Publ. Math. Debrecen
62/1-2 (2003), 213–225

Certain application of an integral formula to CR submanifold of complex projective space

By MIRJANA DJORIĆ (Belgrade) and MASAFUMI OKUMURA (Urawa)

Abstract. Let M be an n-dimensional CR submanifold of CR dimension $\frac{n-1}{2}$ of complex projective space. In this case M is necessarily odd-dimensional and there exists a unit vector field ξ_1 normal to M such that $JT(M) \subset T(M) \oplus \xi_1$. Under the assumption that ξ_1 is parallel with respect to the normal connection, we bring into use an integral formula which leads to an inequality between the Ricci tensor, the scalar curvature and the mean curvature of M. Using this inequality, we provide a sufficient condition for the submanifold M to be a tube over a totally geodesic complex subspace of $P^{\frac{n+k}{2}}(C)$.

0. Introduction

The study of hypersurfaces of complex projective space has been a fertile field for differential geometricians for many years now. Much of this work has involved finding sufficient conditions for a hypersurface to be one of the "standard examples".

However, contrary to the case of hypersurfaces of a Euclidean space where totally geodesic hypersurfaces and totally umbilical hypersurfaces characterize hyperplanes and hyperspheres, respectively, and to the case of a sphere as an ambient space where they characterize great and small spheres, respectively, in complex projective space there exist neither totally geodesic real hypersurfaces nor totally umbilical real hypersurfaces.

Mathematics Subject Classification: 53C15, 53C25, 53C40.
Key words and phrases: CR submanifold, complex projective space, Ricci tensor, scalar curvature, mean curvature.

H. B. Lawson introduced the notion of "generalized equators" $M_{p,q}^C$ in a complex projective space, which naturally generalize the equatorial hypersurfaces of spheres [7]. Following the idea of constructing a circle bundle over a real hypersurface, which is compatible with the Hopf fibration, he introduced the notion of "generalized equators" $M_{p,q}^C$ in a complex projective space. In the same paper H. B. Lawson proved the theorem asserting that if M is an n-dimensional compact, minimal real hypersurface of a complex projective space $P^{\frac{n+1}{2}}(C)$ over which the square of the length of the second fundamental form is less or equal to $n-1$ (or, equivalently, the scalar curvature is greater or equal to $(n+2)(n-1)$), then M is congruent up to isometry to $M_{p,q}^C$ for some p, q satisfying $2p+2q=n-1$.

Recently, Y. W. Choe and the second author of this paper gave one generalization of H. B. Lawson's theorem for higher codimensions [5]. Namely, they proved that if M is an n-dimensional compact, minimal CR-submanifold of CR-dimension $\frac{n-1}{2}$ of $P^{\frac{n+k}{2}}(C)$, such that the normal vector field is parallel with respect to the normal connection and the scalar curvature is greater or equal to $(n+2)(n-1)$, then M is $M_{p,q}^C$ for some p, q satisfying $2p+2q=n-1$.

The main purpose of this paper is to give one more generalization of H. B. Lawson's result, also for higher codimensions, but avoiding the condition of minimality. Namely, using the integral formula established by K. Yano [13], we give certain characterization of the "generalized equators" $M_{p,q}^C\big(1/\sqrt{2},1/\sqrt{2}\big)$, for some p, q satisfying $2p+2q=n-1$.

In Section 1 we deduce for later use a series of fundamental formulas for n-dimensional CR submanifolds of CR dimension $\frac{n-1}{2}$ and in Section 2 we study the model space $M_{p,q}^C\big(1/\sqrt{2},1/\sqrt{2}\big)$ in detail. In Section 3 we give an application of an integral formula which we use to derive our main results. Finally, in Section 4 we characterize $M_{p,q}^C\big(1/\sqrt{2},1/\sqrt{2}\big)$ by certain inequality between the Ricci tensor, the scalar curvature and the mean curvature of an n-dimensional compact CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+k}{2}}(C)$ and we give a sufficient condition for the submanifold M to be a tube over a totally geodesic complex subspace of $P^{\frac{n+k}{2}}(C)$.

1. n-dimensional CR submanifolds of CR dimension $\frac{n-1}{2}$

Let $(\bar{M}, \bar{g}, J)$ be an $(n+k)$-dimensional Hermitian manifold and let M be a connected n-dimensional submanifold of $\bar{M}$ with induced metric g. For $x \in M$ we denote by T_xM and $T_x^{\perp}M$ the tangent space and the normal space of M at x, respectively and by $\imath$ the immersion M in $\bar{M}$ (we denote also by $\imath$ the differential of the immersion). If the maximal holomorphic subspace $J\imath T_x(M) \cap \imath T_x(M)$ of $\imath T_x(M)$ has constant dimension for $x \in M$, then M is called a CR submanifold and the constant is called the CR dimension of M ([8]). Moreover, let us recall that a submanifold M of $\bar{M}$ is also said to be a CR submanifold if it is endowed with a pair of mutually orthogonal and complementary distributions $(\Delta, \Delta^{\perp})$ such that for any $x \in M$ we have $J\Delta_x = \Delta_x$, $J\Delta_x^{\perp} \subset T_x^{\perp}M$ ([1]).

Now, let us assume that the maximal holomorphic subspace of T_xM is $(n-1)$-dimensional for any $x \in M$, that is, $\dim(J\imath T_x(M) \cap \imath T_x(M)) = n-1$. In this case both definitions of CR submanifolds coincide. Moreover, this implies that M is necessarily odd-dimensional manifold and our assumption is equivalent to the condition that M is a CR submanifold with $\dim \Delta^{\perp} = 1$.

Furthermore, our hypothesis implies that there exists a unit vector field ξ_1 normal to M such that $J\imath T(M) \subset \imath T(M) \oplus \operatorname{span}\{\xi_1\}$. Hence, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha, \alpha = 1, 2, \ldots, k\}$ of vectors normal to M, we have the following decomposition into tangential and normal components:

$$J\imath X = \imath FX + u^1(X)\xi_1, \tag{1.1}$$

$$J\xi_\alpha = -\imath U_\alpha + P\xi_\alpha, \quad (\alpha = 1, 2, \ldots, k), \tag{1.2}$$

where U_α, $\alpha = 1, 2, \ldots, k$ are tangent vector fields of M and u^1 is 1-form on M. Here F and P are skew-symmetric endomorphisms acting on $T(M)$ and $T^{\perp}(M)$, respectively. Moreover, using (1.1) and (1.2), the Hermitian property of J implies

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \ldots, k, \tag{1.3}$$

$$F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \tag{1.4}$$

$$u^1(FX) = 0, \quad P\xi_1 = 0, \quad FU_1 = 0. \tag{1.5}$$

Therefore, (1.2) may be written in the form

$$J\xi_1 = -\imath U_1, \quad J\xi_\alpha = P\xi_\alpha, \qquad \alpha = 2, \ldots, k \tag{1.6}$$

and further, we may write

$$P\xi_\alpha = \sum_{\beta=2}^{k} P_{\alpha\beta}\xi_\beta, \qquad \alpha = 2, \ldots, k \tag{1.7}$$

where $(P_{\alpha\beta})$ is a skew-symmetric matrix which satisfies $\sum_\beta P_{\alpha\beta}P_{\beta\gamma} = -\delta_{\alpha\gamma}$.

These results imply that (F, U_1, u^1, g) defines an almost contact metric structure on M (see [12]). We refer to [2], [15] for more details about contact geometry.

Now, let $\overline{\nabla}$ and ∇ denote the Levi Civita connection on $(\bar{M}, \bar{g})$ and (M, g), respectively, and let D denote the normal connection induced from $\overline{\nabla}$ in the normal bundle $T^\perp(M)$ of M. Then they are related by the following Gauss and Weingarten equations

$$\overline{\nabla}_{\imath X}\imath Y = \imath\nabla_X Y + h(X, Y), \tag{1.8}$$

$$\overline{\nabla}_{\imath X}\xi_\alpha = -\imath A_\alpha X + D_X\xi_\alpha, \qquad \alpha = 1, \ldots, k \tag{1.9}$$

for any tangent vectors X, Y to M. Here h denotes the second fundamental form and A_α is a symmetric linear transformation of $T(M)$ which is called the shape operator with respect to the normal ξ_α. They are related by $h(X, Y) = \sum_{\alpha=1}^{k} g(A_\alpha X, Y)\xi_\alpha$. The mean curvature vector field μ of M is defined by

$$\mu = \frac{1}{n}\sum_{\alpha=1}^{k}(\mathrm{tr}A_\alpha)\xi_\alpha \tag{1.10}$$

and it is well known that μ is independent of the choice of ξ_α's. (We refer to [4] for more details about submanifold theory.) Furthermore, if we put

$$D_X\xi_\alpha = \sum_{\beta=1}^{k} s_{\alpha\beta}(X)\xi_\beta, \tag{1.11}$$

it follows that $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D.

Finally, let the ambient manifold $\bar{M}$ be the complex projective space $P^{\frac{n+k}{2}}(C)$ with Fubini–Study metric of constant holomorphic sectional curvature 4. Then, since the curvature tensor $\bar{R}_{\bar{X}\bar{Y}}\bar{Z}$ of $P^{\frac{n+k}{2}}(C)$ satisfies

$$\begin{aligned}\bar{R}_{\bar{X}\bar{Y}}\bar{Z} = \bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(J\bar{Y},\bar{Z})J\bar{X} - \bar{g}(J\bar{X},\bar{Z})J\bar{Y}\\ - 2\bar{g}(J\bar{X},\bar{Y})J\bar{Z},\end{aligned}$$

the Gauss equation becomes ([6])

$$\begin{aligned}R_{XY}Z = g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX\\ - g(FX,Z)FY - 2g(FX,Y)FZ\\ + \sum_{\alpha=1}^{k}\{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\},\end{aligned} \tag{1.12}$$

from which

$$\begin{aligned}\mathrm{Ric}(X,Y) = (n+2)g(X,Y) - 3u^1(X)u^1(Y)\\ + \sum_{\alpha=1}^{k}(trA_\alpha)g(A_\alpha Y,X) - \sum_{\alpha=1}^{k} g(A_\alpha^2 Y,X),\end{aligned} \tag{1.13}$$

$$\varrho = (n+3)(n-1) + \sum_{\alpha=1}^{k}(\mathrm{tr}\,A_\alpha)^2 - \sum_{\alpha=1}^{k}\mathrm{tr}\,A_\alpha^2, \tag{1.14}$$

where Ric and ϱ denote, respectively, the Ricci tensor and the scalar curvature.

Further, by differentiating (1.1) and (1.6) covariantly and making use of the fact that the Riemannian connection $\overline{\nabla}$ of $\bar{M}$ leaves the almost complex structure J invariant, and by comparing the tangential and normal parts, we obtain

$$\nabla_X U_1 = FA_1X, \tag{1.15}$$

$$g(A_\alpha U_1, X) = -\sum_{\beta=2}^{k} s_{1\beta}(X)P_{\beta\alpha}, \qquad \alpha = 2,\ldots,k \tag{1.16}$$

for any $X, Y \in T(M)$.

In what follows, besides the hypothesis that M is an n-dimensional compact submanifold of $P^{\frac{n+k}{2}}(C)$, we suppose that the normal vector field ξ_1 is parallel with respect to the normal connection, that is $D_X\xi_1 = 0$. Consequently, using (1.11), we obtain

$$s_{\alpha 1} = 0, \quad \alpha = 2, \dots, k. \tag{1.17}$$

2. Model space $M_{p,q}^C(1/\sqrt{2}, 1/\sqrt{2})$

By definition, an n-dimensional real hypersurface of a complex manifold is a typical example of CR submanifold of CR dimension $\frac{n-1}{2}$. In this section we describe the model space $M_{p,q}^C$ in a complex projective space $P^{\frac{n+k}{2}}(C)$, which is a real hypersurface of $P^{\frac{n+1}{2}}(C)$, and then consider the relation between the Ricci tensor, the scalar curvature and the mean curvature of the model space.

Let $M_{p,q}$ be the family of generalized Clifford tori, i.e. the standard product $S^{2p+1} \times S^{2q+1}$, $2p + 2q = n - 1$. Therefore, $M_{p,q}$ is defined by the following relations

$$\sum_{j=0}^{p} |z_j|^2 = \cos^2 r, \quad \sum_{j=p+1}^{\frac{n+1}{2}} |z_j|^2 = \sin^2 r, \qquad 0 < r < \frac{\pi}{2}$$

and it is the hypersurface of the odd-dimensional sphere S^{n+2}. By choosing the spheres so that they lie in complex subspaces, we get fibrations $S^1 \to M_{p,q} \to M_{p,q}^C$ compatible with the Hopf fibration $\pi : S^{n+2} \to P^{\frac{n+1}{2}}(C)$ which submerses $M_{p,q}$ onto a remarkable class of real hypersurfaces of $P^{\frac{n+1}{2}}(C)$, denoted by $M_{p,q}^C$. Let us remember that in [3] Cecil and Ryan proved that $M_{p,q}^C$ is a tube of radius r over a totally geodesic $P^p(C)$ with three constant principal curvatures: $\lambda_1 = \cot r$, $\lambda_2 = \cot(r - \frac{\pi}{2})$ and $\mu = 2\cot 2r$ with respective multiplicities $2q$, $2p$, 1.

Further, we consider special generalized Clifford tori in

$$S^{n+2}(1) = \left\{ \left(z_0, \dots, z_{\frac{n+1}{2}}\right) \in C^{\frac{n+1}{2}} : \sum_{i=0}^{\frac{n+1}{2}} |z_i|^2 = 1 \right\}$$

defined by

$$S^{2p+1}\big(1/\sqrt{2}\,\big)\times S^{2q+1}\big(1/\sqrt{2}\,\big)$$

$$=\Big\{\big(z_0,\ldots,z_{\frac{n+1}{2}}\big)\in S^{n+2}(1):\sum_{i=0}^{p}|z_i|^2=\frac{1}{2},\ \sum_{i=p+1}^{\frac{n+1}{2}}|z_i|^2=\frac{1}{2}\Big\},$$

where $2p+2q=n-1$. Then, since $S^{2p+1}\big(1/\sqrt{2}\,\big)\times S^{2q+1}\big(1/\sqrt{2}\,\big)$ is a hypersurface of $S^{n+2}(1)$, its shape operator $\bar{A}$ has the form

$$\bar{A}=\begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

for suitable orthonormal basis. The multiplicities of 1 and -1 are $2p+1$ and $2q+1$ respectively (see for example [11]). By choosing the spheres so that they lie in complex subspaces, we get fibrations

$$S^1\to S^{2p+1}\big(1/\sqrt{2}\,\big)\times S^{2q+1}\big(1/\sqrt{2}\,\big)\to M^C_{p,q}\big(1/\sqrt{2},1/\sqrt{2}\,\big)$$

compatible with the Hopf fibration, where $2p+2q=n-1$. In this special case we easily see that the geodesic distance from $P^p(C)$ to $M^C_{p,q}\big(1/\sqrt{2}$, $1/\sqrt{2}\,\big)$ is $\frac{\pi}{4}$ and that the principal curvatures of $M^C_{p,q}\big(1/\sqrt{2},1/\sqrt{2}\,\big)$ are 1, -1 and 0 with respective multiplicities $n-1-2p$, $2p$ and 1.

Further, let ξ be a unit normal vector field of $M^C_{p,q}\big(1/\sqrt{2},1/\sqrt{2}\,\big)$ and let J be the natural almost complex structure of $P^{\frac{n+1}{2}}(C)$. Then $U=-J\xi$ is a principal vector field corresponding to the principal value 0, that is $AU=0$, where A is the shape operator of $M^C_{p,q}\big(1/\sqrt{2},1/\sqrt{2}\,\big)$ in $P^{\frac{n+1}{2}}(C)$. Applying (1.10), (1.13) and (1.14) to a real hypersurface case, we obtain

$$\operatorname{tr}A=4p-n-1,\quad \operatorname{tr}A^2=n-1, \tag{2.1}$$

$$\operatorname{Ric}(U,U)=n-1, \tag{2.2}$$

$$\rho=(n+2)(n-1)+(4p-n-1)^2. \tag{2.3}$$

Hence, for $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\,\big)$ we have

$$\mathrm{Ric}(U,U) + \rho - n^2|\mu|^2 = (n-1)(n+3), \tag{2.4}$$

where $|\mu|$ is the length of the mean curvature vector field μ.

Remark. Let us remember that in [7] Lawson supposed that $S^{2p+1} \times S^{2q+1}$ is immersed minimally in S^{n+2}, which is not the case in this paper. Moreover, let us remark that for $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\,\big)$ the Gauss–Kronecker curvature, i.e. $\operatorname{div} A$, is zero and that, in a class of $M^C_{p,q}$, only $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\,\big)$ has the vanishing Gauss–Kronecker curvature.

3. An integral formula and its application

To proceed our discussions, in this section we use the following integral formula established by Yano [13], [14]:

$$\begin{aligned}&\int_M \{\operatorname{div}(\nabla_X X) - \operatorname{div}(\operatorname{div} X)X\} *1\\ &= \int_M \Big\{\mathrm{Ric}(X,X) + \frac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (\operatorname{div} X)^2\Big\} *1 = 0,\end{aligned} \tag{3.1}$$

where X is an arbitrary tangent vector field on M, $*1$ is the volume element of M, $|Y|$ denotes the length with respect to the Riemannian metric of a vector field Y on M and $L(X)$ is the operator of Lie derivative with respect to X.

Now we prove the following lemma.

Lemma 3.1. *Let M be an n-dimensional compact CR submanifold of CR dimension $\frac{n-1}{2}$ in $P^{\frac{n+k}{2}}(C)$. If the normal vector field ξ_1 of M is parallel with respect to the normal connection and the inequality*

$$\mathrm{Ric}(U_1,U_1) + \varrho - n^2|\mu|^2 \geq (n+3)(n-1)$$

is satisfied on M, then F and A_1 commute, $A_\alpha = 0$, $(\alpha = 2,\dots,k)$ and $A_1U_1 = 0$.

Proof. First, since ξ_1 is parallel with respect to the normal connection D, using (1.16) and (1.17), we get

$$A_\alpha U_1 = 0, \quad \alpha = 2,\dots,k. \tag{3.2}$$

Now, applying (3.1) to the vector field U_1, we obtain

$$\int_M \Big\{ \operatorname{Ric}(U_1, U_1) + \frac{1}{2}|L(U_1)g|^2 - |\nabla U_1|^2 - (\operatorname{div} U_1)^2 \Big\} *1 = 0. \tag{3.3}$$

Further, from (1.15) it follows

$$\operatorname{div} U_1 = \operatorname{tr}(FA_1) = 0 \tag{3.4}$$

and

$$\begin{aligned}(L(U_1)g)(X, Y) &= g(\nabla_X U_1, Y) + g(\nabla_Y U_1, X)\\ &= g((FA_1 - A_1F)X, Y).\end{aligned} \tag{3.5}$$

Now, using (1.4) and (1.15), we derive

$$|\nabla U_1|^2 = \operatorname{tr} A_1^2 - |A_1U_1|^2. \tag{3.6}$$

Moreover, using (1.10) and (1.14), we calculate

$$\operatorname{tr} A_1^2 = -\rho + (n+3)(n-1) + n^2|\mu|^2 - \sum_{\alpha=2}^{k} \operatorname{tr} A_\alpha^2. \tag{3.7}$$

Finally, using (3.3), (3.4), (3.6) and (3.7), we obtain

$$\begin{aligned}\int_M \Big\{ &\operatorname{Ric}(U_1, U_1) + \frac{1}{2}|L(U_1)g|^2 + \rho - (n+3)(n-1)\\ &- n^2|\mu|^2 + \sum_{\alpha=2}^{k} \operatorname{tr} A_\alpha^2 + |A_1U_1|^2 \Big\} *1 = 0.\end{aligned} \tag{3.8}$$

Applying the hypothesis of the lemma, we get

$$\operatorname{Ric}(U_1, U_1) + \rho - n^2|\mu|^2 = (n+3)(n-1), \tag{3.9}$$

$$|L(U_1)g|^2 = 0, \tag{3.10}$$

$$\sum_{\alpha=2}^{k} \operatorname{tr} A_\alpha^2 = 0, \tag{3.11}$$

$$A_1U_1 = 0. \tag{3.12}$$

Finally, we conclude from (3.5) and (3.10) that $FA_1 = A_1F$, and from (3.11) that $A_\alpha = 0$, for $\alpha = 2, \dots, k$. $\square$

4. One characterization of $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\big)$

Let $N_0(x) = \{\xi \in T^{\perp}_x(M) : A_\xi = 0\}$, where we use the notation $\bar{g}(h(X,Y),\xi) = g(A_\xi X, Y)$, for $\xi \in T^{\perp}_x(M)$, and let $H_0(x)$ be the maximal J-invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$. Then we have the following theorem, proved in [9] by the second author of this paper.

Theorem. *Let M be a real n-dimensional submanifold of the real $(n+k)$-dimensional complex projective space $P^{\frac{n+k}{2}}(C)$. If the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T^{\perp}(M)$ is invariant under the parallel translation with respect to the normal connection and l is the constant dimension of $H_1(x)$, then there exists a real $(n+l)$-dimensional totally geodesic complex projective space $P^{\frac{n+l}{2}}(C)$ such that $M \subset P^{\frac{n+l}{2}}(C)$.*

In our case, $N_0(x) = \operatorname{span}\{\xi_2(x), \dots, \xi_k(x)\}$. In fact, as a consequence of Lemma 3.1, A_α=0, for α=2, …, k and we have $\operatorname{span}\{\xi_2(x), \dots, \xi_k(x)\} \subset N_0(x)$. On the other hand, for any $\xi \in N_0(x)$, we put $\xi = \sum_{\alpha=1}^{k} b^\alpha \xi_\alpha$. Then, using Lemma 3.1, we obtain $A_\xi = \sum_{\alpha=1}^{k} b^\alpha A_\alpha = b^1 A_1 = 0$. Hence, $b^1 = 0$ and

$$\xi = \sum_{\alpha=2}^{k} b^\alpha \xi_\alpha \in \operatorname{span}\{\xi_2(x), \dots, \xi_k(x)\}.$$

Moreover, by the second equation of (1.6), $JN_0(x) = N_0(x)$ and consequently $H_0(x) = \operatorname{span}\{\xi_2(x), \dots, \xi_k(x)\}$. Therefore, the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T^{\perp}(M)$ is spanned by the normal vector field ξ_1. Since, from the assumption of this paper, $H_1(x)$ is invariant under parallel translation with respect to the normal connection, we can apply the codimension reduction theorem which is stated above and obtain the following theorem.

Theorem 4.1. *Let M be an n-dimensional compact CR-submanifold of CR-dimension $\frac{n-1}{2}$ in $P^{\frac{n+k}{2}}(C)$. If the normal vector field ξ_1 is parallel with respect to the normal connection and*

$$\operatorname{Ric}(U_1, U_1) + \varrho - n^2|\mu|^2 \geq (n+3)(n-1)$$

holds, then there exists a real $(n+1)$-dimensional totally geodesic complex projective subspace $P^{\frac{n+1}{2}}(C)$ such that $M \subset P^{\frac{n+1}{2}}(C)$.

Using Theorem 4.1, the submanifold M can be regarded as a real hypersurface of $P^{\frac{n+1}{2}}(C)$ which is a totally geodesic submanifold in $P^{\frac{n+k}{2}}(C)$. In what follows we denote $P^{\frac{n+1}{2}}(C)$ by M', the immersion of M into M' by $\imath_1$ and the totally geodesic immersion of M' into $P^{\frac{n+k}{2}}(C)$ by $\imath_2$. Then, from the Gauss equation (1.8), it follows that

$$\nabla'_{\imath_1 X}\imath_1 Y = \imath_1\nabla_X Y + h'(X,Y) = \imath_1\nabla_X Y + g(A'X,Y)\xi', \tag{4.1}$$

for any tangent vector fields X, Y to M, where ∇' is the Levi Civita connection on M', h' is the second fundamental form on M in M', A' is the corresponding shape operator and ξ' is a unit normal vector field to M in M'.

Similarly, since $\imath = \imath_2 \cdot \imath_1$ and using the Gauss equation (1.8), we obtain

$$\bar{\nabla}_{\imath_2\imath_1 X}\imath_2\cdot\imath_1 Y = \imath_2\nabla'_{\imath_1 X}\imath_1 Y + \bar{h}(\imath_1 X,\imath_1 Y), \tag{4.2}$$

where $\bar{h}$ is the second fundamental form on M' in $P^{\frac{n+k}{2}}(C)$. Therefore, it follows

$$\bar{\nabla}_{\imath_2\imath_1 X}\imath_2\cdot\imath_1 Y = \imath_2(\imath_1\nabla_X Y + g(A'X,Y)\xi'), \tag{4.3}$$

since M' is totally geodesic in $P^{\frac{n+k}{2}}(C)$. Comparison of (1.8) and (4.3) yields

$$\xi_1 = \imath_2\xi', \quad A_1 = A'. \tag{4.4}$$

Since M' is a complex submanifold of $P^{\frac{n+k}{2}}(C)$, the expression

$$J\imath_2 X' = \imath_2 J' X' \tag{4.5}$$

is valid for any $X' \in T(M')$, where J' is the induced complex structure of $M' = P^{\frac{n+1}{2}}(C)$. Consequently, using the relation (1.1), it follows that

$$\begin{aligned} J\imath X &= J\imath_2\cdot\imath_1 X = \imath_2 J'\imath_1 X = \imath_2(\imath_1 F'X + v'(X)\xi') \\ &= \imath F'X + v'(X)\imath_2\xi' = \imath F'X + v'(X)\xi_1. \end{aligned}$$

Now, comparing the last equation with relation (1.1), we conclude

$$F = F', \quad V' = U_1, \quad v' = u^1. \tag{4.6}$$

Furthermore, by Theorem 4.1, we know that M is a real hypersurface of $M' = P^{\frac{n+1}{2}}(C)$ whose fundamental tensor F' of the submersion and shape operator A' commute. Therefore, applying the theorem proved in [10] (p. 363, Theorem 4.4) by the second author of this paper, we may conclude that $M = M^C_{p,q}$, for some p, q, that is, M is a tube over a totally geodesic subspace $P^p(C)$. Moreover, $A'V' = A_1U_1 = 0$ implies that the radius of the tube is $\frac{\pi}{4}$. Thus M is locally isometric to $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\,\big)$.

Theorem 4.2. *Let M be an n-dimensional compact CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+k}{2}}(C)$. If the normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\mathrm{Ric}(U_1, U_1) + \varrho - n^2|\mu|^2 \geq (n-1)(n+3)$$

holds, then the equality $\mathrm{Ric}(U_1, U_1) + \varrho - n^2|\mu|^2 = (n-1)(n+3)$ also holds and, up to isometries of $P^{\frac{n+k}{2}}(C)$, M is $M^C_{p,q}\big(1/\sqrt{2}, 1/\sqrt{2}\,\big)$, for some p, q satisfying $2p + 2q = n - 1$.

References

[1] A. Bejancu, CR-submanifolds of a Kähler manifold I, *Proc. Amer. Math. Soc.* **69** (1978), 135–142.

[2] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics 509, *Springer-Verlag, Berlin – Heidelberg – New York*, 1976.

[3] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* **269** (1982), 481–499.

[4] B. Y. Chen, Geometry of submanifolds, Pure Appl. Math. 22, *Marcel Dekker, New York*, 1973.

[5] Y. W. Choe and M. Okumura, Scalar curvature of a certain CR-submanifold of complex projective space, *Arch. Math.* **68** (1997), 340–346.

[6] S. Kobayashi and K. Nomizu, Foundations of Differential geometry, I, II, *Interscience, New York*, 1963, 1969.

[7] H. B. Lawson, Jr., Rigidity theorems in rank-1 symmetric spaces, *J. Differential Geom.* **4** (1970), 349–357.

[8] R. Nirenberg and R. O. Wells, Jr., Approximation theorems on differentiable submanifolds of a complex manifold, *Trans. Amer. Math. Soc.* **142** (1965), 15–35.

[9] M. Okumura, Codimension reduction problem for real submanifolds of complex projective space, *Colloq. Math. Soc. János Bolyai* **56** (1989), 574–585.

[10] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.* **212** (1975), 355–364.

[11] P. Ryan, Homogeneity and some curvature condition for hypersurfaces, *Tôhoku J. Math.* **21** (1969), 363–388.

[12] Y. Tashiro, Relations between almost complex spaces and almost contact spaces, *Sûgaku* **16** (1964), 34–61 (in *Japanese*).

[13] K. Yano, On harmonic and Killing vector fields, *Ann. of Math.* **55** (1952), 38–45.

[14] K. Yano, Integral formulas in Riemannian geometry, *Dekker*, *New York*, 1970.

[15] K. Yano and M. Kon, Structures on manifolds, Series in Pure Mathematics, 3, *World Scientific Publ. Co.*, *Singapore*, 1984.

MIRJANA DJORIĆ
FACULTY OF MATHEMATICS
UNIVERSITY OF BELGRADE
STUDENTSKI TRG 16, PB 550
11000 BELGRADE
YUGOSLAVIA

E-mail: mdjoric@matf.bg.ac.yu

MASAFUMI OKUMURA
DEPARTMENT OF MATHEMATICS
SAITAMA UNIVERSITY
SHIMO-OKUBO
URAWA, 338
JAPAN

E-mail: mokumura@h8.dion.ne.jp

(Received February 11, 2002; revised July 1, 2002)