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Metrizable linear connections in vector bundles

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Dedicated to Professor Dr. Lajos Tamássy
at his 80th anniversary

Abstract. A linear connection ∇ in a vector bundle is said to be metrizable if the vector bundle admits a Riemannian metric h with the property $\nabla h = 0$. Sufficient conditions for the linear connection ∇ to be metrizable are provided.

Introduction

The problem of the metrizability of a linear connection was treated by many authors in various contexts (see the paper [7] by L. Tamassy and the references therein). When a linear connection ∇ in a vector bundle $\xi = (E, p, M)$ is metrizable, its parallel translations are isometries with respect to any Riemannian metric h in ξ with $\nabla h = 0$. Using a local chart around a point x in M, the holonomy group $\phi(x)$ may be identifed with a subgroup of $GL(m, \mathbb{R})$, where m is the dimension of fibre. With this identification, a necessary condition for ∇ to be metrizable is that the holonomy group be contained in the orthogonal group $O(m)$. We prove two versions of the converse of this fact (Theorems 3.1 and 3.2). Then we are dealing with the same problem when the vector bundle ξ is endowed with a Finsler function. The linear connection ∇ induces a

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nonlinear connection on E and a linear connection D in the vertical vector bundle over E. The Finsler function F defines a Riemannian metric g in the vertical vector bundle over E. We show that if g is covariant constant on horizontal directions, then ∇ is metrizable (Theorem 4.2). When the tangent bundle of a manifold M is endowed with a Finsler function F one says that (M, F) is a Finsler manifold. In this case our result is to be compared with the one due to Z. Szabó ([6]), regarding the metrizability of the Berwald connection.

If the cotangent bundle of a manifold M is endowed with a Finsler function K, then the pair (M, K) is called a *Cartan space*. This notion was introduced and studied by R. Miron in [3]. In this case Theorem 4.1 is to be compared with our previous results on the metrizability of the Berwald–Cartan connection [1].

The first two sections of the paper are devoted to some preliminaries from the theory of vector bundles and linear connections in vector bundles.

1. Vector bundles

Let $\xi = (E, p, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n+m$, and $p : E \to M$ is a smooth submersion. The fibres $E_x = p^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha \in A}$ with the bijections $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha = (p(u), \varphi_{\alpha,p(u)}(u))$, where $\varphi_{\alpha,p(u)} : E_p(u) \to \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $(p^{-1}(U_\alpha), \phi_\alpha)_{\alpha \in A}$ on E. Here $\phi_\alpha : p^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(p(u)), \varphi_{\alpha,p(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^m$ and we take (x^i, y^a) as local coordinates on E. If (U_β, ψ_β) is such that $x \in U_\alpha \cap U_\beta \neq \emptyset$ and $\psi_\beta(x) = (\widetilde{x}^i)$, then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, \ldots, x^n), \quad \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{1.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi_{\alpha,x}^{-1}(e_a) = \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ takes the form $u = y^a \varepsilon_a(x)$. We put $\widetilde{y}^a = M_b^a(x) y^b$ with $\operatorname{rank}(M_b^a(x)) = m$. Then $\phi_\beta \circ \phi_\alpha^{-1}$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, \dots, x^n), \qquad &&\operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{y}^a = M_b^a(x) y^b, &&\operatorname{rank}(M_b^a(x)) = m. \end{aligned} \tag{1.2}$$

The indices $i, j, k, \dots,$ $a, b, c, \dots.$ will take the values $1, 2, \dots, n$ and $1, 2, \dots, m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\mathcal{X}(M)$, resp. $\Gamma(E)$, $\mathcal{X}(E)$ the module of sections of the tangent bundle of M, resp. of the bundle ξ and of the tangent bundle of E. On U_α, the vector fields $\left(\partial_k := \frac{\partial}{\partial x^k}\right)$ provide a local basis for $\mathcal{X}(U_\alpha)$. The sections $\varepsilon_a : U_\alpha \to p^{-1}(U_\alpha)$ given by $\varepsilon_a(x) = \varphi_{\alpha,x}^{-1}(e_a)$ will be taken as canonical basis for $\Gamma(p^{-1}(U_\alpha))$ and a section $A : U_\alpha \to p^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x)\varepsilon_a(x)$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We take as local basis of $\Gamma(E^*)$ on U_α the sections $\theta^a \colon U_\alpha \to p^{*-1}(U_\alpha)$, $x \to \theta^a(x) \in E_x^*$ such that $\theta^a(\varepsilon_b(x)) = \delta_b^a$.

Next, we may consider the tensor bundle of type (r, s), $\mathcal{T}_s^r(E) := E \underbrace{\otimes \cdots \otimes}_{r} E \otimes E^* \underbrace{\otimes \cdots \otimes}_{s} E^*$ over M and its sections. For $g \in \Gamma(E^* \otimes E^*)$ we have the local representation $g = g_{ab}(x)\theta^a \otimes \theta^b$. As $E^* \otimes E^* \cong L_2(E, \mathbb{R})$, we may regard g as a smooth mapping $x \to g(x) : E_x \times E_x \to \mathbb{R}$ with $g(x)$ a bilinear mapping given by $g(x)(s_a, s_b) = g_{ab}(x)$.

If the mapping $g(x)$ is symmetric i.e. $g_{ab} = g_{ba}$ and positive-definite i.e. $g_{ab}(x)\zeta^a\zeta^b > 0$ for every $0 \neq (\zeta^a) \in \mathbb{R}^m$, one says that g defines a Riemannian metric in the vector bundle ξ.

The sets of sections $\Gamma(T_s^r(E))$ are $\mathcal{F}(M)$-modules for any natural numbers r, s. On the sum $\bigoplus_{r,s} \Gamma(T_s^r(E))$ a tensor product can be defined and one gets a tensor algebra $\mathcal{T}(E)$. For the vector bundle (TM, τ, M) this reduces to the tensor algebra of the manifold M.

2. Linear connections in a vector bundle

Definition 2.1. A linear connection in the vector bundle $\xi = (E, p, M)$ is a mapping $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$, $(X, A) \to \nabla_X A$ which is $\mathcal{F}(M)$-linear in the first argument, additive in the second and

$$\nabla_X(fA) = X(f)A + f\nabla_X A, \; f \in \mathcal{F}(M). \tag{2.1}$$

For $X = X^k(x)\partial_k$ and $A = A^a(x)\varepsilon_a(x)$, we get

$$\nabla_X A = X^k(\partial_k A^a + \Gamma^a_{bk}(x)A^b)\varepsilon_a(x), \tag{2.2}$$

where the local coefficients $\Gamma^a_{bk}(x)$ are defined by

$$\nabla_{\partial_k}\varepsilon_b = \Gamma^a_{bk}\varepsilon_a. \tag{2.3}$$

If $\widetilde{\Gamma}^c_{dj}$ are the local coefficients of ∇ on U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$\widetilde{\Gamma}^c_{dj}(\widetilde{x}(x)) = M^c_a(x)(M^{-1})^b_d \frac{\partial x^k}{\partial \widetilde{x}^j}\Gamma^a_{bk}(x) - \frac{\partial M^c_b}{\partial x^k}\,\frac{\partial x^k}{\partial \widetilde{x}^j}(M^{-1})^b_d. \tag{2.4}$$

A section A of ξ is called *parallel* if $\nabla_X A = 0$ for every $X \in \mathcal{X}(M)$.

The linear connection ∇ induces operators of covariant derivative ∇_k in the tensor algebra $\mathcal{T}(E)$ taking $\nabla_k f = \partial_k f$, $\nabla_k\beta_a = \partial_k\beta_a - \Gamma^c_{ak}\beta_c$ and requiring that ∇_k to satisfy the Newton–Leibniz rule with respect to the tensor product and to commute with all contractions.

Let $c : [0, 1] \to M$ be a curve on M and $A : t \to A(t) := A(c(t))$ a section of ξ along the curve c. Then $\nabla_{\dot{c}(t)}A =: \frac{\nabla A}{dt}$ is called the covariant derivative of A along c.

On $U_\alpha \cap c[0, 1]$ if we put $c(t) = (x^i(t))$, we get

$$\frac{\nabla A}{dt} = \left(\frac{dA^a}{dt} + \Gamma^a_{bk}(x(t))A^b\frac{dx^k}{dt}\right)\varepsilon_a. \tag{2.5}$$

The section $t \to A(t)$ is said to be *parallel* on c if $\frac{\nabla A}{dt} = 0$. This means that the functions $(A^a(t))$ have to be solutions of the following system of ordinary linear differential equations:

$$\frac{dA^a}{dt} + \Gamma^a_{bk}(x)A^b\frac{dx^k}{dt} = 0. \tag{2.6}$$

For given initial conditions $A^a(0) = (u^a) \in E_{c(0)}$ the system (2.6) admits a unique solution that can be prolonged beyond U_α providing a parallel section A along c. If we associate to $(u^a) = A^a(0)$ the element $(v^a) = A^a(1) \in E_{c(1)}$ we get a linear isomorphism $P_c : E_{c(0)} \to E_{c(1)}$, called the *parallel translation* of $E_{c(0)}$ to $E_{c(1)}$ along c. The parallel translations can be defined along any curve or segment of curve providing linear isomorphisms between fibres in various points of curves on M. In particular, if one considers the loops with origin in $x \in M$, the corresponding parallel translations as linear isomorphisms $E_x \to E_x$ can be composed and a group $\phi(x)$ called the holonomy group in $x \in M$ is obtained.

When M is connected, the holonomy groups $\phi(x)$, $x \in M$ are isomorphic and one speaks about the holonomy group ϕ associated to or defined by ∇.

The covariant derivative along c can be recovered from parallel translations according to the following known

Lemma 2.1. *Let A be a section of ξ along a curve on M, $c : t \to c(t)$, $t \in \mathbb{R}$, starting from $x = c(0)$. Then*

$$(\nabla_{\dot{c}(0)} A)(x) = \lim_{t \to 0} \frac{1}{t} (P_c(A(t)) - A(0)), \tag{2.7}$$

where $P_c : E_{c(t)} \to E_x$ is the parallel translation along c.

3. A sufficient condition for ∇ to be metrizable

Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$. Assume that the manifold M is connected. One says that ∇ is *metrizable* if there exists a Riemannian metric g in ξ such that $\nabla g = 0$. When ∇ is metrizable, then all parallel translations $P_c : (E_x, g_x) \to (E_y, g_y)$ for any points x, y and for any curve c joining them in M are isometries. In particular, the holonomy group $\phi(x)$ is a subgroup of the orthogonal group of (E_x, g_x). These facts follow from

Lemma 3.1. *Let g be any Riemannian metric in the vector bundle ξ and $c : t \to c(t)$, $t \in \mathbb{R}$, a curve in M with $c(0) = x$. Then*

$$\left(\nabla_{\dot{c}(0)} g\right)(A, B) = \lim_{t \to 0} \frac{1}{t}(g_{c(t)}(P_c A, P_c B) - g_x(A, B)), \tag{3.1}$$

where $A, B \in E_x$ and $P_c : E_x \to E_{c(t)}$ is the parallel translation along c.

Proof. Let $\widetilde{A}$, $\widetilde{B}$ be sections of ξ which are parallel on c, such that $\widetilde{A}(0) = A$, $\widetilde{B}(0) = B$. Then $P_c A = \widetilde{A}(t)$ and $P_c(B) = \widetilde{B}(t)$. By the Taylor theorem and using the condition that $\widetilde{A}$ and $\widetilde{B}$ are parallel sections on c, in the natural basis (ε_a) we get $(P_c A)^a = \widetilde{A}^a(t) = A^a + \frac{d\widetilde{A}}{dt}(\tau)t = A^a - \Gamma^a_{ck}(x(\tau))\widetilde{A}^c(\tau)\frac{dx^k}{dt}t$ and a similar formula for $(P_c B)^b$, $a, b = 1, 2, \ldots, m$. Then, using again the Taylor theorem, omitting the terms which contain t^2, we may write:

$$g_{ab}(t)(P_c A)^a (P_c B)^b - g_{ab}(x) A^a B^b = \left(g_{ab}(x) + \frac{dg_{ab}}{dt}(\theta)t\right)(P_c A)^a (P_c B)^b$$

$$-g_{ab}(x) A^a B^b = \left(\frac{dg_{ab}}{dt} - g_{ac}\Gamma^c_{bk}\frac{dx^k}{dt} - g_{cb}\Gamma^c_{ak}\frac{dx^k}{dt}\right) A^a B^b t, \tag{3.2}$$

where the terms in the last paranthesis are computed for $\tau, \tau', \theta \in (0, t)$.

Dividing in (3.2) by t and taking $t \to 0$, one obtains (3.1).

By Lemma 3.1 we have also that if all parallel translations of ∇ are isometries with respect to g, then $\nabla g = 0$. Thus, in order to prove that ∇ is metrizable we need to find a Riemannian metric g such that all parallel translations of ∇ are isometries with respect to g. Taking an arbitrary bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, using the linear isomorphism $\varphi_{\alpha,x} : E_x \to \mathbb{R}^m$, we may identify $\phi(x)$, $x \in U_\alpha$, with a subgroup of $GL(\mathbb{R}^m)$. When ∇ is metrizable, by Lemma 3.1 it follows that this subgroup is contained in the orthogonal group $O(m)$. Therefore, a necessary condition for ∇ to be metrizable is that its holonomy group is contained in $O(m)$. We show two versions of the converse. □

Theorem 3.1. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that there exists a point $x_0 \in M$ such that the holonomy group $\phi(x_0)$ is contained in the orthogonal group of E_{x_0} when E_{x_0} is regarded as being isomorphic with the Euclidean space $(R^m, \langle\, , \rangle)$ via a fixed bundle chart. Then ∇ is metrizable.*

PROOF. Let h_0 be the inner product on E_{x_0} induced by $\langle\, ,\, \rangle$ via the bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, $x_0 \in U_\alpha$, that is,

$$h_0(u,v) = \langle \varphi_{\alpha,x_0} u, \varphi_{\alpha,x_0} v\rangle. \tag{$*$}$$

By hypothesis this inner product is invariant under the group $\phi(x_0)$. Let x be any point of M. We join x with x_0 using a curve $c : [0,1] \to M$, $c(0) = x$, $c(1) = x_0$, consider the parallel translation $P_c : E_x \to E_{x_0}$ and define an inner product h_x in E_x by

$$h_x(A,B) = h_0(P_c A, P_c B), \quad A, B \in E_x. \tag{3.3}$$

Lemma 3.2. *The inner product h_x does not depend on the curve c.*

Indeed, if $\widetilde{c}$ is another curve joining x with x_0, then we consider the reverse c_- of c and the loop $\widetilde{c}\circ c_-$ in x_0. It follows that $h_0(P_{\widetilde{c}\circ c_-} u, P_{\widetilde{c}\circ c_-} v) = h_0(u,v)$, $u, v \in E_{x_0}$. Inserting here $u = P_c A$ and $v = P_c B$ and taking into account (3.3), the lemma follows.

The mapping $x \to h_x$ is smooth since P_c smoothly depends on x according to the general theory of differential equations. Thus we obtain a Riemannian metric h in ξ. The parallel translations of ∇ are isometries with respect to h. Indeed, for a point y of M different from x, any parallel translation from E_x to E_y has the form $P_{\sigma_-\circ c} = P_{\sigma_-} \circ P_c$, for σ_- the reverse of a curve σ joining y with x_0. As a product of isometries this is an isometry. Therefore, using Lemma 3.1 we may conclude that $\nabla h = 0$. □

The following version of Theorem 3.1 extends to the vector bundle setting a result of B. G. SCHMIDT [5].

Theorem 3.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that for a fixed $x_0 \in M$, the holonomy group $\phi(x_0)$ leaves invariant a given positive-definite quadratic form h_0 on E_{x_0}. Then there exists a Riemannian metric h in ξ such that $\nabla h = 0$.*

PROOF. Let us denote by the same letter h_0 the inner product in E_{x_0} defined by the quadratic form h_0. This inner product could be obtained by transferring one from $\mathbb{R}^m$ using a bundle chart. By hypothesis the inner product h_0 is invariant under $\phi(x_0)$. From now on the reasoning proving Theorem 3.1 can be repeated in its entirety in order to find h such that $\nabla h = 0$. $\square$

Remark 3.1. The Riemannian metric h found in Theorem 3.1 is not unique and is not canonical in any way. The same applies for h found in Theorem 3.2.

4. Another condition for ∇ to be metrizable

We are to deal with the problem of the metrizability of a linear connection ∇ in a vector bundle endowed with a Finsler function.

Definition 4.1. Let $\xi = (E, p, M)$ be a vector bundle of rank m. A *Finsler function* on E is a nonnegative real function F on E with the properties

1) F is smooth on $E \setminus \{(x,0), x \in M\}$,
2) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,
3) The matrix with the entries $g_{ab}(x,y) = \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^a \partial y^b}$ is positive definite.

On the manifold E we have the vertical distribution $u \to V_uE = \ker p_{*,u}$ where p_* denotes the differential of p. This is spanned by $\dot{\partial}_a := \frac{\partial}{\partial y^a}$. A distribution $u \to H_uE$ which is supplementary to the vertical distribution is called a *horizontal* distribution or a *nonlinear connection* on E. This is usually taken as spanned by $\delta_i = \partial_i - N_i^a(x,y)\dot{\partial}_a$, where the functions $(N_i^a(x,y))$ are called the *coefficients* of the given nonlinear connection. Under a change of coordinates they behave as follows:

$$\widetilde{N}_j^a\,\frac{\partial \widetilde{x}^j}{\partial x^k} = M_b^a(x) N_k^b(x,y) - \frac{\partial M_b^a}{\partial x^k} y^b, \tag{4.1}$$

a fact which is equivalent to

$$\delta_i = \frac{\partial \widetilde{x}^k}{\partial x^i}\widetilde{\delta}_k. \tag{4.1'}$$

Introducing the horizontal distribution we have

$$T_uE = H_uE \oplus V_uE, \quad u \in E. \tag{4.2}$$

It is convenient to decompose the geometrical objects on E according to (4.2) using the adapted basis $(\delta_i, \dot{\partial}_a)$ and its dual $(dx^i, \delta y^a = dy^a + N_i^a(x,y)dx^i)$.

The linear connection ∇ in ξ defines a nonlinear connection on E if we take $N_i^a(x,y) = \Gamma_{bi}^a(x)y^b$. Indeed, using (2.4) it is easy to check that these functions satisfy (4.1). From now on we shall use only the decomposition (4.2) provided by these functions. Furthermore, the linear connection ∇ induces a linear connection D in the vertical bundle over E as follows: $D : \mathcal{X}(E) \times \Gamma(VE) \to \Gamma(VE)$, $(X,Z) \to D_X Z$ is given for $Z = Z^a\dot{\partial}_a$ by

$$D_{\delta_k}\dot{\partial}_a = \Gamma_{bk}^a(x)\dot{\partial}_a, \quad D_{\dot{\partial}_b}\dot{\partial}_a = 0. \tag{4.3}$$

We call D the vertical lift of ∇ and we use D_{δ_k} for defining a *horizontal* covariant derivative operator in the tensor algebra of the vertical bundle, denoted by $_{|k}$, setting

$$\begin{gathered} f_{|k} = \delta_k f \quad \text{for any function on } E, \\ X^a_{|k} = \delta_k X^a + \Gamma_{bk}^a(x)X^b. \end{gathered} \tag{4.4}$$

For a fixed $x \in E$, the pair (E_x, F_x) is a Minkowski space. Here F_x denotes the restriction of F to E_x and it is obvious that this is a Minkowski norm on E_x.

Now we show that under certain conditions the parallel translations of ∇ are isometries of Minkowski spaces.

Theorem 4.1. *Let $\xi = (E,p,M)$ be a vector bundle of rank m with M connected, endowed with a Finsler function F and with a linear connection ∇ as well. Let $_{|k}$ be the horizontal covariant derivative operator defined by the vertical lift D of ∇. If $F_{|k} = 0$, then the parallel translation defined by ∇, $P_c : (E_x, F_x) \to (E_y, F_y)$ is an isometry of Minkowski spaces for any points $x, y \in M$ and any curve $c : [0,1] \to M$ joining them.*

Proof. Let be $u \in E_x$, and $t \to A(t)$, $t \in [0,1]$ a section of ξ which is parallel along c, and $A(0) = u$. Its local components A^a are solutions of the system of differential equations (2.6), and $P_c(u) = A(1) := v$.

We know already that P_c is a linear isomorphism. Let us write out the condition $F_{|k} = 0$ for the points $(x(t), A(t))$ of E where $t \to x(t)$ is the local representation of the curve c. We obtain:

$$0 = \left(\frac{\partial F}{\partial x^k} - A^b \Gamma^a_{bk} \frac{\partial F}{\partial y^a}\right) \frac{dx^k}{dt} \stackrel{(2.6)}{=} \frac{\partial F}{\partial x^k} \frac{dx^k}{dt} + \frac{\partial F}{\partial y^a} \frac{dA^a}{dt} = \frac{dF(x(t), A(t))}{dt}.$$

Thus the function $F(x(t), A(t))$ is constant. It follows $F(x, u) = F(y, P_c u)$, that is, $F_x(u) = F_y(P_c u)$. In other words, P_c is an isometry of Minkowski spaces (E_x, F_x) and (E_y, F_y). $\square$

Corollary 4.1. *Under the hypothesis of Theorem 4.1, the holonomy group $\phi(x)$ consists of isometries of the Minkowski space (E_x, F_x).*

The functions $g_{ab}(x, y)$ define a Riemannian metric in the vertical bundle over E by $g = g_{ab}(x, y)\delta y^a \otimes \delta y^b$. We call $(g_{ab}(x, y))$ the Finsler metric associated with F.

The condition $F_{|k} = 0$ from the hypothesis of Theorem 4.1 can be replaced by $g_{ab|k} = 0$, because of

Lemma 4.1. *$F_{|k} = 0$ is equivalent to $g_{ab|k} = 0$.*

Proof. The homogeneity of F implies $F^2(x, y) = g_{ab}(x, y)y^a y^b$. Then $F^2_{|k} = 2FF_{|k} = g_{ab|k}y^a y^b + 2g_{ab}y^a_{|k}y^b = g_{ab|k}y^a y^b$ since $y^a_{|k} = 0$. Thus if $g_{ab|k} = 0$, then $F_{|k} = 0$. In order to prove the converse, we notice that $\dot{\partial}_a(H_{|k}) = (\dot{\partial}_a H)_{|k}$ for any function H on E. This follows by a direct calculation taking into account that $\dot{\partial}_a H$ is a vertical 1-form. Using this "commutation" formula we get $g_{ab|k} = \frac{1}{2}\dot{\partial}_a\dot{\partial}_b(F^2_{|k}) = \dot{\partial}_a\dot{\partial}_b(FF_{|k}) = 0$. $\square$

Now we are ready to prove the main result of this section.

Theorem 4.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Suppose that E is endowed with a Finsler function F having the associated Finsler metric $g_{ab}(x, y)$. Let $_{|k}$ be the h-covariant derivative operator induced by ∇. If $g_{ab|k} = 0$, then ∇ is metrizable.*

Proof. For a fixed $x_0 \in M$ we have the Minkowski space (E_{x_0}, F_{x_0}). Let G be the group of all linear isomorphisms of E_{x_0} which preserve the set $S_{x_0} = \{u \in E_{x_0}, F_{x_0}(u) = 1\}$. This G is a compact Lie group since S_{x_0} is compact. In our hypothesis, according to Lemma 4.1 and Corollary 4.1, the holonomy group $\phi(x_0)$ is a Lie subgroup of G. Let $\langle\,,\,\rangle$ be any inner product on E_{x_0}. Define a new inner product on E_{x_0} by

$$h_{x_0}(u,v) = \frac{1}{\mathrm{vol}(G)} \int_G \langle gu, gv\rangle \mu_G, \tag{4.5}$$

for $u, v \in E_{x_0},\ g \in G$ and μ_G the bi-invariant Haar measure on G.

It follows that for every $a \in G$ we have

$$h_{x_0}(au, av) = h_{x_0}(u,v), \quad u, v \in E_{x_0}. \tag{4.6}$$

In particular, (4.6) holds for any element of $\phi(x_0) \subset G$. Thus $\phi(x_0)$ leaves invariant the inner product h_{x_0} in E_{x_0}. The inner product h_{x_0} is extended by parallel translations to a Riemannian metric h in ξ. Furthermore, this metric verifies $\nabla h = 0$ since all parallel translations of ∇ become isometries with respect to h. Thus ∇ is metrizable. $\square$

Remark 4.1. The Riemannian metric h is not unique and it is not canonical in any way.

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