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Lightlike foliations of codimension one

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We would like to dedicate this paper to Prof. L. Tamássy on the occasion of his 80th birthday

Abstract. We construct the null transversal bundle of a lightlike foliation of codimension one and define a second fundamental form of the foliation. We show that the second fundamental form is degenerate and does not depend on the null transversal bundle. Finally, we prove that any lightlike foliation of codimension one on a 3-dimensional semi-Riemannian manifold is either totally geodesic or totally umbilical.

Introduction

The geometry of foliations on a Riemannian manifold has been intensively studied and many interesting results have been obtained (cf. Ph. Tondeur [5]). However, as far as we know, only a small number of papers dealt with foliations on semi-Riemannian manifolds. In this case non-degenerate foliations and degenerate foliations must be treated separately. The first category of foliations was investigated in the special case when the ambient manifold is a space-time (cf. G. Walschap [7] and A. D. Rendall [4]). The geometry of degenerate foliations is

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still very little understood. With some additional geometric structures, M. T. Calapso and R. Rosca [2] obtained interesting results on the so called coisotropic foliations.

The purpose of our paper is to investigate the geometry of a lightlike (degenerate) foliation of codimension one. The main difficulty in this study is that for a lightlike foliation we cannot construct a transversal distribution that is orthogonal to the distribution of the foliation. We overcome this difficulty by considering the so called screen distribution which enables us to construct a null transversal bundle to a lightlike foliation. It is noteworthy that the second fundamental form of the lightlike foliation does not depend on the choice of the screen distribution.

Thus we may claim that we have developed the theory of fundamental tools for studying lightlike foliations of codimension one. As an application of this theory, we prove that any lightlike foliation of codimension one on a 3-dimensional Lorentz manifold is either totally geodesic or totally umbilical.

1. Definitions and examples

Let $\tilde{M}$ be a real $(m+2)$-dimensional smooth manifold with $m > 0$, and $\tilde{g}$ a symmetric non-degenerate tensor field of type $(0,2)$ on $\tilde{M}$. We assume that the bilinear form

$$\tilde{g}_x : T_x\tilde{M} \times T_x\tilde{M} \to \mathbb{R};\ \tilde{g}_x(X,Y) = \tilde{g}(x)(X,Y); \quad X, Y \in T_x\tilde{M}$$

has the same index q for all $x \in \tilde{M}$. Then $(\tilde{M}, \tilde{g})$ is a *semi-Riemannian manifold* with metric tensor field $\tilde{g}$ (cf. O'Neill [3], p. 55). In the present paper we suppose that $(\tilde{M}, \tilde{g})$ is never a Riemannian manifold, that is, $q \in \{1, \ldots, m+1\}$.

Next, we consider on $\tilde{M}$ a foliation $\mathcal{F}$ of codimension one given by the integrable distribution D on $\tilde{M}$. Then $\tilde{g}$ induces on D a field of symmetric bilinear forms which we denote by g. Hence

$$g(X,Y) = \tilde{g}(X,Y), \quad \forall X, Y \in \Gamma(D),$$

where $\Gamma(D)$ is the module of smooth sections of D. Since $\tilde{g}$ is not a Riemannian metric on $\tilde{M}, g$ might be either degenerate or non-degenerate. The latter case was investigated for foliations in space-time (cf. G. Walschap [7] and A. D. Rendall [4]). The geometry of a foliation with degenerate g is little known so far. Using some additional geometric structures, M. T. Calapso and R.Rosca [2] obtained interesting results on the so called coisotropic foliations whose g is degenerate. In the present paper we investigate foliations $\mathcal{F}$ of codimension one for which g is degenerate. The precise definitions are given below.

To define the class of foliations we are dealing with, consider $\mathcal{F}$ as a foliation of codimension one given by the integrable distribution D and denote by $D^{\perp}$ the vector bundle on $\tilde{M}$ whose fibres are defined by

$$D_x^{\perp} = \left\{ Y_x \in T_x\tilde{M};\ \tilde{g}_x(Y_x, X_x) = 0,\ \forall X_x \in D_x \right\}.$$

Clearly, $D^{\perp}$ is a distribution of rank 1 on $\tilde{M}$. The *null distribution* of D is $\mathcal{N} = D \cap D^{\perp}$. Then we say that $\mathcal{F}$ is a *lightlike (degenerate) foliation* if $\mathcal{N}$ has nonzero fibres at any point of $\tilde{M}$. As the fibres of $D^{\perp}$ are one-dimensional, it follows that $\mathcal{N}$ has constant rank 1 and therefore it coincides with $D^{\perp}$. Thus $\mathcal{F}$ is a lightlike foliation if and only if $D^{\perp}$ is a subbundle of D. On the other hand, if $\mathcal{F}$ is a lightlike foliation then g must be of rank m since the null distribution is of rank 1. The converse is also true. Finally, we see that $\mathcal{F}$ is a lightlike foliation if and only if any leaf of $\mathcal{F}$ is a lightlike hypersurface (cf. A. Bejancu [1]).

Next, we consider a coordinate system $(x^0, \ldots, x^{m+1})$ on $\tilde{U} \subset \tilde{M}$. Then the local components of the metric tensor $\tilde{g}$ on $\tilde{M}$ are

$$\tilde{g}_{ij} = \tilde{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j \in \{0, \ldots, m+1\}.$$

At each point $x \in \tilde{U}$ the matrix $[\tilde{g}_{ij}(x)]$ is invertible and we denote its inverse matrix by $[\tilde{g}^{ij}(x)]$. Consider a leaf M of the lightlike foliation $\mathcal{F}$ given locally by the equation

$$F(x^0, \ldots, x^{m+1}) = c. \tag{1.1}$$

Then

$$\xi = \operatorname{grad} F = \tilde{g}^{ij}(x) \frac{\partial F}{\partial x^j} \frac{\partial}{\partial x^i} \tag{1.2}$$

is a null vector field, that is, $\tilde{g}(\xi, \xi) = 0$ or equivalently

$$\tilde{g}^{ij}(x) \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = 0. \tag{1.3}$$

Conversely, if (1.3) is satisfied for any leaf of the foliation $\mathcal{F}$ then $\mathcal{F}$ is a lightlike foliation. We remark that $D^{\perp}$ is locally spanned by ξ.

Thus, summing up the above discussion we may state the following

Theorem 1.1. *Let $\mathcal{F}$ be a foliation of codimension one on $\widetilde{M}$ given by the integrable distribution D. Then the following assertions are equivalent:*

(i) *$\mathcal{F}$ is a lightlike foliation.*

(ii) *$D^{\perp}$ is a vector subbundle of D.*

(iii) *The induced tensor field g is of rank m.*

(iv) *Any leaf of D is a lightlike hypersurface of $\widetilde{M}$.*

(v) *Locally, on any leaf M of $\mathcal{F}$ given by* (1.1), *the condition* (1.3) *is satisfied.*

Now we give some examples of lightlike foliations.

Example 1.1. Let $\mathbb{R}^{m+2}_q = (\mathbb{R}^{m+2}, \tilde{g})$ be the $(m+2)$-dimensional semi-Euclidean space with the metric tensor field

$$\tilde{g}(x, y) = -\sum_{\alpha=0}^{q-1} x^{\alpha} y^{\alpha} + \sum_{a=q}^{m+1} x^a y^a. \tag{1.4}$$

Consider $m+2$ fixed real numbers $\lambda_0, \ldots, \lambda_{m+1}$ satisfying

$$\sum_{\alpha=0}^{q-1} (\lambda_{\alpha})^2 = \sum_{a=q}^{m+1} (\lambda_a)^2.$$

Then the foliation by hyperplanes

$$\sum_{i=0}^{m+1} \lambda_i x^i = c, \; c \in R$$

is a lightlike foliation on $\mathbb{R}^{m+2}_q$ with

$$\xi = -\sum_{\alpha=0}^{q-1} \lambda_\alpha \frac{\partial}{\partial x^\alpha} + \sum_{a=q}^{m+1} \lambda_a \frac{\partial}{\partial x^a}.$$

Example 1.2. Let $\mathbb{R}^{m+2}_1 = (\mathbb{R}^{m+2}, \tilde{g})$ be the $(m+2)$-dimensional Lorentz space with $\tilde{g}$ given by

$$\tilde{g}(x, y) = -x^0 y^0 + \sum_{a=1}^{m+1} x^a y^a. \tag{1.5}$$

Denote by L the x^0-axis of $\mathbb{R}^{m+2}_1$ and consider the open submanifold $\tilde{M} = \mathbb{R}^{m+2}_1 \backslash L$ of $\mathbb{R}^{m+2}_1$. Then denote by $\mathcal{F}^+$ and $\mathcal{F}^-$ the foliations on $\tilde{M}$ given by

$$x^0 = \left(\sum_{a=1}^{m+1} (x^a)^2 \right)^{\frac{1}{2}} + c, \quad c \in \mathbb{R} \tag{1.6}$$

and

$$x^0 = -\left(\sum_{a=1}^{m+1} (x^a)^2 \right)^{\frac{1}{2}} + c, \quad c \in \mathbb{R} \tag{1.7}$$

respectively. By using (v) of Theorem 1.1 it is easy to check that both $\mathcal{F}^+$ and $\mathcal{F}^-$ are lightlike foliations on $\mathbb{R}^{m+2}_1$. According to the terminology in physics under which leaves for $c = 0$ are known, we call $\mathcal{F}^+$ and $\mathcal{F}^-$ the *future cones foliation* and the *past cones foliation* respectively. In the cases of $\mathcal{F}^+$ and $\mathcal{F}^-$, the distributions $D^\perp$ are spanned by

$$\xi^+ = \frac{\partial}{\partial x^0} + \frac{1}{\alpha} \sum_{a=1}^{m+1} x^a \frac{\partial}{\partial x^a}, \tag{1.8}$$

and

$$\xi^- = \frac{\partial}{\partial x^0} - \frac{1}{\alpha} \sum_{a=1}^{m+1} x^a \frac{\partial}{\partial x^a}, \tag{1.9}$$

respectively, where we set

$$\alpha = \left(\sum_{a=1}^{m+1} (x^a)^2 \right)^{\frac{1}{2}}.$$

Example 1.3. Let $\tilde{M}$ be the $(m+1)$-dimensional submanifold of $\mathbb{R}_1^{m+2}$, situated in the half space $x^{m+1} > 0$ and given by the equation

$$x^{m+1} = \left(1 - \sum_{a=2}^{m} (x^a)^2\right)^{\frac{1}{2}}.$$

Consider the distribution D on $\tilde{M}$ spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}; \quad X_a = \frac{\partial}{\partial x^a} - \frac{x^a}{x^{m+1}} \frac{\partial}{\partial x^{m+1}}, \; a \in \{2, \ldots, m\}.$$

It is easy to see that D is an integrable distribution and $D^\perp$ is spanned by

$$\xi = \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}.$$

Hence $D^\perp$ is a vector subbundle of D, and by (ii) of Theorem 1.1 we conclude that D defines a lightlike foliation on $\tilde{M}$.

2. The null transversal bundle for a lightlike foliation

Let $\mathcal{F}$ be a lightlike foliation defined by the integrable distribution D on $\tilde{M}$. Since $D^\perp$ is a vector subbundle of D we may consider a complementary distribution $S(D)$ to $D^\perp$ in D. Thus we have the decomposition

$$D = S(D) \perp D^\perp, \tag{2.1}$$

where, here and in the sequel, the sign "$\perp$" between the vector bundles indicates that $S(D)$ and $D^\perp$ are complementary orthogonal vector subbundles of D. As $\tilde{M}$ is supposed to be paracompact, there exists such a distribution $S(D)$ on $\tilde{M}$. Moreover, it is easy to see that any distribution $S(D)$ satisfying (2.1) is non-degenerate with respect to $\tilde{g}$. Thus the tangent bundle of $\tilde{M}$ has the decomposition

$$T\tilde{M} = S(D) \perp S(D)^\perp, \tag{2.2}$$

where $S(D)^\perp$ is the vector bundle that is orthogonal to $S(D)$ in $T\tilde{M}$. We call $S(D)$ a *screen distribution* of the foliation $\mathcal{F}$.

The study of the geometry of a foliation is mainly based on the existence of a transversal distribution (cf. VAISMAN [6]), which in the case of a Riemannian foliation is orthogonal and complementary to the distribution that defines the foliation. As we have seen in Theorem 1.1, the distribution $D^\perp$ that is orthogonal to D lies in D, so it cannot be taken as transversal distribution to the lightlike foliation. However, we show in this section that a screen distribution induces a transversal distribution which is going to play an important role in studying the lightlike foliation. First, we prove the following

Theorem 2.1. *Let $S(D)$ be a screen distribution of the lightlike foliation $\mathcal{F}$ on $\tilde{M}$. Then for any nonzero section ξ of $D^\perp$ on a coordinate neighborhood $\mathcal{U} \subset \tilde{M}$, there exists on $\mathcal{U}$ a unique null section N of $S(D)^\perp$ satisfying*

$$\tilde{g}(N, \xi) = 1. \tag{2.3}$$

PROOF. $S(D)^\perp$ is a non-degenerate vector bundle of rank 2 since both $S(D)$ and $T\tilde{M}$ are non-degenerate vector bundles. $D^\perp$ is orthogonal to $S(D)$, so it is a null vector subbundle of $S(D)^\perp$. Now, consider a complementary vector bundle $\tilde{D}$ to $D^\perp$ in $S(D)^\perp$ and take a nonzero $V \in \Gamma(\tilde{D}_{|\mathcal{U}})$. Then we have $\tilde{g}(\xi, V) \neq 0$ on $\mathcal{U}$. Indeed, if at one point $x \in \mathcal{U}$ we have $\tilde{g}_x(\xi_x, V_x) = 0$, then $S(D)^\perp$ would be degenerate at x, which is a contradiction. Next, the vector field N we look for, is written as follows:

$$N = \alpha\xi + \beta V,$$

where α and β are smooth functions to be determined on $\mathcal{U}$. Taking into account that N is a null section of $S(D)^\perp$ satisfying (2.3), we deduce that

$$N = \frac{1}{\tilde{g}(\xi, V)} \left\{ V - \frac{\tilde{g}(V, V)}{2\tilde{g}(\xi, V)}\xi \right\}. \tag{2.4}$$

It is easy to check that when we change the complementary vector bundle $\tilde{D}$ we obtain the same N given by (2.4). Thus the section N is unique. □

Let E and F be two vector subbundles of $T\tilde{M}$ that are not orthogonal to each other with respect to $\tilde{g}$. Then their direct sum is denoted by $E \oplus F$.

Theorem 2.2. *Let $\mathcal{F}$ be a lightlike foliation on $\tilde{M}$ and $S(D)$ a screen distribution for $\mathcal{F}$. Then there exists a unique null line bundle* $\mathrm{tr}(D)$ *satisfying the conditions:*

(i) *The tangent bundle of $\tilde{M}$ has the decomposition*

$$T\tilde{M} = D \oplus \mathrm{tr}(D). \tag{2.5}$$

(ii) $\mathrm{tr}(D)$ *is orthogonal to $S(D)$ but it is not orthogonal to $D^{\perp}$.*

Proof. Locally, on a coordinate neighborhood $\mathcal{U} \subset \tilde{M}$ we define $\mathrm{tr}(D)_{|\mathcal{U}} = \mathrm{span}\{N\}$, where N is given by (2.4). Now consider another coordinate neighborhood $\mathcal{U}^* \subset \tilde{M}$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \phi$. Then, by Theorem 2.1, for $\xi^* \in \Gamma(D^{\perp}_{|\mathcal{U}^*})$ there exists an N^* given by a similar formula as (2.4). Taking into account that $\xi^* = f\xi$, where f is a smooth nonzero function on $\mathcal{U} \cap \mathcal{U}^*$, we deduce by direct calculation using (2.4) for both N and N^* that $N^* = (1/f)N$. Hence $\mathrm{span}\{N^*\} = \mathrm{span}\{N\}$. Next we put $D_x = (D^{\perp}_x)^{\perp}$ for any $x \in \tilde{M}$, that is, $\tilde{g}_x(X_x, \xi_x) = 0$ for any $X_x \in D_x$. Thus by (2.3) it follows that N_x does not lie in D_x. We have constructed a null line bundle $\mathrm{tr}(D)$ that is a bundle complementary (but not orthogonal) to D in $T\tilde{M}$. Moreover, (2.4) and (2.3) imply that $\mathrm{tr}(D)$ is orthogonal to $S(D)$ but it is not orthogonal to $D^{\perp}$. Hence $\mathrm{tr}(D)$ satisfies conditions (i) and (ii). Finally, suppose there exists another null line bundle $\mathrm{tr}(D)'$ satisfying (i) and (ii). Then, for any section N' of $\mathrm{tr}(D)'$ on $\mathcal{U} \subset \tilde{M}$, we have $\tilde{g}(N', \xi) = h \neq 0$ on $\mathcal{U}$. It follows that $\tilde{g}((1|h)N', \xi) = 1$, that is, $(1|h)N'$ satisfies the same conditions as N from Theorem 2.1. By the uniqueness of N we conclude that $(1|h)N' = N$. Hence $\mathrm{tr}(D)' = \mathrm{tr}(D)$, that is, $\mathrm{tr}(D)$ is the only null vector bundle satisfying (i) and (ii). □

In this way, for any screen distribution $S(D)$ we have a unique null line bundle $\mathrm{tr}(D)$ which is locally represented by the section N given by (2.4) and satisfies the conditions (i) and (ii) of Theorem 2.2. Since $\mathrm{tr}(D)$ is complementary to D in $T\tilde{M}$ we call it the *null transversal bundle* to the lightlike foliation $\mathcal{F}$ associated to the screen distribution $S(D)$. Also, by the construction we performed in Theorem 2.2 it follows that $\mathrm{tr}(D)$ is complementary (but not orthogonal) to $D^{\perp}$ in $S(D)^{\perp}$. Hence we have

$$S(D)^{\perp} = D^{\perp} \oplus \mathrm{tr}(D). \tag{2.6}$$

Finally, taking into account (2.1), (2.2), (2.5) and (2.6), we may decompose $T\tilde{M}$ as follows:

$$T\tilde{M} = D \oplus \mathrm{tr}(D) = S(D)\perp(D^{\perp} \oplus \mathrm{tr}(D)). \tag{2.7}$$

3. The second fundamental form of a lightlike foliation

Let $\mathcal{F}$ be a lightlike foliation on $\tilde{M}$ with screen distribution $S(D)$ and associate null transversal bundle $\mathrm{tr}(D)$. Denote by $\tilde{\nabla}$ the Levi–Civita connection on $\tilde{M}$. Then according to the decomposition (2.5) on the coordinate neighborhood $U \subset \tilde{M}$ we write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N, \quad \forall X,Y \in \Gamma(D_{|\mathcal{U}}), \tag{3.1}$$

where N is the local section of $\mathrm{tr}(D)$ given by (2.4) and $\nabla_X Y \in \Gamma(D_{|\mathcal{U}})$. It is easy to see that h is an $F(M)$-bilinear symmetric form on $\Gamma(D_{|\mathcal{U}}) \times \Gamma(D_{|\mathcal{U}})$. We call h the second fundamental form of the foliation $\mathcal{F}$ with respect to the null transversal bundle $\mathrm{tr}(D)$.

In order to study the dependence of h on both $S(D)$ and $\mathrm{tr}(D)$ we consider another screen distribution $S(D)'$ and denote by $\mathrm{tr}(D)'$ the associate null transversal bundle. Then on $\mathcal{U}$ we have

$$\tilde{\nabla}_X Y = \nabla'_X Y + h'(X,Y)N', \quad \forall X,Y \in \Gamma(D_{|\mathcal{U}}), \tag{3.2}$$

where N' is given by a similar formula as (2.4), but the construction is done in $S(D)'^{\perp}$. As both N' and N satisfy (2.3), from (3.1) and (3.2) we deduce that

$$h(X,Y) = h'(X,Y) = \tilde{g}(\tilde{\nabla}_X Y, \xi), \quad \forall X,Y \in \Gamma(D_{|\mathcal{U}}). \tag{3.3}$$

Therefore we may state the following result that plays an important role in studying the geometry of a lightlike foliation.

Theorem 3.1. *The second fundamental form of a lightlike foliation is independent of the choice of the screen distribution.*

However, from (3.3) it follows that $h(X, Y)$ depends upon the choice of ξ. More precisely if $\xi^* = f\xi$ is another local section of $D^\perp$ then we get

$$h^* = fh, \tag{3.4}$$

where h^* is the second fundamental form corresponding to ξ^* and f is a nonzero smooth function on $\mathcal{U}$.

In view of these results, we denote by h the second fundamental form with respect to $\mathrm{tr}(D)$ locally corresponding to $\xi \in \Gamma(D^\perp)$ and call it the *second fundamental form* of $\mathcal{F}$.

Now we put $Y = \xi$ in (3.3) and taking into account that ξ is a null vector field and $\tilde{g}$ is parallel with respect to $\tilde{\nabla}$, we obtain

$$h(X, \xi) = 0, \quad \forall X \in \Gamma(D). \tag{3.5}$$

Thus we may state the following

Proposition 3.1. *The second fundamental form of a lightlike foliation $\mathcal{F}$ is degenerate on D.*

When h vanishes identically on D, we say that the lightlike foliation $\mathcal{F}$ is *totally geodesic*. Also, we say that $\mathcal{F}$ is *totally umbilical* if on any coordinate neighborhood $\mathcal{U} \subset \tilde{M}$ there exists a smooth function ρ such that

$$h(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(D_{|\mathcal{U}}). \tag{3.6}$$

It is easy to see from (3.4) that the property that $\mathcal{F}$ is totally geodesic (totally umbilical) is independent of the choice of ξ.

Example 3.1. The lightlike foliation from Example 1.1 is totally geodesic because its leaves are lightlike hyperplanes which are totally geodesic immersed in $\mathbb{R}^{m+2}_q$ (cf. A. Bejancu [1]).

Example 3.2. Consider the foliation $\mathcal{F}^+$ from Example 1.2. Then the distribution D is spanned by the vector fields

$$X_a = \frac{\partial}{\partial x^a} + \frac{x^a}{\alpha}\frac{\partial}{\partial x^0}, \quad a \in \{1, \dots, m+1\}. \tag{3.7}$$

By direct calculation using (3.3), (3.7), (1.5) and (1.8) we obtain

$$h(X_a, X_b) = \tilde{g}\big(\tilde{\nabla}_{X_a} X_b, \xi^+\big) = \frac{1}{\alpha^3}(x^a x^b - \alpha^2 \delta_{ab}).$$

Also, we have

$$g(X_a, X_b) = \frac{1}{\alpha^2}(\alpha^2 \delta_{ab} - x^a x^b).$$

Thus the future cones foliation $\mathcal{F}^+$ is totally umbilical with $\rho = -\frac{1}{\alpha}$. Similarly, it follows that $\mathcal{F}^-$ is also totally umbilical with the same function ρ.

Finally, we state the following important result for lightlike foliations on manifolds of low dimension.

Theorem 3.2. *Let $\tilde{M}$ be a 3-dimensional Lorentz manifold. Then any lightlike foliation on $\tilde{M}$ is either totally geodesic or totally umbilical.*

Proof. Suppose that locally $D = \mathrm{span}\{E, \xi\}$ where ξ spans $D^\perp$. Then by (3.5) we have

$$h(\xi, \xi) = h(E, \xi) = 0.$$

As $\tilde{g}(\xi, \xi) = \tilde{g}(E, \xi) = 0$, we see that (3.6) is satisfied with $\rho = h(E, E)/g(E, E)$. Hence $\mathcal{F}$ is either totally geodesic or totally umbilical, depending on whether $h(E, E) = 0$ or $h(E, E) \neq 0$. □

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