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On the number of simple zeros of certain polynomials

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Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

The purpose of the present note is to study the number of simple zeros of a large class of integer-valued polynomials. Combining our result with a well-known theorem on superelliptic equations we obtain an effective finiteness statement for a general diophantine equation. The proof of our Theorem is based on some properties of the canonical mapping $\mathbb{Z}[X] \to \mathbb{Z}_p[X]$ (p is a prime). We remark that this method has been applied fruitfully by several authors, especially to Bernoulli polynomials; see e.g. [2],[3],[4],[5],[9],[10].

2. The Theorem

Let n be a positive integer, and set $\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$. Furthermore, let $f(X)$ be an integer-valued polynomial with $\deg f(X) \leq n-1$, and let $g(X) \in \mathbb{Z}[X]$.

Theorem. *Suppose that $n \geq 6$ and let p denote a prime for which*

$$\frac{2}{3}n < p \leq n.$$

If a_n is an integer not divisible by p then the polynomial

$$F(X) = a_n \binom{X}{n} + f(X) + g(X)$$

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has at least $\left[\frac{n}{3}\right]+1$ *simple zeros.*

Proof. Put $f_i(X)=X(X-1)\cdots(X-i+1)$ for $i=1,\dots,n$ and $f_0(X)=1$. Since $f(X)$ is an integer-valued polynomial, thus (cf. [7])

$$f(X)=a_{n-1}\binom{X}{n-1}+\ldots+a_1\binom{X}{1}+a_0,$$

where the coefficients $a_{n-1},\dots,a_1,a_0$ are rational integers. We get

$$n!F(X)=a_nf_n(X)+\ldots+a_pn(n-1)\cdots(p+1)f_p(X)+\ldots+n!a_0+n!g(X)\in\mathbb{Z}[X]\,.$$

For $S(X)\in\mathbb{Z}[X]$, we denote by $(S(X))_p$ the image of S in $\mathbb{Z}_p[X]$ under the canonical homomorphism $\mathbb{Z}\to\mathbb{Z}_p$. There is a $h(X)\in\mathbb{Z}[X]$ such that

$$(n!F(X))_p=(f_p(X))_p(h(X))_p$$

and $\deg(h(X))_p=n-p$. Since all the zeros of $(f_p(X))_p$ are simple, the polynomial $(n!F(X))_p$ as well as the polynomial $n!F(X)$ has at least $p-(n-p)=2p-n>\frac{n}{3}$ simple zeros.

3. An application to diophantine equations

Let $F(X)$ be as above, and let a be a non-zero integer.

Corollary. *All the solutions of the equation*

(1) $\quad F(x)=ay^m \qquad$ *in integers* x,y,m *with* $|y|>1$ *and* $m>1$

satisfy

$$\max(x,|y|,m)<c,$$

where c is an effectively computable constant depending only on $F(X)$ *and* a.

Our Corollary is a consequence of our Theorem and of the following powerful result from the theory of diophantine equations.

Lemma. *Let* $t(X)\in\mathbb{Q}[X]$ *and suppose that the polynomial* $t(X)$ *possesses at least three simple zeros. Then the equation*

$$t(x)=y^m \qquad \text{in integers } x,y,m \text{ with } |y|>1 \text{ and } m>1$$

implies that

$$\max(|x|,|y|,m)<c_1,$$

where c_1 *is an effectively computable constant depending only on the polynomial* $t(X)$.

Proof. See Theorem 9.1 in [8].

Remark. Several special cases of the equation (1) have been considered and have been applied to certain combinatorial diophantine problems. The equations

$$\binom{x}{n} = \binom{y}{2} \quad \text{and} \quad \binom{x}{n} = \binom{y}{4} \tag{2}$$

lead to equations

$$8\binom{x}{n} + 1 = (2y-1)^2 \quad \text{and} \quad 24\binom{x}{n} + 1 = (y^2 - 3y - 1)^2, \text{ respectively.}$$

Applying the Corollary to the polynomials $F(X) = 8\binom{x}{n} + 1$ and $F(X) = 24\binom{x}{n} + 1$, respectively, we have that all the solutions of the equations (2) are bounded by an effectively computable constant depending only on n. By using another approach Kiss [6] (in the case n is a prime number) and Brindza [1] have proved that all the zeros of the polynomial $8\binom{x}{n} + 1$ are simple, thus the first equation of (2) has only finitely many solutions.

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