = U_i$ on Γ_U, and that on the rest of its boundary $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on Γ_F, where

$$\begin{gathered} \langle E, \sigma\rangle = \int_\Omega \epsilon_{ij}\sigma_{ij} d\Omega, \quad \langle f, \sigma\rangle = \int_{\Gamma_U} U_i S_i d\Gamma, \\ Z = \{\tau : \tau_{i_j,j} + f_i = 0 \text{ on } \Omega,\ i,j = 1,2,3, \\ T_i = F_i \text{ on } \Gamma_F,\ i = 1,2,3\} \end{gathered} \tag{2.9}$$

is the set of statically admissible stresses and Ω is the structure of the body. Let us assume that the material of the structure Ω is hyperelastic such that

$$\langle E^e, \tau - \sigma\rangle \leq \langle W'_m(\sigma), \tau - \sigma\rangle \quad \text{for all } \tau \in R^6, \tag{2.10}$$

where W_m is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasidifferentiable, that is, there exist convex and compact subsets $\underline{\partial}' W_m$ and $\overline{\partial}' W_m$ such that

$$\langle W_m'(\sigma), \tau - \sigma\rangle = \max_{W_1^e \in \underline{\partial}' W_m} \langle W_1^e, \tau - \sigma\rangle + \min_{W_2^e \in \overline{\partial}' W_m} \langle W_2^e, \tau - \sigma\rangle. \quad (2.11)$$

We also introduce the generally nonconvex yield function $P \subset Z$, which is defined by means of the general quasidifferentiable function $F(\sigma)$, that is,

$$P = \{\sigma \in Z : F(\sigma) \leq 0\}. \quad (2.12)$$

Here W_m is a generally nonconvex and nonsmooth, but quasidifferentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining (2.7)–(2.12), we can obtain the following problem: find $\sigma \in P(\sigma)$ such that $W_1^e \in \underline{\partial}' W_m(\sigma)$, $W_2^e \in \overline{\partial}' W_m(\sigma)$ and

$$\langle N(W_1^e, W_2^e), \tau - \sigma\rangle \geq \langle f, \tau - \sigma\rangle \quad \text{for all } \tau \in P(\sigma),$$

which is exactly problem (2.1) with $C = h$, $D = g = I$, $N(x,y,z) = hz - N(x,y) + 2f$, $u = \sigma$, $K(u) = P(\sigma)$, $W_1^e \in \underline{\partial}' W_m(\sigma) = Au$ and $W_2^e \in \overline{\partial}' W_m(\sigma) = Bu$.

Definition 2.1. A mapping $h : H \to H$ is said to be *strongly monotone* and *Lipschitz continuous*, if there exist constants $r, s > 0$ such that

$$\langle hu - hv, u - v\rangle \geq r\|u - v\|^2 \quad \text{and} \quad \|hu - hv\| \leq s\|u - v\|$$

for all $u, v \in H$, respectively.

Definition 2.2. A multivalued mapping $A : H \to CB(H)$ is said to be *H-Lipschitz continuous*, if there exists a constant $r > 0$ such that

$$H(Au, Av) \leq r\|u - v\| \quad \text{for all } u, v \in H,$$

where $H(\cdot,\cdot)$ denote the Hausdorff metric on $CB(H)$.

Definition 2.3. A multivalued mapping $A : H \to 2^H$ is said to be

(i) *relaxed Lipschitz with respect to the first argument* of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(x,a,b) - N(y,a,b), u - v\rangle \leq -r\|u - v\|^2$$

for all $a, b, u, v \in H$, $x \in Au$ and $y \in Av$;

(ii) *generalized pseudocontractive with respect to the second argument* of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(a, x, b) - N(a, y, b), u - v\rangle \le r\|u - v\|^2$$

for all $a, b, u, v \in H$, $x \in Au$ and $y \in Av$;

(iii) *relaxed monotone with respect to the third argument* of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(a, b, x) - N(a, b, y), u - v\rangle \ge -r\|u - v\|^2$$

for all $a, b, u, v \in H$, $x \in Au$ and $y \in Av$.

Definition 2.4. A mapping $N : H \times H \times H \to H$ is said to be *Lipschitz continuous with respect to the first argument* if there exists a constant $t > 0$ such that

$$\|N(x, a, b) - N(y, a, b)\| \le t\|x - y\| \quad \text{for all } x, y, a, b \in H.$$

In a similar way, we can define the Lipschitz continuity of the mapping $N(\cdot, \cdot, \cdot)$ with respect to the second or third argument.

3. Main results

Lemma 3.1 ([4])**.** *Let K be a closed convex set in H. Then, given $z \in H$, $u = P_K z$ if and only if $u \in K$ satisfies*

$$\langle u - z, v - u\rangle \ge 0 \quad \textit{for all } v \in K.$$

Furthermore, the projection operator P_K is nonexpensive, that is, $\|P_K u - P_K v\| \le \|u - v\|$ for all $u, v \in H$.

It follows from (2.1), (2.2) and Lemma 3.1 that

Lemma 3.2. *The completely generalized multivalued strongly quasi-variational inequality (2.1) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $gu \in K(w)$ if and only if there exist $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that*

$$gu = P_{K(w)}[gu - \rho(hgu - N(x, y, z) + f)],$$

where $\rho > 0$ is a constant.

Lemma 3.3. *The completely generalized multivalued quasivariational inequality* (2.2) *has a solution* $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ *with* $gu \in K(w)$ *if and only if there exist* $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ *such that*

$$gu = P_{K(w)}[(1-\rho)gu + \rho N(x,y,z) - \rho f],$$

where $\rho > 0$ *is a constant.*

Remark 3.1. Lemmas 3.2 and 3.3 extend Lemma 3.3 in [1], Proposition 3.1 in [2], Lemma 3.1 in [8], Lemma 3.2 in [11], [12], Theorem 2.1 in [14] and Lemma 2.2 in [15].

Based on Lemmas 3.2, 3.3 and Nadler's result [5], we are now in a position to propose the following algorithms for the completely generalized multivalued strongly quasivariational inequality (2.1) and the completely generalized multivalued quasivariational inequality (2.2).

Algorithm 3.1. *Let* $A, B, C, D : H \to CB(H)$, $K : H \to CC(H)$ *and* $g, h : H \to H$. *Given* $u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, $w_0 \in Du_0$, *compute* u_{n+1} *by the iterative scheme*

$$gu_{n+1} = P_{K(w_n)}[gu_n - \rho(hgu_n - N(x_n,y_n,z_n) + f)], \tag{3.1}$$

$$\begin{aligned} \|x_n - x_{n+1}\| &\le (1+(n+1)^{-1})H(Au_n, Au_{n+1}), \quad x_n \in Au_n, \\ \|y_n - y_{n+1}\| &\le (1+(n+1)^{-1})H(Bu_n, Bu_{n+1}), \quad y_n \in Bu_n, \\ \|z_n - z_{n+1}\| &\le (1+(n+1)^{-1})H(Cu_n, Cu_{n+1}), \quad z_n \in Cu_n, \\ \|w_n - w_{n+1}\| &\le (1+(n+1)^{-1})H(Du_n, Du_{n+1}), \quad w_n \in Du_n \end{aligned} \tag{3.2}$$

for all $n \ge 0$, *where* $\rho > 0$ *is a constant.*

Algorithm 3.2. *Let* $A, B, C, D : H \to CB(H)$, $K : H \to CC(H)$ *and* $g : H \to H$. *Given* $u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, $w_0 \in Du_0$, *compute* u_{n+1} *by the iterative scheme*

$$gu_{n+1} = P_{K(w_n)}[(1-\rho)gu_n + \rho N(x_n,y_n,z_n) - \rho f] \tag{3.3}$$

for all $n \ge 0$, *where* $\rho > 0$ *is a constant and* $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$, $\{z_n\}_{n\ge0}$, $\{w_n\}_{n\ge0}$ *are defined in* (3.2).

Remark 3.2. Algorithms 3.1 and 3.2 include several known algorithms of [1], [11], [12] as special cases.

Now we establish the existence of solutions of the completely generalized multivalued strongly quasivariational inequality (2.1) and the completely generalized multivalued quasivariational inequality (2.2), and the convergence of iterative sequences generated by Algorithms 3.1 and 3.2.

Theorem 3.1. *Let $g, h : H \to H$ be Lipschitz continuous with constants p, q, respectively, and g be strongly monotone with constant δ. Let $N : H \times H \times H \to H$ be Lipschitz continuous with constants σ, η, ζ with respect to the first, second and third arguments, respectively. Let $K : H \to CC(H)$ be a multivalued mapping such that*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \le \mu\|x - y\| \quad \text{for all } x, y, z \in H, \tag{3.4}$$

where $\mu > 0$ is a constant. Suppose that $A, B, C, D : H \to CB(H)$ are H-Lipschtiz continuous with H-Lipschitz constants α, β, γ, ξ, respectively, and A is relaxed Lipschitz with constant τ with respect to the first argument of N, and B is generalized pseudocontractive with constant υ with respect to the second argument of N. Let $k = \sqrt{1 - 2\delta + p^2} + \mu\xi$ and $j = qp + \zeta\gamma$. If there exists a constant $\rho > 0$ satisfying

$$k + \rho j < \delta, \tag{3.5}$$

and at least one of the following conditions

$$\begin{aligned} &\sigma\alpha + \eta\beta > j, \quad |\tau - \upsilon - (\delta - k)j| > \sqrt{(1 - (\delta - k)^2)((\sigma\alpha + \eta\beta)^2 - j^2)}, \\ &\left|\rho - \frac{\tau - \upsilon - (\delta - k)j}{(\sigma\alpha + \eta\beta)^2 - j^2}\right| \\ &\quad < \frac{\sqrt{(\tau - \upsilon - (\delta - k)j)^2 - (1 - (\delta - k)^2)((\sigma\alpha + \eta\beta)^2 - j)}}{(\sigma\alpha + \eta\beta)^2 - j^2}; \end{aligned} \tag{3.6}$$

$$\sigma\alpha + \eta\beta = j, \quad \tau - \upsilon < (\delta - k)j, \quad \rho > \frac{1 - (\delta - k)^2}{2(\tau - \upsilon - (\delta - k)j)}; \tag{3.7}$$

$$\begin{aligned} &\sigma\alpha + \eta\beta < j, \\ &\left|\rho - \frac{(\delta - k)j - \tau + \upsilon}{j^2 - (\sigma\alpha + \eta\beta)^2}\right| \end{aligned} \tag{3.8}$$

$$> \frac{\sqrt{(1-(\delta-k)^2)(j^2-(\sigma\alpha+\eta\beta)^2)+((\delta-k)j-\tau+\upsilon)^2}}{j^2-(\sigma\alpha+\eta\beta)^2},$$

then for every $f \in H$, the completely generalized multivalued strongly quasivariational inequality (2.1) *has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $hu \in K(w)$ and $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$, where $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ are defined in Algorithm 3.1.*

Proof. Since g is Lipschitz continuous and strongly monotone, it follows that

$$\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \leq \sqrt{1-2\delta+p^2}\|u_n - u_{n-1}\|. \tag{3.9}$$

Since A and B are H-Lipschitz continuous, relaxed Lipschitz and generalized pseudocontractive with respect to the first and second arguments of N, respectively, and N is Lipschitz continuous with respect to the first and second arguments, by (3.2) we know that

$$\begin{aligned}
&\|u_n - u_{n-1} + \rho[N(x_n,y_n,z_n) - N(x_{n-1},y_{n-1},z_n)]\|^2 \\
&\quad = \|u_n - u_{n-1}\|^2 + 2\rho\langle N(x_n,y_n,z_n) - N(x_{n-1},y_n,z_n), u_n - u_{n-1}\rangle \\
&\quad + 2\rho\langle N(x_{n-1},y_n,z_n) - N(x_{n-1},y_{n-1},z_n), u_n - u_{n-1}\rangle \\
&\quad + \rho^2\|N(x_n,y_n,z_n) - N(x_{n-1},y_n,z_n) + N(x_{n-1},y_n,z_n) \\
&\quad - N(x_{n-1},y_{n-1},z_n)\|^2 \\
&\leq (1-2\rho(\tau-\upsilon))\|u_n - u_{n-1}\|^2 + \rho^2(\sigma\|x_n - x_{n-1}\| + \eta\|y_n - y_{n-1}\|)^2 \\
&\leq (1-2\rho(\tau-\upsilon) + \rho^2(\sigma\alpha+\eta\beta)^2(1+n^{-1})^2)\|u_n - u_{n-1}\|^2
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\|N(x_{n-1},y_{n-1},z_n) - N(x_{n-1},y_{n-1},z_{n-1})\| \\
&\quad \leq \xi\|z_n - z_{n-1}\| \leq \xi\gamma(1+n^{-1})\|u_n - u_{n-1}\|.
\end{aligned} \tag{3.11}$$

It follows from (3.1), (3.4), (3.9)–(3.11), Lemma 3.2 and the strong monotonicity of g that

$$\|u_{n+1} - u_n\| \leq \delta^{-1}\|gu_{n+1} - gu_n\|$$

$$
\begin{aligned}
&\le \delta^{-1}(\|P_{K(w_n)}(gu_n - \rho(hgu_n - N(x_n, y_n, z_n) + f)) \\
&\quad - P_{K(w_n)}(gu_{n-1} - \rho(hgu_{n-1} - N(x_{n-1}, y_{n-1}, z_{n-1}) + f))\| \\
&\quad + \|P_{K(w_n)}(gu_{n-1} - \rho(hgu_{n-1} - N(x_{n-1}, y_{n-1}, z_{n-1}) + f)) \\
&\quad - P_{K(w_{n-1})}(gu_{n-1} - \rho(hgu_{n-1} - N(x_{n-1}, y_{n-1}, z_{n-1}) + f))\|) \\
&\le \delta^{-1}(\|gu_n - gu_{n-1} - \rho(hgu_n - hgu_{n-1} - N(x_n, y_n, z_n) \\
&\quad + N(x_{n-1}, y_{n-1}, z_{n-1}))\| + \mu\|w_n - w_{n-1}\|) \\
&\le \delta^{-1}(\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| + \rho\|hgu_n - hgu_{n-1}\| \\
&\quad + \|u_n - u_{n-1} + \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n))\| \\
&\quad + \rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\| \\
&\quad + \mu(1 + n^{-1})H(Du_n, Du_{n-1})) \\
&\le \delta^{-1}(\sqrt{1 - 2\delta + p^2} \\
&\quad + \rho qp + \sqrt{1 - 2\rho(\tau - \upsilon) + \rho^2(\sigma\alpha + \eta\beta)^2(1 + n^{-1})^2} \\
&\quad + \rho\zeta\gamma(1 + n^{-1}) + \mu\xi(1 + n^{-1}))\|u_n - u_{n-1}\| \\
&= \theta_n\|u_n - u_{n-1}\|,
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\theta_n &= \delta^{-1}(\sqrt{1 - 2\delta + p^2} + \rho qp + \sqrt{1 - 2\rho(\tau - \upsilon) + \rho^2(\sigma\alpha + \eta\beta)^2(1 + n^{-1})^2} \\
&\qquad + \rho\zeta\gamma(1 + n^{-1}) + \mu\xi(1 + n^{-1})) \to \theta \\
&= \delta^{-1}(k + \rho j + \sqrt{1 - 2\rho(\tau - \upsilon) + \rho^2(\sigma\alpha + \eta\beta)^2})
\end{aligned}
$$

as $n \to \infty$. It is clear that (3.5) yields that

$$
\begin{aligned}
\theta < 1 &\Leftrightarrow \sqrt{1 - 2\rho(\tau - \upsilon) + \rho^2(\sigma\alpha + \eta\beta)^2} < \delta - k - \rho j \\
&\Leftrightarrow ((\sigma\alpha + \eta\beta)^2 - j^2)\rho^2 - 2\rho(\tau - \upsilon - (\delta - k)j) < (\delta - k)^2 - 1.
\end{aligned}
\tag{3.13}
$$

It follows from (3.13) and one of (3.6), (3.7) and (3.8) that $\theta < 1$. Thus $\theta_n < 1$ for n sufficiently large and (3.12) yields that $\{u_n\}_{n\ge 0}$ is a Cauchy sequence in H. Let $u_n \to u \in H$ as $n \to \infty$. Using (3.2) and the H-Lipschitz continuity of A, B, C, D, we have

$$\begin{aligned}
\|x_n - x_{n+1}\| &\le (1+(n+1)^{-1})H(Au_n, Au_{n+1}) \\
&\le (1+(n+1)^{-1})\alpha\|u_n - u_{n+1}\|, \\
\|y_n - y_{n+1}\| &\le (1+(n+1)^{-1})H(Bu_n, Bu_{n+1}) \\
&\le (1+(n+1)^{-1})\beta\|u_n - u_{n+1}\|, \\
\|z_n - z_{n+1}\| &\le (1+(n+1)^{-1})H(Cu_n, Cu_{n+1}) \\
&\le (1+(n+1)^{-1})\gamma\|u_n - u_{n+1}\|, \\
\|w_n - w_{n+1}\| &\le (1+(n+1)^{-1})H(Du_n, Du_{n+1}) \\
&\le (1+(n+1)^{-1})\xi\|u_n - u_{n+1}\|,
\end{aligned}$$

which mean that $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$, $\{z_n\}_{n\ge0}$, $\{w_n\}_{n\ge0}$ are Cauchy sequences in H. Let $x_n \to x$, $y_n \to y$, $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Notice that

$$\begin{aligned}
d(x, Au) &= \inf\{\|x-t\| : t \in Au\} \le \|x - x_n\| + d(x_n, Au) \\
&\le \|x - x_n\| + H(Au_n, Au) \le \|x - x_n\| + \alpha\|u_n - u\| \to 0
\end{aligned}$$

as $n \to \infty$. Hence $x \in Au$. Similarly, we have $y \in Bu$, $z \in Cu$ and $w \in Du$. Using Lemma 3.1, (3.1), and the Lipschitz continuity of h, g, and the H-Lipschitz continuity of N with respect to the first, second and third arguments, respectively, we get that

$$gu = P_{K(w)}(gu - \rho(hgu - N(x,y,z) + f)). \tag{3.14}$$

In view of (3.14 and Lemma 3.2, we obtain that the completely generalized multivalued strongly quasivariational inequality (2.1) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $gu \in K(w)$. This completes the proof. □

Theorem 3.2. *Let g, N, K, A, B, C, D, k be as in Theorem 3.1 and*

$$j = \zeta\gamma - \sqrt{1 - 2\delta + p^2} \ge 0.$$

Suppose that there exists a constant $\rho \in (0,1]$ satisfying (3.5) and one of the following conditions:

$$\begin{aligned} &1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2 > j^2,\\ &|1+\tau-\upsilon-(\delta-k)j|\\ &\quad > \sqrt{(1-(\delta-k)^2)(1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2-j^2)}, \\ &\left|\rho-\frac{1+\tau-\upsilon-(\delta-k)j}{1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2-j^2}\right|\\ &\quad < \frac{\sqrt{(1+\tau-\upsilon-(\delta-k)j)^2-(1-(\delta-k)^2)(1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2-j^2)}}{1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2-j^2}; \end{aligned} \tag{3.15}$$

$$\begin{gathered} 1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2 = j^2, \quad 1+\tau-\upsilon > (\delta-k)j,\\ \rho > \frac{1-(\delta-k)^2}{2(1+\tau-\upsilon-(\delta-k)j)}; \end{gathered} \tag{3.16}$$

$$\begin{aligned} &1+2(\tau-\upsilon)+(\sigma\alpha+\eta\beta)^2 < j^2,\\ &\left|\rho-\frac{(\delta-k)j-1-\tau+\upsilon}{j^2-1-2(\tau-\upsilon)-(\sigma\alpha+\eta\beta)^2}\right|\\ &\quad < \frac{\sqrt{((\delta-k)j-1-\tau+\upsilon)^2+(1-(\delta-k)^2)(j^2-1-2(\tau-\upsilon)-(\sigma\alpha+\eta\beta)^2)}}{j^2-1-2(\tau-\upsilon)-(\sigma\alpha+\eta\beta)^2}. \end{aligned} \tag{3.17}$$

Then for every $f \in H$, the completely generalized multivalued quasivariational inequality (2.2) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $gu \in K(w)$ and $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$, where $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ are defined in Algorithm 3.2.

PROOF. Since A and B are H-Lipschitz continuous, relaxed Lipschitz and generalized pseudocontractive with respect to the first and second arguments of N, respectively, and N is Lipschitz continuous with respect to the first and second arguments, respectively, by (3.2) we infer that

$$\|(1-\rho)(u_n-u_{n-1})+\rho[N(x_n,y_n,z_n)-N(x_{n-1},y_{n-1},z_n)]\|^2 \tag{3.18}$$

$$
\begin{aligned}
&= (1-\rho)^2\|u_n - u_{n-1}\|^2 + 2(1-\rho)\rho\langle N(x_n, y_n, z_n) \\
&\quad - N(x_{n-1}, y_n, z_n), u_n - u_{n-1}\rangle + 2(1-\rho)\rho\langle N(x_{n-1}, y_n, z_n) \\
&\quad - N(x_{n-1}, y_{n-1}, z_n), u_n - u_{n-1}\rangle + \rho^2\|N(x_n, y_n, z_n) \\
&\quad - N(x_{n-1}, y_n, z_n) + N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\|^2 \\
&\leq ((1-\rho)^2 - 2(1-\rho)\rho(\tau-\upsilon) + \rho^2(\sigma\alpha+\eta\beta)^2(1+n^{-1})^2)\|u_n - u_{n-1}\|^2.
\end{aligned}
$$

In view of (3.2)–(3.4), (3.9), (3.11), (3.18), Lemma 3.1 and the strong monotonicity of g, we conclude that

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \leq \delta^{-1}\|gu_{n+1} - gu_n\| \\
&\quad \leq \delta^{-1}(\|P_{K(w_n)}((1-\rho)gu_n + \rho(N(x_n, y_n, z_n) - f)) \\
&\qquad - P_{K(w_n)}((1-\rho)gu_{n-1} + \rho(N(x_{n-1}, y_{n-1}, z_{n-1}) - f))\| \\
&\qquad + \|P_{K(w_n)}((1-\rho)gu_{n-1} + \rho(N(x_{n-1}, y_{n-1}, z_{n-1}) - f)) \\
&\qquad - P_{K(w_{n-1})}((1-\rho)gu_{n-1} + \rho(N(x_{n-1}, y_{n-1}, z_{n-1}) - f))\|) \\
&\quad \leq \delta^{-1}(\|(1-\rho)(gu_n - gu_{n-1}) + \rho(N(x_n, y_n, z_n) \\
&\qquad - N(x_{n-1}, y_{n-1}, z_{n-1}))\| + \mu\|w_n - w_{n-1}\|) \\
&\quad \leq \delta^{-1}((1-\rho)\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\
&\qquad + \|(1-\rho)(u_n - u_{n-1}) + \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n))\| \\
&\qquad + \rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\| \\
&\qquad + \mu(1+n^{-1})H(Du_n, Du_{n-1})) = \theta_n\|u_n - u_{n-1}\|,
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
\theta_n &= \delta^{-1}((1-\rho)\sqrt{1-2\delta+p^2} \\
&\quad + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho(\tau-\upsilon) + \rho^2(\sigma\alpha+\eta\beta)^2(1+n^{-1})^2} \\
&\quad + \rho\zeta\gamma(1+n^{-1}) + \mu\xi(1+n^{-1})) \to \theta \\
&= \delta^{-1}(k + \rho j + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho(\tau-\upsilon) + \rho^2(\sigma\alpha+\eta\beta)^2})
\end{aligned}
$$

as $\to \infty$. The remaining portion of the proof can be derived as in Theorem 3.1. This completes the proof. $\square$

Theorem 3.3. *Let g, N, K, A, B, C, D, k be as in Theorem 3.1. Suppose that C is relaxed monotone with constant φ with respect to the third argument of N and $j = \sqrt{1+2\varphi+\zeta^2\gamma^2} + \sqrt{1+2\upsilon+\eta^2\beta^2} - \sqrt{1-2\delta+p^2} \geq 0$. If there exists a constant $\rho \in (0,1]$ satisfying (3.5) and one of the following conditions:*

$$\begin{aligned}
&1+2\tau+\sigma^2\alpha^2 > j^2,\\
&|1+\tau-(\delta-k)j| > \sqrt{(1-(\delta-k)^2)(1+2\tau+\sigma^2\alpha^2-j^2)},\\
&\left|\rho - \frac{1+\tau-(\delta-k)j}{1+2\tau+\sigma^2\alpha^2-j^2}\right|\\
&\quad < \frac{\sqrt{(1+\tau-(\delta-k)j)^2-(1-(\delta-k)^2)(1+2\tau+\sigma^2\alpha^2-j^2)}}{1+2\tau+\sigma^2\alpha^2-j^2};
\end{aligned}\tag{3.20}$$

$$\begin{gathered}
1+2\tau+\sigma^2\alpha^2 = j^2, \quad 1+\tau > (\delta-k)j,\\
\rho > \frac{1-(\delta-k)^2}{2(1+\tau-(\delta-k)j)};
\end{gathered}\tag{3.21}$$

$$\begin{aligned}
&1+2\tau+\sigma^2\alpha^2 < j^2,\\
&\left|\rho - \frac{(\delta-k)j-1-\tau}{j^2-1-2\tau-\sigma^2\alpha^2}\right|\\
&\quad < \frac{\sqrt{((\delta-k)j-1-\tau)^2+(1-(\delta-k)^2)(j^2-1-2\tau-\sigma^2\alpha^2)}}{j^2-1-2\tau-\sigma^2\alpha^2},
\end{aligned}\tag{3.22}$$

then for every $f \in H$, the completely generalized multivalued quasivariational inequality (2.2) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $gu \in K(w)$ and $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$, where $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ are defined in Algorithm 3.2.

Proof. Because N is Lipschitz continuous with respect to the first, second and third arguments, respectively, A, B and C are relaxed Lipschitz, generalized pseudocontractive and relaxed monotone with respect to the first, second and third arguments of N, respectively, we immediately conclude that

$$\begin{aligned}
&\|(1-\rho)(u_n-u_{n-1})+\rho[N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n)]\|^2\\
&\quad=(1-|,\rho)^2\|u_n-u_{n-1}\|^2+2(1-\rho)\rho\langle N(x_n,y_n,z_n)\\
&\qquad-N(x_{n-1},y_n,z_n),u_n-u_{n-1}\rangle+\rho^2\|N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n)\|^2\\
&\quad\le((1-\rho)^2-2(1-\rho)\rho\tau+\rho^2\sigma^2\alpha^2(1+n^{-1})^2)\|u_n-u_{n-1}\|^2, \qquad (3.23)
\end{aligned}$$

$$\begin{aligned}
&\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)+(u_n-u_{n-1})\|^2\\
&\quad=\|u_n-u_{n-1}\|^2+2\langle N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n),u_n-u_{n-1}\rangle\\
&\qquad+\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)\|^2\\
&\quad\le(1+2\upsilon+\eta^2\beta^2(1+n^{-1})^2)\|u_n-u_{n-1}\|^2 \qquad (3.24)
\end{aligned}$$

and

$$\begin{aligned}
&\|N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1})-(u_n-u_{n-1})\|^2\\
&\quad=\|u_n-u_{n-1}\|^2-2\langle N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1}),u_n-u_{n-1}\rangle\\
&\qquad+\|N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1})\|^2\\
&\quad\le(1+2\varphi+\zeta^2\gamma^2(1+n^{-1})^2)\|u_n-u_{n-1}\|^2. \qquad (3.25)
\end{aligned}$$

As in the proofs of Theorems 3.1 and 3.2, by (3.23)–(3.25) we have

$$\begin{aligned}
\|u_{n+1}-u_n\|&\le\delta^{-1}\|gu_{n+1}-gu_n\|\\
&\le\delta^{-1}(\|(1-\rho)(gu_n-gu_{n-1})+\rho(N(x_n,y_n,z_n)\\
&\quad-N(x_{n-1},y_{n-1},z_{n-1}))\|+\mu\|w_n-w_{n-1}\|)\\
&\le\delta^{-1}((1-\rho)\|gu_n-gu_{n-1}-(u_n-u_{n-1})\|\\
&\quad+\|(1-\rho)(u_n-u_{n-1})+\rho(N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n))\|\\
&\quad+\rho\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)+u_n-u_{n-1}\|
\end{aligned}$$

$$+ \rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\|$$
$$+ \mu(1 + n^{-1})H(Du_n, Du_{n-1})) = \theta_n\|u_n - u_{n-1}\|,$$

where

$$\begin{aligned}\theta_n &= \delta^{-1}((1-\rho)\sqrt{1-2\delta+p^2} \\ &\quad + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho\tau + \rho^2\sigma^2\alpha^2(1+n^{-1})^2} \\ &\quad + \rho\sqrt{1+2\upsilon+\eta^2\beta^2(1+n^{-1})^2} + \rho\sqrt{1+2\varphi+\zeta^2\gamma^2(1+n^{-1})^2}) \to \theta \\ &= \delta^{-1}(k + \rho j + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho\tau + \rho^2\sigma^2\alpha^2})\end{aligned}$$

as $n \to \infty$. The rest of the proofis exactly the same as that of Theorem 3.1. This completes the proof. □

Remark 3.3. We claim that $\delta - k \leq 1$ under the assumptions of one of Theorem 3.1, Theorem 3.2 and Theorem 3.3. Otherwise $\delta - k > 1$. Notice that

$$\begin{aligned}\delta - k > 1 &\Leftrightarrow \delta - 1 > k \\ &\Leftrightarrow \delta^2 - 2\delta + 1 > k^2 = 1 - 2\delta + p^2 + 2\mu\xi\sqrt{1-2\delta+p^2} + \mu^2\xi^2,\end{aligned}$$

which imply that $\delta > p$. Since g is Lipschitz continuous with constant p and strongly monotone with constant δ, it follows that

$$p\|x-y\|^2 \geq \|gx - gy\|\|x-y\| \geq \langle gx - gy, x - y\rangle \geq \delta\|x-y\|^2$$

for all $x, y \in H$, which means that $p > \delta$. This is a contradiction.

Remark 3.4. Theorems 3.1–3.3 extend, improve and unify Theorems 3.1–3.3 in [1], Theorem 3.1 in [7], [10]–[12] and Theorem 3.6 in [16].

Remark 3.5. In many important applications, the set $K(u)$ is of the form

$$K(u) = m(u) + K,$$

where $m : H \to H$ is a mapping and $K \in CC(H)$. Using (3.5), Noor [6] established the following

$$P_{K(u)}(z) = m(u) + P_K(z - m(u)) \quad \text{for all } z, u \in H. \tag{3.26}$$

Now we point out that (3.4) holds with $\mu = 2b$ if m is Lipschitz continuous with constant b and (3.5) is fulfilled. In fact, by (3.26), the Lipschitz continuity of m and the nonexpansitivity of P_K, we know that

$$\begin{aligned}\|K_{K(x)}(z) - K_{K(y)}(z)\| &\leq \|m(x) - m(y)\| \\ &\quad + \|P_K(z - m(x)) - P_K(z - m(y))\| \\ &\leq 2\|m(x) - m(y)\| \leq 2b\|x - y\|\end{aligned}$$

for all $x, y, x \in H$.

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