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Stability of the Fischer–Muszély functional equation

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Abstract. We show that the Fischer–Muszély functional equation

$$\|f(xy)\| = \|f(x) + f(y)\|$$

is Hyers–Ulam stable in the class of surjective functions.

Let $(G, \cdot)$ be a semigroup and let X be a Banach space. We are going to deal with the stability of the Fischer–Muszély functional equation:

$$\|f(xy)\| = \|f(x) + f(y)\| \quad \text{for } x, y \in G. \tag{1}$$

The investigation of this functional equation was motivated by the paper of M. Hosszú [5], who studied the so called alternative equation: $e(x)f(x+y) = f(x) + f(y)$.

If a function f satisfies equation (1) then we say that it is *norm-additive*.

One of the reasons why this equation is important is that it characterizes strictly convex spaces (see [3]) – every solution f of the Fischer–Muszély functional equation is additive iff the target space X is strictly convex.

One can show even more in Hilbert spaces [6]: every solution of the inequality $\|f(xy)\| \geq \|f(x) + f(y)\|$ is necessarily additive. If this is also the case in strictly convex spaces is an open problem.

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This functional equation is also connected to the isometry equation. As one can easily check, every odd isometry $f : X \to X$ satisfies it. Moreover, every continuous solution $f : \mathbb{R} \to X$ such that $f(0) = 0$ is proportional to an odd isometry (see [4]).

Let $f : G \to X$ be arbitrary and let $\varepsilon > 0$. If

$$\big| \|f(xy)\| - \|f(x) + f(y)\| \big| \leq \varepsilon \quad \text{for } x, y \in G$$

then we say that f is *ε-norm-additive*. If f is ε-norm-additive with certain $\varepsilon > 0$, then we write that f is approximately norm-additive.

In the case when $X = \mathbb{R}$ and G is commutative, Fischer–Muszély equation is stable [1], even if we replace $\mathbb{R}$ with a lattice and the norm with absolute value (see [2]).

As shows the following example, it is clearly not stable in general.

Example 1. In $\mathbb{R}^2$ we take the euclidean norm $\|(x_1, x_2)\| := \sqrt{x_1^2 + x_2^2}$. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by the formula

$$f(x) := \big(x, \operatorname{sgn}(x)\sqrt{|x|}\,\big) \quad \text{for } x \in \mathbb{R}.$$

To show that f is approximately norm-additive, we first verify that it is an approximate isometry. Clearly

$$\|f(x) - f(y)\| \geq \|x - y\| \quad \text{for } x, y \in \mathbb{R}.$$

Let $x, y \in \mathbb{R}$ be of the same sign. Then

$$\begin{aligned}\|f(x) - f(y)\| &= \Big\|\big(x - y, \sqrt{|x|} - \sqrt{|y|}\,\big)\Big\| = \sqrt{|x - y|^2 + \big(\sqrt{|x|} - \sqrt{|y|}\,\big)^2} \\ &\leq \sqrt{(x - y)^2 + |x - y|} \leq |x - y| + \tfrac{1}{2}.\end{aligned}$$

If $x, y \in \mathbb{R}$ are of different sign then

$$\begin{aligned}\|f(x) - f(y)\| &= \Big\|\big(|x| + |y|, \sqrt{|x|} + \sqrt{|y|}\,\big)\Big\| \\ &= \sqrt{(|x| + |y|)^2 + \big(\sqrt{|x|} + \sqrt{|y|}\,\big)^2} \\ &\leq \sqrt{(|x| + |y|)^2 + 2(|x| + |y|)} \leq |x - y| + 1.\end{aligned}$$

This yields that f is an odd function which is an approximate isometry, and therefore it is approximately norm-additive. However, since every norm-additive function is additive (see [4]), f can not be approximated by a norm-additive function.

We need the notion of nearsurjective functions. Let S be a set, let X be a Banach space, and let $\delta \geq 0$ be arbitrary. We say that a function $f : S \to X$ is *δ-nearsurjective* if

$$\forall\, x \in X\, \exists\, s \in S : \|x - f(s)\| \leq \delta.$$

Morever, a function is nearsurjective if it is δ-nearsurjective for some $\delta > 0$.

Clearly a given function is 0-nearsurjective function if and only if it is a surjection.

Let X, Y be Banach spaces. A function $f : X \to Y$ is said to be an *ε-isometry* if

$$\big|\, \|f(x) - f(y)\| - \|x - y\| \,\big| \leq \varepsilon$$

for $x, y \in X$.

We will need the following result on the stability of approximate isometries (Theorem 3.5 from [9]) (for a weaker result of this type see [8]).

Theorem ŠV. *Let X, Y be Banach spaces, and let $f : X \to Y$ be an ε-isometry which is nearsurjective and such that $f(0) = 0$.*

Then there exists a unique linear isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq 2\varepsilon \quad \text{for } x \in X. \tag{2}$$

As we know (see Example 1) isometry equation is not stable under no assumptions – to obtain stability we need surjectivity. The same situation occurs when one considers the stability of the Fischer–Muszély functional equation.

Theorem 1. *Let $(G, \cdot)$ be a group with unit e, let X be a Banach space and let $f : G \to X$ be a surjection such that*

$$\big|\, \|f(ab)\| - \|f(a) + f(b)\| \,\big| \leq \varepsilon \quad \text{for } a, b \in G. \tag{3}$$

Then

$$\|f(ab) - f(a) - f(b)\| \leq 13\varepsilon \quad \text{for } a, b \in G,$$

Before proceeding to the proof we would like to mention that if G is abelian (or in general amenable) by the Hyers Theorem and Theorem 1 we obtain stability of the Fischer–Muszély functional equation in the class of surjective functions.

Proof. For a metric space M and $a, b \in M$ we write $a \overset{r}{\approx} b$ to denote $d(a,b) \leq r$. Inserting $a = b = e$ in (3) we obtain

$$f(e) \overset{1\varepsilon}{\approx} 0. \tag{4}$$

If we put in (3) $b = a^{-1}$ and apply (4) we get

$$f(a) \overset{2\varepsilon}{\approx} -f(a^{-1}) \quad \text{for } a \in G. \tag{5}$$

Making use of (3) and (5) we get

$$\|f(ab^{-1})\| \overset{1\varepsilon}{\approx} \|f(a) + f(b^{-1})\| \overset{2\varepsilon}{\approx} \|f(a) - f(b)\| \quad \text{for } a, b \in G,$$

which means that

$$\|f(ab^{-1})\| \overset{3\varepsilon}{\approx} \|f(a) - f(b)\| \quad \text{for } a, b \in G. \tag{6}$$

For arbitrary $a \in G$ we define the set-valued function $i_a : X \rightsquigarrow X$ by the formula

$$i_a(x) := \{f(g_x a^{-1}) + f(a) : g_x \in f^{-1}(x)\}.$$

Since f is surjective i_a is a set-valued function with nonempty images.

Let $x, y \in X$ and let $g_x \in f^{-1}(x), g_y \in f^{-1}(y)$. Then applying (6) repeatedly we get

$$\begin{aligned}
&\|f(g_x a^{-1}) + f(a) - f(g_y a^{-1}) - f(a)\| = \|f(g_x a^{-1}) - f(g_y a^{-1})\| \\
&\overset{3\varepsilon}{\approx} \|f(g_x a^{-1}(g_y a^{-1})^{-1})\| = \|f(g_x a^{-1} a g_y^{-1})\| \\
&= \|f(g_x g_y^{-1})\| \overset{3\varepsilon}{\approx} \|f(g_x) - f(g_y)\| = \|x - y\|.
\end{aligned}$$

This yields that

$$\|z_x - z_y\| \overset{6\varepsilon}{\approx} \|x - y\| \quad \text{for } x, y \in X,\ z_x \in i_a(x),\ z_y \in i_a(y). \tag{7}$$

This means that i_a is a 6ε-approximate isometry (or in other way its every selection is a 6ε-approximate isometry). Moreover, putting $x = y$ in (7) we obtain that

$$\|z_1 - z_2\| \leq 6\varepsilon \quad \text{for } z_1, z_2 \in i_a(x). \tag{8}$$

Now we show that i_a is surjective. Let $y \in X$. Let $g \in f^{-1}(y-f(a))$ be arbitrarily chosen and let $x = f(ga)$. Clearly $ga \in f^{-1}(x)$, and therefore by the definition of i_a we obtain that

$$y = (y - f(a)) + f(a) = f(g) + f(a) = f((ga)a^{-1}) + f(a) \in i_a(x).$$

Let s_a be an arbitrary selection of i_a. Then $\|s_a(0)\| \in \|f(f^{-1}(0)a^{-1})+ f(a)\| \overset{\varepsilon}{\approx} 0$, and therefore

$$\|s_a(0)\| \leq \varepsilon. \tag{9}$$

By (7) we obtain that s_a is a 6ε-approximate isometry. By the surjectivity of i_a and (8) we obtain that s_a is 6ε-nearsurjective.

Applying Theorem ŠV we obtain that there exists a linear isometry $I_{s_a} : X \to X$ such that

$$\|s_a(x) - s_a(0) - I_{s_a}(x)\| \leq 2 \cdot 6\varepsilon \quad \text{for } x \in X.$$

By (9) this yields that

$$\|s_a(x) - I_{s_a}(x)\| \leq 13\varepsilon \quad \text{for } x \in X. \tag{10}$$

Let t_a be another selection of i_a. Then by the above inequality applied for s_a and t_a and (8) we obtain that

$$\|I_{s_a}(x) - I_{t_a}(x)\| \leq 32\varepsilon \quad \text{for } x \in X,$$

which by the linearity of I_{s_a} and I_{t_a} implies that $I_{s_a} = I_{t_a}$ and therefore $I_a := I_{s_a}$ does not depend on the selection s_a of i_a. Now by the definition of i_a the equation (10) means that

$$\|f(ba^{-1}) + f(a) - I_a(f(b))\| \leq 13\varepsilon \quad \text{for } a, b \in G. \tag{11}$$

Exchanging in the above inequality the roles of a, b, adding up and applying (5) we get

$$\|f(a) + f(b) - I_a(f(b)) - I_b(f(a))\| \leq \|f(ba^{-1}) + f(a) - I_a(f(b))\|$$

$$+\|f(ab^{-1}) + f(b) - I_b(f(a))\| + \|f(ba^{-1}) + f(ab^{-1})\| \leq 28\varepsilon \quad \text{for } a, b \in G.$$

Let us fix a in the above inequality. Then

$$\|f(b) - I_a(f(b))\| \leq 28\varepsilon + 2\|f(a)\| \quad \text{for } b \in G.$$

Since f is surjective this means that

$$\|x - I_a(x)\| \le 28\varepsilon + 2\|f(a)\| \quad \text{for } x \in X,$$

and therefore we obtain that $I_a(x) = x$ for all $x \in X$. Now (11) becomes

$$\|f(ba^{-1}) + f(a) - f(b)\| \le 13\varepsilon \quad \text{for } a, b \in G. \qquad \square$$

As a direct corollary we obtain the following result concerning surjective solutions to the Fischer–Muszély functional equation.

Corollary 1. *Let $(G, \cdot)$ be a group, X be Banach space, and $f : G \to X$ be surjection such that*

$$\|f(x) + f(y)\| = \|f(xy)\| \quad \text{for } x, y \in G.$$

Then f is additive.

At the end of the paper we would like to pose the problem if the assumption that G is a group (and not a semigroup) is essential in Theorem 1.

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