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Classification of symmetric-like contact metric (k, μ)-spaces

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Dedicated to Professor L. Tamássy on the occasion of his eightieth birthday

Abstract. We determine the non-Sasakian contact metric (k, μ)-spaces which have volume-preserving geodesic symmetries up to sign (i.e., are D'Atri spaces) or which satisfy the condition that their Jacobi operators have constant eigenvalues or parallel eigenspaces along the corresponding geodesics, respectively (i.e., are $\mathfrak{C}$- or $\mathfrak{P}$-spaces, respectively).

1. Introduction

Locally symmetric spaces have a lot of interesting geometric properties. In particular, they are *D'Atri spaces* (i.e., their local geodesic symmetries are volume-preserving up to sign) and they are also $\mathfrak{C}$- and $\mathfrak{P}$-*spaces* (i.e., their Jacobi operators have, respectively, constant eigenvalues and parallel eigenspaces along the corresponding geodesics). D'Atri spaces have been introduced in [11] while the study of $\mathfrak{C}$- and $\mathfrak{P}$-spaces goes back to [3]. Since then, these classes of symmetric-like spaces [1] have been studied extensively. A number of geometric properties have been derived and many non-trivial (i.e., non-symmetric) examples are found. On the other hand, several problems still remain unsolved. It is intriguing

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that only locally homogeneous D'Atri and $\mathfrak{C}$-spaces are known and it is yet unknown whether local homogeneity holds in general for these two classes. We refer to the survey papers [2], [4] and [12] for more details and further information.

The main purpose of this paper is to study these spaces in the framework of contact geometry and in particular for a special class of contact metric spaces where we have good knowledge of the curvature tensor which is, as is well-known, needed for the analytic treatment of the three types of spaces. These contact metric spaces are the so-called (k, μ)-*spaces* which are introduced in [7] and which are characterized as contact metric spaces satisfying the curvature condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)$$

where k, μ are constants and $2h$ is the Lie derivative of the structure tensor ϕ in the direction of the unit characteristic vector ξ. For a contact metric structure (ϕ, ξ, η, g), the contact form η is the metric dual one-form of ξ. Sasakian spaces $(k = 1, h = 0)$ are trivial examples. In [7], non-Sasakian examples are provided by the unit tangent sphere bundles of spaces of constant curvature $c \neq 1$ (for $c = 1$, we get a Sasakian unit sphere bundle [16]) and from there, new examples are derived by means of D-homothetic transformations [15] since the above curvature condition is invariant under such transformations of the contact metric structure. Furthermore, in the same paper, a classification of non-Sasakian three-dimensional (k, μ)-spaces is given. They are locally isometric to some Lie groups. In [8], it was proved that this is not a surprising fact because all non-Sasakian (k, μ)-spaces are locally homogeneous. Moreover, a classification of these spaces has been derived in [9] where also new examples, not belonging to the former classes, are discovered.

In [13], [14], it was proved that locally symmetric Sasakian spaces have constant curvature 1. Furthermore, in [10], the first author showed that locally symmetric non-Sasakian (k, μ)-spaces of dimension $2n + 1$ are locally isometric to the product of a flat $(n + 1)$-dimensional space and an n-dimensional manifold of constant curvature 4. Thus local symmetry is a rather strong condition and hence, it is natural to consider the weaker conditions imposed by the defining ones for the D'Atri, $\mathfrak{C}$- and $\mathfrak{P}$-spaces, respectively. In this paper, we treat this problem and in particular, we shall prove the following results:

Theorem A. *Let M be a non-Sasakian contact (k, μ)-space. Then M is a D'Atri- or a $\mathfrak{C}$-space if and only if it is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of constant curvature 4, or it is 3-dimensional and locally isometric to a unimodular Lie group $SU(2)(\mu < 0)$, $SL(2, \mathbb{R})(\mu > 0)$ or the group $E(2)(\mu = 0)$ of rigid motions of the Euclidean 2-space, each with a special left-invariant metric.*

Theorem B. *Let M be a non-Sasakian contact (k, μ)-space. Then M is a $\mathfrak{P}$-space if and only if it is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of constant curvature 4 or is locally flat if* $\dim M = 3$.

2. Preliminaries

We start by collecting some basic material about contact metric geometry and refer to [5], [6] for further details. All manifolds in the present paper are assumed to be connected and of class C^∞.

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be *a contact manifold* if it admits a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exists an associated Riemannian metric g and a $(1,1)$-type tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \quad (2.1)$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\phi\xi = 0, \ \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \qquad (2.2)$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact metric manifold M, we consider the $(1,1)$-type tensor field h given by $h = \frac{1}{2}L_\xi\phi$, where L denotes the Lie differentiation. The tensor h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \tag{2.3}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$

where ∇ is the Levi Civita connection. From (2.3) and (2.4), we see that each trajectory of ξ is a geodesic.

A contact Riemannian manifold for which ξ is a Killing vector field, is called a *K-contact manifold*. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, $(M;\eta,g)$ is said to be *normal* or *Sasakian*. It is known that M is normal if and only if M satisfies

$$[\phi,\phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is also characterized by the condition

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.5}$$

for all vector fields X and Y on the manifold. Moreover, if we denote by R the Riemannian curvature tensor of M defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z on M, then it follows that M is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.6}$$

for all vector fields X and Y.

Note that for a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as the direct sum $T_pM = D_p \oplus \{\xi\}$, where $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution which is orthogonal to ξ. This $2n$-dimensional distribution D is called the *contact distribution*.

The following useful result is proved in [5], [6].

Theorem 2.1. *Let $M = (M; \eta, g)$ be a $(2n+1)$-dimensional contact Riemannian manifold and suppose that $R(X, Y)\xi = 0$ for all vector fields X, Y on M. Then M is locally the product of an $(n+1)$-dimensional flat manifold and an n-dimensional manifold of constant curvature 4 for $n > 1$ and it is flat for $n = 1$.*

Next, we consider the (k, μ)-spaces. A contact metric space $(M; \eta, g)$ is said to be a (k, μ)-*space* [7] if the curvature tensor satisfies

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

where k, μ are constant. Furthermore, on a (k, μ)-space we have

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{2.8}$$

and

$$\begin{aligned}(\nabla_Z h)X = &\big\{(1-k)g(Z, \phi X) + g(Z, h\phi X)\big\}\xi \\ &+ \eta(X)(h\phi + h\phi h)Z - \mu\eta(Z)\phi hX\end{aligned} \tag{2.9}$$

for all vector fields X, Z on M. In [7], it is also proved that $k \leq 1$ and that $(M; \eta, g)$ is Sasakian if and only if $k = 1$ (since then $h = 0$). Moreover, if $k < 1$, then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where for the eigenvalue λ we have $\lambda = \sqrt{1-k}$. As concerns the Ricci operator Q of a (k, μ)-space, we find the following result in [7]:

Theorem 2.2. *The Ricci operator Q of a non-Sasakian contact (k, μ)-space is given by*

$$\begin{aligned}Q = &\{2(n-1) - n\mu\}I + \{2(n-1) + \mu\}h \\ &+ \{2(1-n) + n(2k+\mu)\}\eta \otimes \xi.\end{aligned} \tag{2.10}$$

Based on the results in [7], the following explicit expression for the curvature tensor R is derived in [8].

Theorem 2.3. *Let $M = (M^{2n+1}; \eta, g)$ be a non-Sasakian contact (k, μ)-space. Then its Riemannian curvature tensor R is given explicitly by*

$$\begin{aligned}
R(X,Y)Z = &\left(1-\frac{\mu}{2}\right)(g(Y,Z)X - g(X,Z)Y)\\
&+ g(Y,Z)hX - g(X,Z)hY - g(hX,Z)Y + g(hY,Z)X\\
&+ \frac{1-(\mu/2)}{1-k}(g(hY,Z)hX - g(hX,Z)hY)\\
&- \frac{\mu}{2}(g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y)\\
&+ \frac{k-(\mu/2)}{1-k}(g(\phi hY,Z)\phi hX - g(\phi hX,Z)\phi hY)\\
&+ \mu g(\phi X,Y)\phi Z\\
&+ \eta(X)((k-1+(\mu/2))g(Y,Z) + (\mu-1)g(hY,Z))\xi\\
&- \eta(Y)((k-1+(\mu/2))g(X,Z) + (\mu-1)g(hX,Z))\xi\\
&- \eta(X)\eta(Z)((k-1+(\mu/2))Y + (\mu-1)hY)\\
&+ \eta(Y)\eta(Z)((k-1+(\mu/2))X + (\mu-1)hX). \qquad (2.11)
\end{aligned}$$

Finally, we recall some facts about the curvature for the three special classes of Riemannian manifolds we have mentioned in the Introduction. First, we note that when (M,g) is a $\mathfrak{P}$-space, then R_x and R'_x are simultaneously diagonalizable, where $R_x = R(\cdot,x)x$ is the Jacobi operator corresponding to the vector x and $R'_x = (\nabla_x R)(\cdot,x)x$ [3]. When (M,g) is a D'Atri space or a $\mathfrak{C}$-space, then the Ricci tensor ρ of type $(0,2)$ and the curvature tensor R satisfy the so-called Ledger conditions L_3 and L_5 of order three and five (see [3] and [12], for example):

$$L_3 : (\nabla_X \rho)(X,X) = 0,$$

$$L_5 : \sum_{a,b} g(R(e_a,X,X),e_b)g((\nabla_X R)(e_a,X)X,e_b) = 0$$

where $\{e_a,\ a = 1,\ldots,\dim M\}$ is a local orthonormal frame field on (M,g).

3. Proof of Theorem A

We now turn to the proof of Theorem A and put $R_{XYZW} = g(R(X,Y)Z,W)$, $\nabla_V R_{XYZW} = g((\nabla_V R)(X,Y)Z,W)$, $\rho_{XY} = \rho(X,Y)$ and $\nabla_V \rho_{XY} = (\nabla_V \rho)(X,Y)$.

Let M be a non-Sasakian (k,μ)-space. Then, from (2.4), (2.9) and (2.10), we obtain

$$\begin{aligned}\nabla_V \rho_{XY} &= [2(n-1)+\mu][(1-k)g(V,\phi X)\eta(Y) - g(V,\phi hX)\eta(Y)\\ &\quad - \eta(X)g(\phi hV + \phi h^2 V, Y) - \mu\eta(V)g(\phi hX, Y)] \qquad (3.1)\\ &\quad - [2(1-n)+n(2k+\mu)][g(\phi V + \phi hV, X)\eta(Y) + g(\phi V + \phi hV, Y)\eta(X)].\end{aligned}$$

From (3.1), we easily see that M satisfies the Ledger condition of order three, i.e., $\nabla_X \rho_{XX} = 0$ for any vector field X on M, if and only if (k,μ) satisfies

$$\frac{1}{n}\mu^2 - 4\lambda^2 + 4\mu + 4 = 0. \qquad (3.2)$$

Also, we have from (2.11)

$$\begin{aligned}(\nabla_V R)(X,Y)Z &= g(Y,Z)(\nabla_V h)X - g(X,Z)(\nabla_V h)Y\\ &- g((\nabla_V h)X,Z)Y + g((\nabla_V h)Y,Z)X\\ &+ \frac{1-(\mu/2)}{1-k}(g((\nabla_V h)Y,Z)hX + g(hY,Z)(\nabla_V h)X\\ &- g((\nabla_V h)X,Z)hY - g(hX,Z)(\nabla_V h)Y)\\ &- \frac{\mu}{2}(g((\nabla_V \phi)Y,Z)\phi X + g(\phi Y,Z)(\nabla_V \phi)X\\ &- g((\nabla_V \phi)X,Z)\phi Y - g(\phi X,Z)(\nabla_V \phi)Y)\\ &+ \frac{k-(\mu/2)}{1-k}(g((\nabla_V \phi h)Y,Z)\phi hX + g(\phi hY,Z)(\nabla_V \phi h)X\\ &- g((\nabla_V \phi h)X,Z)\phi hY - g(\phi hX,Z)(\nabla_V \phi h)Y)\\ &+ \mu(g((\nabla_V \phi)X,Y)\phi Z + g(\phi X,Y)(\nabla_V \phi)Z)\\ &+ (\nabla_V \eta)(X)((k-1+(\mu/2))g(Y,Z) + (\mu-1)g(hY,Z))\xi\\ &+ \eta(X)(\mu-1)g((\nabla_V h)Y,Z)\xi\end{aligned}$$

$$
\begin{aligned}
&+ \eta(X)((k-1+(\mu/2))g(Y,Z)+(\mu-1)g(hY,Z))\nabla_V\xi \\
&- (\nabla_V\eta)(Y)((k-1+(\mu/2))g(X,Z)+(\mu-1)g(hX,Z))\xi \\
&- \eta(Y)(\mu-1)g((\nabla_V h)X,Z)\xi \\
&- \eta(Y)((k-1+(\mu/2))g(X,Z)+(\mu-1)g(hX,Z))\nabla_V\xi \\
&- (\nabla_V\eta)(X)\eta(Z)((k-1+(\mu/2))Y+(\mu-1)hY) \\
&- \eta(X)(\nabla_V\eta)(Z)((k-1+(\mu/2))Y+(\mu-1)hY) \\
&- \eta(X)\eta(Z)(\mu-1)(\nabla_V h)Y \\
&+ (\nabla_V\eta)(Y)\eta(Z)((k-1+(\mu/2))X+(\mu-1)hX) \\
&+ \eta(Y)(\nabla_V\eta)(Z)((k-1+(\mu/2))X+(\mu-1)hX) \\
&+ \eta(Y)\eta(Z)(\mu-1)(\nabla_V h)X. \qquad (3.3)
\end{aligned}
$$

Next, we shall take into account the Ledger condition L_5. Then we see, by means of a linearization procedure, that M satisfies

$$
\begin{aligned}
\sum_{a,b=1}^{2n+1} \big\{ &(R_{aXYb}+R_{aYXb})[\mathfrak{S}_{Z,W,V}\nabla_Z R_{aWVb}+\mathfrak{S}_{Z,V,W}\nabla_Z R_{aVWb}] \\
&+ (R_{aXZb}+R_{aZXb})[\mathfrak{S}_{Y,W,V}\nabla_Y R_{aWVb}+\mathfrak{S}_{Y,V,W}\nabla_Y R_{aVWb}] \\
&+ (R_{aXWb}+R_{aWXb})[\mathfrak{S}_{Y,Z,V}\nabla_Y R_{aZVb}+\mathfrak{S}_{Y,V,Z}\nabla_Y R_{aVZb}] \\
&+ (R_{aXVb}+R_{aVXb})[\mathfrak{S}_{Y,Z,W}\nabla_Y R_{aZWb}+\mathfrak{S}_{Y,W,Z}\nabla_Y R_{aWZb}] \\
&+ (R_{aYZb}+R_{aZYb})[\mathfrak{S}_{X,V,W}\nabla_X R_{aVWb}+\mathfrak{S}_{X,W,V}\nabla_X R_{aWVb}] \qquad (3.4) \\
&+ (R_{aYWb}+R_{aWYb})[\mathfrak{S}_{X,V,Z}\nabla_X R_{aVZb}+\mathfrak{S}_{X,Z,V}\nabla_X R_{aZVb}] \\
&+ (R_{aYVb}+R_{aVYb})[\mathfrak{S}_{X,Z,W}\nabla_X R_{aZWb}+\mathfrak{S}_{X,W,Z}\nabla_X R_{aWZb}] \\
&+ (R_{aZWb}+R_{aWZb})[\mathfrak{S}_{X,Y,V}\nabla_X R_{aYVb}+\mathfrak{S}_{X,V,Y}\nabla_X R_{aVYb}] \\
&+ (R_{aZVb}+R_{aVZb})[\mathfrak{S}_{X,Y,W}\nabla_X R_{aYWb}+\mathfrak{S}_{X,W,Y}\nabla_X R_{aWYb}] \\
&+ (R_{aWVb}+R_{aVWb})[\mathfrak{S}_{X,Y,Z}\nabla_X R_{aYZb}+\mathfrak{S}_{X,Z,Y}\nabla_X R_{aZYb}]\big\} = 0
\end{aligned}
$$

for all vector fields V, W, X, Y and Z on M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z.

In (3.4) we now put $X = Y = Z = \xi$, $W = \phi V$, and assume that $hV = \lambda V$, $\|V\| = 1$. Then we have the following :

$$\begin{aligned}\sum_{a,b=1}^{2n+1} \big\{ & R_{a\xi\xi b}(\nabla_\xi R_{a(\phi V)Vb} + \nabla_{\phi V} R_{aV\xi b} + \nabla_V R_{a\xi(\phi V)b} \\ & + \nabla_\xi R_{aV(\phi V)b} + \nabla_{\phi V} R_{a\xi Vb} + \nabla_V R_{a(\phi V)\xi b}) \\ & + (R_{a\xi(\phi V)b} + R_{a(\phi V)\xi b})(\nabla_\xi R_{a\xi Vb} + \nabla_\xi R_{aV\xi b} + \nabla_V R_{a\xi\xi b}) \\ & + (R_{a\xi Vb} + R_{aV\xi b})(\nabla_\xi R_{a\xi(\phi V)b} + \nabla_\xi R_{a(\phi V)\xi b} + \nabla_{\phi V} R_{a\xi\xi b}) \\ & + (R_{a(\phi V)Vb} + R_{aV(\phi V)b})\nabla_\xi R_{a\xi\xi b}\big\} = 0.\end{aligned} \tag{3.5}$$

Using (2.3), (2.4), (2.7), (2.11), (3.2), (3.3) and the fundamental properties of the curvature tensor R, lengthy but routine computations yield

$$\begin{aligned}\sum_{a,b=1}^{2n+1} & R_{a\xi\xi b}(\nabla_\xi R_{a(\phi V)Vb} + \nabla_{\phi V} R_{aV\xi b} + \nabla_V R_{a\xi(\phi V)b} + \nabla_\xi R_{aV(\phi V)b} \\ & + \nabla_{\phi V} R_{a\xi Vb} + \nabla_V R_{a(\phi V)\xi b}) = -2k\nabla_\xi R_{\xi(\phi V)V\xi} \\ & + 2\mu \sum_{a=1}^{2n+1} (\nabla_\xi R_{a(\phi V)V(he_a)} + \nabla_{\phi V} R_{aV\xi(he_a)} + \nabla_V R_{a\xi(\phi V)(he_a)}) \\ = {} & 2k\lambda\mu^2 + 2\mu\Big(-2(n-1)\lambda^3\mu + 4\lambda^3 + 2\lambda^3(2n(1-\mu) + (\mu - 2)) \\ & - 2\lambda(\mu(2-n) + 2n)\Big).\end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\sum_{a,b=1}^{2n+1} & (R_{a\xi(\phi V)b} + R_{a(\phi V)\xi b})(\nabla_\xi R_{a\xi Vb} + \nabla_\xi R_{aV\xi b} + \nabla_V R_{a\xi\xi b}) \\ & = -2(k - \lambda\mu)(\lambda\mu^2),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\sum_{a,b=1}^{2n+1} & (R_{a\xi Vb} + R_{aV\xi b})(\nabla_\xi R_{a\xi(\phi V)b} + \nabla_\xi R_{a(\phi V)\xi b} + \nabla_{\phi V} R_{a\xi\xi b}) \\ & = -2(k + \lambda\mu)(\lambda\mu^2),\end{aligned} \tag{3.8}$$

and

$$\sum_{a,b=1}^{2n+1} (R_{a(\phi V)Vb} + R_{aV(\phi V)b})\nabla_\xi R_{a\xi\xi b} = -\mu^2\big(3\lambda\mu + (2n-1)\lambda(2k-\mu)\big). \tag{3.9}$$

Summing up (3.6)–(3.8) and (3.9), we obtain the condition

$$-2(1-\lambda^2)\lambda\mu^2 + 2\mu\big(-2(n-1)\lambda^3\mu + 4\lambda^3 + 2\lambda^3(2n(1-\mu)+(\mu-2)) -2\lambda(\mu(2-n)+2n)\big) - \mu^2\big(3\lambda\mu + (2n-1)\lambda(2k-\mu)\big) = 0, \tag{3.10}$$

where we have used $k = 1-\lambda^2$.

Now, we note that the conic, given by (3.2), is decomposed into two factors $\mu+2+2\lambda$ and $\mu+2-2\lambda$ only in dimension three. For that reason, we divide our further arguments into two cases: (i) $n=1$, (ii) $n>1$. Solving the non-linear system given by (3.2) and (3.10), this leads to

(i) $n=1$, $\lambda = \pm\frac{1}{2}(\mu+2)$. Due to the classification table of three-dimensional (k,μ)-spaces (see [7]), we conclude that M is locally isometric to a unimodular Lie group $SU(2)(\mu<0)$, $SL(2,\mathbb{R})(\mu>0)$ or the (flat) group $E(2)(\mu=0)$ of rigid motions of Euclidean 2-space, each with a special left-invariant metric. Conversely, it is known that the above unimodular Lie groups appear in the classification of three-dimensional $\mathfrak{C}$-spaces (or equivalently, D'Atri spaces) given in [3]

(ii) $n>1$, $\lambda=\pm 1$, $\mu=0$. Then $k=\mu=0$, i.e., $R(X,Y)\xi = 0$. Hence, by Theorem 2.1, M is locally the product of an $(n+1)$-dimensional flat manifold and an n-dimensional manifold of constant curvature 4. The converse is trivial since the product is symmetric.

This concludes the proof of Theorem A.

4. Proof of Theorem B

In this section, we prove Theorem B. Let M be a contact (k,μ)-space and suppose that M is of $\mathfrak{P}$-type. Then the Jacobi operator $R_x = R(\cdot,x)x$

and its covariant differential operator $R'_x = (\nabla_x R)(\cdot, x)x$ are simultaneously diagonalizable for all x. First, from (2.7) it follows immediately that

$$R(\cdot, \xi)\xi = k(I - \eta \otimes \xi) + \mu h$$

and since from (2.9) we get $\nabla_\xi h = \mu h\phi$, we obtain

$$(\nabla_\xi R)(\cdot, \xi)\xi = \mu^2 h\phi.$$

From these relations we then derive

$$\begin{aligned} R_\xi \cdot R'_\xi &= k\mu^2 h\phi + \mu^3 h^2 \phi, \\ R'_\xi \cdot R_\xi &= k\mu^2 h\phi + \mu^3 h\phi h. \end{aligned} \tag{4.1}$$

Since M is a non-Sasakian $\mathfrak{P}$-space, from (4.1), we obtain $\mu = 0$. Furthermore, for $V \in D(\lambda)$ ($\|V\| = 1$) and for any vector field X tangent to M, we have from (2.11)

$$\begin{aligned} R_V X = R(X, V)V &= (1 + \lambda)X - (1 + 2\lambda)g(X, V)V + hX \\ &+ \frac{1}{1-k}(\lambda hX - \lambda^2 g(X, V)V) \\ &+ \frac{k}{1-k}\lambda^2 g(\varphi X, V)\phi V + (k - 1 - \lambda)\eta(X)\xi. \end{aligned} \tag{4.2}$$

From this, it then follows that

$$R_V \xi = R(\xi, V)V = k\xi. \tag{4.3}$$

We easily get from (2.9) that $(\nabla_V h)\xi = (k - 1 - \lambda)\phi V$ where $\lambda = \sqrt{1-k}$, and making use of this, we have from (3.3):

$$R'_V \xi = (\nabla_V R)(\xi, V)V = -2k(\lambda + 1)\phi V. \tag{4.4}$$

Therefore, from (4.3) and (4.4), we obtain

$$R'_V(R_V \xi) = -2k^2(\lambda + 1)\phi V. \tag{4.5}$$

On the other hand, from (4.2) and (4.4), we have

$$R_V(R'_V \xi) = 2k^2(\lambda + 1)\varphi V. \tag{4.6}$$

Since M is a $\mathfrak{P}$-space, from (4.5) and (4.6), we deduce $k = 0$. Thus, we have proved Theorem B.

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