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Multiple positive solutions for boundary value problems on a measure chain

By CHEN-HUANG HONG (Chung-Li), HUEI-LIN HONG (Chung-Li)
and CHEH-CHIH YEH (Kueishan Taoyuan)

Abstract. Under some suitable conditions on a positive function $f(t,u)$, we get the boundary value problem of the form:

$$\begin{cases} \text{(E)} & u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ \text{(BC)} & \begin{cases} \alpha u(0) - \beta u^{\Delta}(0) = 0, \\ \gamma u(\sigma(1)) + \delta u^{\Delta}(\sigma(1)) = 0, \end{cases} \end{cases} \tag{BVP}$$

has at least three positive solutions by using a fixed point theorem of Legget and Williams.

1. Introduction

In this paper, we consider the existence of three positive solutions of the following boundary value problem on measure chain of the form:

$$\begin{cases} \text{(E)} & u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ \text{(BC)} & \begin{cases} \alpha u(0) - \beta u^{\Delta}(0) = 0, \\ \gamma u(\sigma(1)) + \delta u^{\Delta}(\sigma(1)) = 0, \end{cases} \end{cases} \tag{BVP}$$

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where α, β, γ, δ are nonnegative real numbers and $f \in C\big([0, \sigma(1)] \times \Re\big)$. Here $\Re = (-\infty, \infty)$.

There has recently been increasing interest in studying the existence of solutions for the following continuous-discrete boundary value problems

$$\begin{cases} (\mathrm{E}_1) & u''(t) + \lambda f(t, u((t)) = 0, \quad 0 < t < 1, \\ (\mathrm{BC}_1) & \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \end{cases} \tag{$\mathrm{BVP_A}$}$$

and

$$\begin{cases} (\mathrm{E}_2) & \Delta^2 u(i) + \lambda f(i, u(i) = 0, \quad 0 < i < T, \\ (\mathrm{BC}_2) & \begin{cases} \alpha u(0) - \beta \Delta u(0) = 0, \\ \gamma u(T+1)) + \delta \Delta u(T+1) = 0 \end{cases} \end{cases} \tag{$\mathrm{BVP_B}$}$$

in the last twenty-five years, see, for example, Agarwal and Wong [1], [2], Erbe and Wong [6], Henderson and Thompson [8], Lian, Wong and Yeh [14]. In 1990, S. Hilger [9] introduced the theory of measure chain. Recently, some authors, see for example, Chyan and Henderson [3], Erbe and Peterson [4], [5], Hong and Yeh [10], Lian, Chou, Liu and Wong [13], dealt with the existence of one or two positive solutions for the boundary value problem (BVP) on a measure chains.

In this paper, we study the existence of three solutions for the nonlinear boundary value problem (BVP) by using a fixed point theorem of Leggett and Williams [12]. We will state this result and give some definitions concerning measure chains and useful lemmas in Section 2.

2. Definitions and lemmas

In this section, we provide some background material from measure chain and the theory of cones in Banach spaces. We also state a fixed point theorem due to Leggett and Williams [12] for multiple fixed points of a cone preserving operator.

First, we give definitions of a measure chain and a cone, see [7], [9], [11].

Definition 2.1. A measure chain $\mathcal{T}$ is a closed subset of the set $\Re$ of all real numbers. We assume throughout this paper that $\mathcal{T}$ has the topology that it inherits from the standard topology on $\Re$. For $t < \sup \mathcal{T}$, define the forward jump operator $\sigma : \mathcal{T} \to \mathcal{T}$ by

$$\sigma(t) := \inf\{\tau \in \mathcal{T} : \tau > t\}$$

and for $t > \inf \mathcal{T}$ define the backward jump operator $\rho : \mathcal{T} \to \mathcal{T}$ by

$$\rho(t) := \sup\{\tau \in \mathcal{T} : \tau < t\}$$

for all $t \in \mathcal{T}$.

If $\sigma(t) > t$, $t \in \mathcal{T}$, we say t is *right-scattered.* If $\rho(t) < t$, $t \in \mathcal{T}$, we say t is *left-scattered.* If $\sigma(t) = t$, $t \in \mathcal{T}$ we say t is *right-dense.* If $\rho(t) = t$, $t \in \mathcal{T}$, we say t is *left-dense.*

Definition 2.2. If r, $s \in \mathcal{T} \cup \{+\infty, -\infty\}$, $r < s$, then an open interval (r, s) in $\mathcal{T}$ is defined by

$$(r, s) := \{t \in \mathcal{T} : \ r < t < s\}.$$

Other types of intervals are defined similarly.

Throughout this paper we make the assumption that $[a, b]$ as $[a, b] \cap \mathcal{T}$ if $a, b \in \Re$, $a \leq b$.

Definition 2.3. Assume that $x : \mathcal{T} \to \Re$ and fix $t \in \mathcal{T}$ (if $t = \sup \mathcal{T}$, we assume t is not left-scattered). Then x is called differentiable at $t \in \mathcal{T}$ if there exists a $\theta \in \Re$ such that for any given $\epsilon > 0$, there is an open neighborhood U of t such that

$$|x(\sigma(t)) - x(s) - \theta|\sigma(t) - s|\,| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, θ is called the Δ-*derivative* of x at $t \in \mathcal{T}$ and denote it by $\theta = x^{\Delta}(t)$. It can be shown that if $x : \mathcal{T} \to \Re$ is continuous at $t \in \mathcal{T}$, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} \quad \text{if } t \text{ is right-scattered}$$

and

$$x^{\Delta}(t) = \lim_{s \to t} \frac{x(t) - x(s)}{t - s} \quad \text{if } t \text{ is right-dense.}$$

Definition 2.4. Let P be a cone on a Banach space E. A map $\psi \in C(P;[0,\infty))$ is said to be a nonnegative continuous concave functional in P if

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Definition 2.5. Let P be a cone on a Banach space E, $r > 0$, $0 < a < b$ and ψ a nonnegative continuous concave functional on P. Define two cones P_r and $P(\psi, a, b)$ by

$$P_r := \{y \in P : \|y\| < r\}$$

and

$$P(\psi, a, b) := \{y \in P : a \leq \psi(y),\ \mid y\mid\ \leq b\},$$

respectively.

In order to abbreviate our discussion, throughout this paper, we suppose that the following assumptions hold:

(C_1) $G(t,s)$ is the Green's function of the differential equation

$$-u^{\Delta\Delta}(t) = 0, \quad t \in (0,1)$$

subject to the boundary condition (BC).

(C_2) $f \in C([0,\sigma(1)] \times [0,\infty);[0,\infty))$.

(C_3) $\rho := \gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$.

(C_4) $\xi := \min\left\{t \in T : t \geq \frac{\sigma(1)}{4}\right\}$ and $\omega := \max\left\{t \in T : t \leq \frac{3\sigma(1)}{4}\right\}$ both exist and satisfy

$$\frac{\sigma(1)}{4} \leq \xi < \omega \leq \frac{3\sigma(1)}{4}.$$

(C_5) $M = \min\{M_1, M_2\}$, where

$$M_1 := \min\left\{\frac{\gamma\sigma(1)+4\delta}{4(\gamma\sigma(1)+\delta)}, \frac{\alpha\sigma(1)+4\beta}{4(\alpha\sigma(1)+\beta)}\right\} \in (0,1)$$

and

$$M_2 = \min_{s\in[0,\sigma(1)]} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)}.$$

(C_6) $D_1 := \left(\int_0^{\sigma(1)} G(\sigma(s),s)\Delta s\right)^{-1}$ and $D_2 := \left(\int_0^{\sigma(1)} G(\theta,s)\Delta s\right)^{-1}$,
here $\theta \in [\xi, \omega]$.

In order to prove our main result, we need the following two useful lemmas. The first is due to Erbe and Peterson [5] and the second due to Leggett and Williams [12].

Lemma 2.5 (Erbe and Peterson [5]). *Suppose that $G(t,s)$ is defined as in (C_2). Then $G(t,s)$ can be written as*

$$G(t,s) = \begin{cases} \dfrac{1}{\rho}(\beta+\alpha t)\big[\delta+\gamma(\sigma(1)-\sigma(s))\big], & 0 \le t \le s \le \sigma(1), \\ \dfrac{1}{\rho}(\beta+\alpha\sigma(s))\big[\delta+\gamma(\sigma(1)-t)\big], & 0 \le \sigma(s) \le t \le \sigma(1). \end{cases}$$

and satisfies

$$\text{(i)} \qquad \frac{G(t,s)}{G(\sigma(s),s)} \le 1 \qquad \text{for } t \in [0,\sigma(1)] \quad \text{and } s \in [0,1];$$

$$\text{(ii)} \qquad \frac{G(t,s)}{G(\sigma(s),s)} \ge M \quad \text{for } t \in [\xi,\sigma(\omega)] \quad \text{and } s \in [0,1],$$

where M is defined as in condition (C_3).

Lemma 2.6 (Leggett and Williams [12]). *Suppose there exist $0 < a < b < d \le c$ such that*

(A_1) *$\{u \in P(\psi,a,b) : \psi(u) > b\}$ is nonempty and $\psi(\Phi u) > b$ for $u \in P(\psi,b,d)$,*

(A_2) *$\|\Phi u\| < a$ for $\|u\| \le a$,*

(A_3) *$\psi(\Phi u) > b$ for $u \in P(\psi,b,c)$ with $\|\Phi u\| > d$,*

where $\Phi : \overline{P_c} \to \overline{P_c}$ be a completely continuous and ψ be a nonnegative continuous concave functional on P such that $\psi(u) \le \|u\|$ for all $y \in \overline{P_c}$. Then Φ has at least three fixed points u_1, u_2 and u_3 satisfying

$$\|u_1\| < a,\ b < \psi(u_2) \quad \text{and} \quad \|u_3\| > a \quad \text{with } \psi(u_3) < b.$$

3. Main results

The set $E = C([0, \sigma(1)], \Re)$ is a Banach space with the supremum norm $\|u\| = \sup_{0 \le t \le \sigma(1)} |u(t)|$, where $u \in E$. Let $u, v \in E$, then the ordering $u \le v$ means $u(t) \le v(t)$ for all $t \in [0, \sigma(1)]$. Define

$$P = \{u \in E : u(t) \ge 0,\ t \in [0, \sigma(1)]\}.$$

Clearly, P is a cone of E. Finally, define a function $\psi : P \to [0, \infty)$ by

$$\psi(u) = \min_{t \in [\xi, \sigma(\omega)]} u(t), \quad u \in P. \tag{1}$$

Then ψ is a nonnegative continuous concave functional and

$$\psi(u) \le \|u\|.$$

Clearly, $u \in E$ is a solution of (BVP) if and only if

$$u(t) = \int_0^{\sigma(1)} G(t, s) f(s, u(\sigma(s)))\Delta s, \quad t \in [0, \sigma(1)].$$

Now, we can state and prove our main result.

Theorem 3.1. *Let $a, b, c \in \Re$ with $0 < a < b < Mc$, where M is defined as in (C_4). Suppose f satisfies*

(i) $f(t, u) < D_1 a$ *for* $(t, u) \in [0, \sigma(1)] \times [0, a]$,

(ii) $f(t, u) \ge \frac{D_2}{M} b$ *for* $(t, u) \in [\xi, \omega] \times [b, \frac{b}{M}]$,

(iii) $f(t, u) \le D_1 c$ *for* $(t, u) \in [0, \sigma(1)] \times [0, c]$.

Then the boundary value problem (BVP) *has at least three solutions u_1, u_2 and u_3 satisfying*

$$\|u_1\| < a,\ b < \psi(u_2) \quad \textit{and} \quad \|u_3\| > a \quad \textit{with} \ \ \psi(u_3) < b,$$

where ψ is defined as in (1).

Proof. It is clear that (BVP) has a solution $u = u(t)$ if, and only if, $u(t)$ is a solution of the operator equation

$$\Phi u(t) := \int_0^{\sigma(1)} G(t, s) f(s, u(\sigma(s)))\Delta s = u(t).$$

We note that for $u \in P$, $\psi(u) \leq \|u\|$. Now choose $u \in \overline{P_c}$, that is $||u|| \leq c$. Then $f(t,u) \leq D_1 c$ for $t \in [0, \sigma(1)]$ by condition (iii). It follows from (i) of Lemma 2.5 that

$$\begin{aligned}\Phi u(t) &= \int_0^{\sigma(1)} G(t,s) f(s, u(\sigma(s)))\Delta s \leq \int_0^{\sigma(1)} G(\sigma(s),s) f(s,u(\sigma(s)))\Delta s \\ &\leq \int_0^{\sigma(1)} G(\sigma(s),s) D_1 c \Delta s = c, \qquad t \in (0,1).\end{aligned}$$

Thus, $\|\Phi u\| \leq c$ for $u \in \overline{P_c}$. Hence, $\Phi(\overline{P_c}) \subseteq \overline{P_c}$. And Φ satisfies the condition (A_2) of Lemma 2.B. That is, if $u \in \overline{P_a}$, then $f(t,u) < D_1 a$ for $t \in [0, \sigma(1)]$ by condition (i). Thus, $\Phi(\overline{P_a}) \subseteq P_a$.

To fulfill property (A_1) of Lemma 2.6, we note that $x(t) = \frac{b}{M} \in P(\psi, b, \frac{b}{M})$ for $t \in [0, \sigma(1)]$. Then

$$\psi(x) = \psi\left(\frac{b}{M}\right) = \frac{b}{M} > b \text{ and } \left\{u \in P\Big(\psi, b, \frac{b}{M}\Big) : \psi(u) > b\right\} \text{ is nonempty.}$$

In addition, if $u \in P(\psi, b, \frac{b}{M})$, then

$$\psi(u) = \min_{t \in [\xi, \sigma(\omega)]} u(t) \geq b$$

and hence

$$b \leq u(\sigma(t)) \leq \frac{b}{M}, \quad \text{for } t \in [\xi, \omega].$$

Thus, for any $u \in P(\psi, b, \frac{b}{M})$, it follows from condition (ii) that

$$f(t,u) \geq \frac{D_2}{M} b \quad \text{for } t \in [\xi, \sigma(\omega)],$$

and it follows from (ii) of Lemma 2.5 that

$$\begin{aligned}\psi(\Phi u) &= \min_{t \in [\xi, \sigma(\omega)]} \Phi u(t) = \min_{t \in [\xi, \sigma(\omega)]} \int_0^{\sigma(1)} G(t,s) f(s, u(\sigma(s)))\Delta s \\ &\geq M \int_0^{\sigma(1)} G(\sigma(s),s) f(s,u(\sigma(s)))\Delta s > M \int_0^{\sigma(1)} G(\theta, s) \frac{D_2}{M} b \Delta s = b.\end{aligned}$$

Hence, condition (A_1) of Lemma 2.6 is satisfied. We finally claim that (A_3) of Lemma 2.6 is also satisfied. Clearly, it is enough to show that $\psi(\Phi u) > b$

if $u \in P(\psi, b, c)$ and $\|\Phi u\| > \frac{b}{M}$. In fact, if we choose $u \in P(\psi, b, c)$ satisfying $\|\Phi u\| > \frac{b}{M}$, then

$$\begin{aligned}\psi(\Phi u) &= \min_{t\in[\xi,\sigma(\omega)]} \int_0^{\sigma(1)} G(t,s) f(s, u(\sigma(s)))\Delta s \\ &\geq M \int_0^{\sigma(1)} G(\sigma(s), s) f(s, u(\sigma(s)))\Delta s \\ &\geq M \int_0^{\sigma(1)} G(t,s) f(s, u(\sigma(s)))\Delta s = M\Phi u(t) \qquad \text{for } t \in [0, \sigma(1)].\end{aligned}$$

Thus,

$$\psi(\Phi u) \geq M\|\Phi u\| > b,$$

and (A_3) of Lemma 2.6 is satisfied. Hence, an application of Lemma 2.6 completes the proof. $\square$

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CHEN-HUANG HONG
DEPARTMENT OF MATHEMATICS
NATIONAL CENTRAL UNIVERSITY
CHUNG-LI, 320 TAIWAN
REPUBLIC OF CHINA

E-mail: hongch@math.ncu.edu.tw

HUEI-LIN HONG
DEPARTMENT OF INFORMATION MANAGEMENT
LUNGHWA UNIVERSITY OF SCIENCE AND TECHNOLOGY
KUEISHAN TAOYUAN, 333 TAIWAN
REPUBLIC OF CHINA

E-mail: hong@mail.lhu.edu.tw

CHEH-CHIH YEH
DEPARTMENT OF INFORMATION MANAGEMENT
LUNGHWA UNIVERSITY OF SCIENCE AND TECHNOLOGY
KUEISHAN TAOYUAN, 33306 TAIWAN
REPUBLIC OF CHINA

E-mail: ccyeh@mail.lhu.edu.tw

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