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Annihilators of derivations with Engel conditions on one-sided ideals

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Abstract. Let R be a noncommutative prime ring with extended centroid C, two-sided Martindale quotient ring Q and λ a nonzero left ideal of R. Suppose that D is a nonzero derivation of R and $0 \neq a \in R$ such that $a[D(u^k), u^k]_n = 0$ for all $u \in \lambda$, where k and n are fixed positive integers. Then $D = \mathrm{ad}\,(b)$ for some $b \in Q$ such that $\lambda b = 0$ and $ab = 0$. We also prove an analogous result for right ideals.

Throughout this paper, unless specially stated, R always denotes a prime ring with extended centroid C and two-sided Martindale quotient ring Q. For $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_n = [[x, y]_{n-1}, y]$ for $n > 1$. For a subset S of R we denote by $\ell_R(S)$ the left annihilator of S in R, that is, $\ell_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$. By a derivation of R, we mean an additive map D from R into itself satisfies the rule $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. For $b \in Q$, we denote $\mathrm{ad}(b)$ to be the inner derivation induced by b; that is, $\mathrm{ad}(b)(x) = bx - xb$ for $x \in R$. In [2] Brešar proved the theorem: Let R be a semiprime $(n-1)!$ torsion-free ring. If D is a nonzero derivation of R such that $aD(x)^n = 0$ for all $x \in R$, where $a \in R$, then $aD(R) = 0$. In particular, if R is prime then $\ell_R(S) = 0$, where $S = \{D(x)^n \mid x \in R\}$. In [8] Lee and Lin proved Brešar's result without the $(n-1)!$ torsion-free assumption on R, where n is a fixed positive integer. In fact, they studied the Lie ideal case as given by Lanski [5] and then obtained Brešar's result as a corollary to their main

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result. On the other hand, in [3] BREŠAR and VUKMAN showed that if R is a noncommutative prime ring of characteristic not 2, then U, the subring of R generated by the subset $\{[D(x),x] \mid x \in R\}$, contains a nonzero left ideal of R. In particular, $\ell_R(S)=0$ where $S=\{[D(x),x] \mid x \in R\}$. The goal of this paper is to extend above results to the case of one-sided ideals. More precisely, we shall prove the following two theorems

Theorem 1. *Let R be a noncommutative prime ring with a nonzero left ideal λ. Suppose that D is a nonzero derivation of R and $0 \neq a \in R$ such that $a[D(u^k),u^k]_n = 0$ for all $u \in \lambda$, where k and n are fixed positive integers. Then $D=\mathrm{ad}(b)$ for some $b \in Q$ such that $\lambda b = 0$ and $ab=0$.*

Theorem 2. *Let R be a noncommutative prime ring with a nonzero right ideal ρ. Suppose that D is a nonzero derivation of R and $a \in R$ such that $a[D(u^k),u^k]_n = 0$ for all $u \in \rho$, where k and n are fixed positive integers. Then $aD(\rho)=0=a\rho$.*

We first prove the special case when $\lambda = R$.

Proposition 3. *Let R be a noncommutative prime ring and $0 \neq a \in R$. Suppose that D is a derivation of R such that $a[D(x^k),x^k]_n = 0$ for all $x \in R$, where k and n are fixed positive integers. Then $D=0$.*

PROOF. Suppose on the contrary that $D \neq 0$. Assume first that D is Q-inner. Thus there exists $b \in Q \setminus C$ such that $D=\mathrm{ad}(b)$. This implies $a[[b,x^k],x^k]_n = a[b,x^k]_{n+1} = 0$ for all $x \in R$. Hence $a[b,X^k]_{n+1}$ is a nontrivial generalized polynomial identity (GPI) for R because $aX^{k(n+1)}b$ occurs nontrivially in $a[b,X^k]_{n+1}$. By [1, Theorem 6.4.1], $a[b,X^k]_{n+1}$ is also a GPI for Q. Replacing R by Q, we may assume that R is a centrally closed prime ring having a nonzero socle H. If R is a domain, since $a \neq 0$, then $[b,x^k]_{n+1}=0$ for all $x \in R$. By [6], this implies $b \in C$, a contradiction. So we may assume that R is not a domain. Let e be a nontrivial idempotent of R. By hypothesis, we have $a[b,(xe)^k]_{n+1} = 0$ for all $x \in R$. Right-multiplying by $1-e$ yields that $a(xe)^{k(n+1)}b(1-e)=0$. By [7], this implies $axeb(1-e)=0$ for all $x \in R$. Since $a \neq 0$, so we have $eb(1-e)=0$. Replacing e by $1-e$, we get $(1-e)be=0$. This implies $[b,e]=0$ for every nontrivial idempotent e of R. Hence $[b,E]=0$, where E is the additive subgroup generated by all idempotents of R. Since E is a noncentral Lie ideal of R, this implies $b \in C$, a contradiction.

Suppose next that D is not Q-inner. To continue the proof we set $g(Y,X)=\sum_{i=0}^{k-1}X^iYX^{k-1-i}$, a noncommuting polynomial in variables X and Y. Note that $D(x^k)=g(D(x),x)$. Hence
$a[D(x^k),x^k]_n=a[g(D(x),x),x^k]_n=0$ for all $x\in R$. Applying Kharchenko's theorem [4] yields that $a[g(y,x),x^k]_n=0$ for all $x,y\in R$. For $u\in R$, replacing y by $[u,x]$ and applying the fact that $[u,x^k]=g([u,x],x)$ we see that $a[[u,x^k],x^k]_n=a[u,x^k]_{n+1}=0$ for all $u,x\in R$. The Q-inner case implies that $u\in C$ for all $u\in R$. Thus R is commutative, a contradiction. This proves the proposition. □

To continue our proof we need a technical result.

Lemma 4. *Let $a,b,e\in R$ with e an idempotent. Suppose that $a[b,e]_n=0$, where n is a fixed positive integer. Then $a[b,e]=0$.*

Proof. Since $e^2=e$, we have $[b,e]_3=[b,e]$. Thus $0=a[b,e]_n=a[b,e]$ if n is odd and so we are done in this case. So we may assume that n is even. Using the fact that $[b,e]_3=[b,e]$, this implies that $0=a[b,e]_n=a[b,e]_2=a(be-2ebe+eb)$. Right-multiplying by $1-e$ yields that $aeb(1-e)=0$ and so $aeb=aebe$. This implies $0=a(be-2ebe+eb)=a(be-2eb+eb)=a[b,e]$, proving the lemma. □

We are now ready to prove the case of left ideals.

Proof of Theorem 1. Suppose first that D is Q-inner. Thus there exists $b\in Q\setminus C$ such that $D=\mathrm{ad}(b)$.This implies

$$a\left[[b,u^k],u^k\right]_n=a[b,u^k]_{n+1}=0 \tag{1}$$

for all $u\in\lambda$. It is enough to show that $\lambda b=0$. Indeed, in this case, (1) becomes $abu^{k(n+1)}=0$ for all $u\in\lambda$. By [7], this implies $ab\lambda=0$ and so $ab=0$, as asserted. Suppose on the contrary that $\lambda b\neq 0$. We first claim that R satisfies a nontrivial GPI. For $r\in R$ and $x\in\lambda$, setting $u=rx$ in (1), we have $a[b,(rx)^k]_{n+1}=0$. If x and xb are linearly dependent over C for all $x\in\lambda$, then $[xb,x]=x[b,x]=0$ for all $x\in\lambda$. By [9, Lemma 3], we have $\lambda(b-\mu)=0$ for some $\mu\in C$. Since $ad(b)=\mathrm{ad}(b-\mu)$, replacing b by $b-\mu$, this implies $\lambda b=0$, a contradiction. So we may assume that there exists some $v\in\lambda$ such that v and vb are C-independent. This implies that $a[b,(Xv)^k]_{n+1}$ is a nontrivial GPI for R and hence for Q [1, Theorem 6.4.1].

By MARTINDALE's Theorem [10], Q is a primitive ring having a nonzero socle H and we have $\lambda b = 0$ if and only if $H\lambda b = 0$. Replacing R, λ by Q, $H\lambda$ respectively, we may assume that R is a primitive ring having a nonzero socle H and $\lambda \subseteq H$. Let e be an idempotent in λ. We claim that $eb \in Ce$. Suppose $eb \notin Ce$. For any $x \in R$, we have $xe \in \lambda$. Setting $u = xe$ in (1), we get $a[b, (xe)^k]_{n+1} = 0$. This implies $a[b, (xe)^k]_{n+1}(1-e) = 0$ and so $a(xe)^{k(n+1)}b(1-e) = 0$. By [7], we have $axeb(1-e) = 0$ for all $x \in R$. Since R is prime and $a \neq 0$, this implies $eb = ebe$. Next setting $u = e$ in (1), we have $a[b, e]_{n+1} = 0$. By Lemma 4, we have $a[b, e] = 0$. Since $e+(1-e)xe$ is also an idempotent in λ, this implies $a[b, e+(1-e)xe] = 0$ and so $a[b, (1-e)xe] = 0$. Hence $ab(1-e)xe = a(1-e)xeb$ for all $x \in R$. Since $eb \notin Ce$, by [1, Theorem 2.3.4], this implies $ab(1-e) = a(1-e) = 0$. So $ab = abe$ and $a = ae$. Setting $u = exe$ in (1), we have $a[b, (exe)^k]_{n+1} = 0$. This implies $ea[b, (exe)^k]_{n+1}e = eae[ebe, (exe)^k]_{n+1} = 0$ because $a = ae$. By Proposition 3, this implies either $eae = 0$ or $ebe \in Ce$. If $ebe \in Ce$, then $eb = ebe \in Ce$, a contradiction. So we have $eae = 0$. But for every $r \in R$, we have $ra[b, (exe)^k]_{n+1} = 0$. By the same argument of above, we have $erae = 0$ for all $r \in R$. This implies $a = ae = 0$, a contradiction. We have proved that $eb \in Ce$ for any idempotent $e \in \lambda$. For $u \in \lambda$, since $\lambda \subseteq H$ and H is completely reducible, there exist $x \in R$ and an idempotent $e \in \lambda$ such that $u = xe$. This implies $ub = xeb \in xCe = Cu$ for all $u \in \lambda$. By a standard argument, there exists $\mu \in C$ such that $ub = \mu u$ for all $u \in \lambda$. This implies $\lambda(b-\mu) = 0$. Since $\mathrm{ad}(b) = \mathrm{ad}(b-\mu)$, replacing b by $b-\mu$, this implies $\lambda b = 0$, a contradiction.

Suppose next that D is not Q-inner. Let $x \in R$ and $u \in \lambda$, then $xu \in \lambda$. By assumption, $a[D((xu)^k), (xu)^k]_n = 0$, implying that $a[g(D(x)u + xD(u), xu), (xu)^k]_n = 0$ for all $x \in R$ and $u \in \lambda$, where $g(Y, X)$ is the polynomial defined in the proof of Proposition 3. Applying KHARCHENKO's theorem [4] yields that

$$a\left[g(yu + xD(u), xu), (xu)^k\right]_n = 0$$

for all $x, y \in R$, and $u \in \lambda$. By the linearity of $g(Y, X)$ in Y, this implies that $a[g(yu, xu), (xu)^k]_n = 0$ for all $x, y \in R$, $u \in \lambda$. Replacing y by $[u, x]$ yields that $a[g([u, x]u, xu), (xu)^k]_n = a[g([u, xu], xu), (xu)^k]_n = 0$. So $a[u, (xu)^k]_{n+1} = 0$ for all $x \in R$. Applying the inner case to the left ideal

Ru yields that $au = 0$ for all $u \in \lambda$. This implies $a = 0$, a contradiction. This proves the theorem. $\square$

We next prove the result about right ideals.

PROOF of Theorem 2. It is enough to show that $a\rho = 0$. If $a\rho = 0$, then $0 = a[D(u^k), u^k]_n = aD(u^k)u^{kn}$ for all $u \in \rho$. For $r \in R$, we have $ur \in \rho$ and so $aD((ur)^k)(ur)^{kn} = 0$. This implies $aD(u)r(ur)^s = 0$ for all $r \in R$, where $s = kn + k - 1$. Hence $aD(u)(ru)^{s+1} = 0$ for all $r \in R$. By [7], we have $aD(u)ru = 0$ for all $r \in R$. This implies, for every $u \in \rho$, either $aD(u) = 0$ or $u = 0$. In any case we have $aD(\rho) = 0$. Suppose on the contrary that $a\rho \neq 0$. We first assume that D is Q-inner, thus there exists $b \in Q \setminus C$ such that $D = \mathrm{ad}(b)$. This implies

$$a\left[[b, u^k], u^k\right]_n = a[b, u^k]_{n+1} = 0 \tag{2}$$

for all $u \in \rho$. Since $a\rho \neq 0$, there exists some $v \in \rho$ such that $av \neq 0$. This implies $a[b, (vX)^k]_{n+1}$ is a nontrivial GPI for R and hence for Q [1, Theorem 6.4.1]. By the same argument in the proof of Theorem 1, we may assume that R is a primitive ring having a nonzero socle H and $\rho \subseteq H$. Since $a\rho \neq 0$ and $\rho \subseteq H$, there exists some idempotent $e \in \rho$ such that $ae \neq 0$. Setting $u = e$ in (2) and by Lemma 4, we have $a[b, e] = 0$. Since $e + ex(1-e)$ is also an idempotent in ρ, this implies $a[b, e + ex(1-e)] = 0$ and so $a[b, ex(1-e)] = 0$. Hence $a[b, ex(1-e)]e = -aex(1-e)be = 0$ for all $x \in R$. Since $ae \neq 0$, this implies $(1-e)be = 0$ and so $be = ebe$. Next setting $u = exe$ in (2), we have $a[b, (exe)^k]_{n+1} = 0$. This implies $a[b, (exe)^k]_{n+1}e = ae[be, (xe)^k]_{n+1} = 0$ for all $x \in R$ because $be = ebe$. Applying Theorem 1 to the left ideal Re yields that $Re(be - \mu) = 0$ for some $\mu \in C$. By the same argument in the proof of Theorem 1, we may assume that $\mu = 0$. This implies $Rebe = 0$ and so $be = ebe = 0$. Setting $u = ex$ in (2), we have $a[b, (ex)^k]_{n+1} = (-1)^{n+1}a(ex)^{k(n+1)}b = 0$ for all $x \in R$. By [7], we have $aexb = 0$ for all $x \in R$. This implies either $ae = 0$ or $b = 0$, a contradiction.

Suppose next that D is not Q-inner. Let $x \in R$ and $u \in \rho$, then $ux \in \rho$. By assumption, $a[D((ux)^k), (ux)^k]_n = 0$, implying that $a[g(D(u)x + uD(x), ux), (ux)^k]_n = 0$ for all $x \in R$ and $u \in \rho$, where $g(Y, X)$ is the polynomial defined in the proof of Proposition 3. Applying

KHARCHENKO's theorem [4] yields that $a[g(D(u)x+uy, ux), (ux)^k]_n = 0$ for all $x, y \in R$, $u \in \rho$. It follows the linearity of $g(Y, X)$ in Y that $a[g(uy, ux), (ux)^k]_n = 0$ for all $x, y \in R$, $u \in \rho$. Replacing y by $[u, x]$ yields that $a[g(u[u, x], ux), (ux)^k]_n = a[g([u, ux], ux), (ux)^k]_n = 0$. This implies $a[u, (ux)^k]_{n+1} = 0$ for all $x \in R$. Applying the inner case to the right ideal uR yields that $au = 0$ for all $u \in \rho$. This means that $a\rho = 0$, a contradiction. This proves the theorem. □

Remark. For the case of Lie ideals, the author [11] proved the following result:

Theorem 5. *Let R be a prime ring, L a noncentral Lie ideal of R and $a \in R$. Suppose that D is a nonzero derivation of R such that $a[D(u), u]_n = 0$ for all $u \in L$, where n is a fixed positive integer. Then $a = 0$ except when* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

By the same argument of Theorem 5, we get the similar result for noncentral Lie ideals.

Theorem 6. *Let R be a prime ring, L a noncentral Lie ideal of R and $a \in R$. Suppose that D is a nonzero derivation of R such that $a[D(u^k), u^k]_n = 0$ for all $u \in L$, where k and n are fixed positive integers. Then $a = 0$ except when* $\dim_C RC = 4$.

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