Publ. Math. Debrecen
62/3-4 (2003), 363–377

On functional equations involving means

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Dedicated to the 80th birthday of Professor Lajos Tamássy

Abstract. The main results of the paper concern functional equations of the form

$$f(M(x,y))(g(y)-g(x)) = \mu(f(x)g(y)-f(y)g(x)) \qquad (x,y \in I),$$

where f and g are continuous functions defined on an open interval I and M is a strict two variable mean on I. As an application, a generalization of the so-called Matkowski–Sutô problem for weighted two variable quasi-arithmetic means is solved under first-order continuous differentiability assumptions.

1. Introduction

Let $I \subset \mathbb{R}$ be a nonvoid open interval. We say that a function $M : I^2 \to I$ is a *pre-mean* on I if

$$\min\{x,y\} \le M(x,y) \le \max\{x,y\} \tag{1}$$

Mathematics Subject Classification: 39B22, 39B12, 26A18.
Key words and phrases: functional equation, weighted quasi-arithmetic mean, Stolarsky means, Matkowski–Sutô problem.
This research has been supported by the Hungarian National Research Science Foundation (OTKA) Grant T-0300082 and by the Higher Education, Research, and Development Fund (FKFP) Grant 0310/1997.

for all $x, y \in I$. If M is a pre-mean on I and the inequalities in (1) are strict for all $x \neq y$ $(x, y \in I)$ then M is called a *strict pre-mean* on I. If M is a pre-mean on I and continuous on I^2 then M is said to be a *mean* on I. Finally, M is called a *strict mean* on I if M is a strict pre-mean on I and continuous on I^2.

In this paper we deal with functional equations of the form

$$f(M(x,y)) = N(f(x), f(y)) \quad \text{if } f(x) \neq f(y) \qquad (x, y \in I) \tag{2}$$

and

$$f(M(x,y))(g(y) - g(x)) = \mu(f(x)g(y) - f(y)g(x)) \qquad (x, y \in I) \tag{3}$$

under the following conditions:

(A) In (2) M is a strict mean on I, N is strict pre-mean on $J(\subset \mathbb{R}$ is a nonvoid open interval) and the unknown function $f : I \to J$ is continuous;

(B) In (3) M is a mean (or a strict mean) on I and the unknown functions $f, g : I \to \mathbb{R}_+$ are continuous (here and in the sequel $\mathbb{R}_+$ denotes the set of positive real numbers).

As an application, we shall consider the Matkowski–Sutô type problem

$$\lambda x + (1-\lambda) y = \mu M(x,y) + (1-\mu) N(x,y) \qquad (x, y \in I) \tag{4}$$

where $0 < \lambda < 1$ $\mu \neq 0, 1$ are constants, M and N are weighted quasi-arithmetic means on I with weight λ. The case $\lambda = \mu = \frac{1}{2}$ is the original Matkowski–Sutô problem (cf. [1], [2], [4], [3], [7], [13], [14]). In [5], the case $\lambda = \mu$ was treated. The aim of this paper is to solve the above problem if the *generator functions* of M and N are continuously differentiable with nonvanishing derivatives on I. Our results offer the same set of solutions that was found by Głazowska, Jarczyk, and Matkowski [8] under twice continuous differentiability assumptions in the case $\lambda = 1/2$.

2. Injectivity Theorem

The following result, concerning the functional equation (2) is an injectivity theorem of independent interest.

Theorem 1. *Let $I, J \subset \mathbb{R}$ be nonvoid open intervals and let a strict mean $M : I^2 \to I$ and a strict pre-mean $N : J^2 \to J$ be given. If $f : I \to J$ is a continuous and nonconstant solution of the (conditional) functional equation*

$$f(M(x,y)) = N(f(x), f(y)) \quad \text{if } f(x) \neq f(y) \qquad (x, y \in I) \tag{2}$$

then f is injective on I.

Proof. Suppose that $f : I \to J$ is a *continuous* solution of (2) which is *nonconstant* on I and *not injective*. Since f is continuous, there exist $a < b$ $(a, b \in I)$ such that $f(a) = f(b) =: k \in J$ and $f(x) \neq k$ if $x \in \;]\,a, b\,[$. We show that there *exist* numbers $A, B \in [a, b]$ such that

$$M(a, B) = A \quad \text{and} \quad M(A, b) = B \tag{5}$$

hold. For, consider the function

$$\varphi(x, y) := (M(a, y), M(x, b)) \qquad (x, y \in [a, b]).$$

The function M being continuous, $\varphi : [a, b]^2 \to [a, b]^2$ is a continuous self-mapping of $[a, b]^2$, hence, by Brouwer's Fixed Point Theorem, it has a fixed point $(A, B) \in [a, b]^2$, that is,

$$\varphi(A, B) = (A, B).$$

This means that the equations (5) hold. Since M is a strict mean, we necessarily have

$$a < A < B < b.$$

This implies $k = f(a) \neq f(B)$ and $k = f(b) \neq f(A)$, and by (5) and (2),

$$f(A) = f(M(a, B)) = N(f(a), f(B)) = N(k, f(B))$$

and

$$f(B) = f(M(A, b)) = N(f(A), f(b)) = N(f(A), k).$$

Since N is a strict pre-mean, the previous equations yield that $f(A)$ belongs to the open interval joining k and $f(B)$, and similarly, $f(B)$ belongs to the open interval joining $f(A)$ and k. Since $f(x) \neq k$ if $x \in \;]\,a, b\,[$, we have reached a contradiction. Therefore, if f is a continuous and nonconstant solution of (2) then f is injective on I. $\square$

3. Equation (3)

The following result is of basic importance.

Theorem 2. *Let $M : I^2 \to I$ be a mean on I and let $\mu \neq 0, 1$ be a real constant. If $f, g : I \to \mathbb{R}_+$ are continuous solutions of the functional equation*

$$f(M(x,y))(g(y)-g(x)) = \mu(f(x)g(y)-f(y)g(x)) \quad (x, y \in I) \tag{3}$$

then there exists $c \in \mathbb{R}_+$ such that

$$f(x)^\mu g(x)^{1-\mu} = c \tag{6}$$

for all $x \in I$.

Proof. If $g(x) = g(y)$ (> 0) then (3) implies $f(x) = f(y)$, thus there exists a function $F : g(I) \to \mathbb{R}_+$ such that

$$f(x) = F(g(x)) \qquad (x \in I). \tag{7}$$

If g is constant on I then f is constant on I, too, and clearly (6) holds. Therefore we can assume that g is *nonconstant* on I. Since g is continuous, then $J := g(I)$ is a proper interval and $J \subset \mathbb{R}_+$.

In what follows, we intend to show that the function

$$F : J \to \mathbb{R}_+$$

is *differentiable* on J.

Let $u \in J$ and let $\{u_n \mid n \in \mathbb{N}\} \subset J$ be a sequence such that $u_n \to u$ from the left $(u_n < u)$ (or from the right $u_n > u$). It is sufficient to show that

$$\frac{F(u_n) - F(u)}{u_n - u}$$

tends to the same limit $(n \to \infty)$ depending only on u.

Let

$$u_0 := \inf\{u_n \mid n \in \mathbb{N}\} = \min\{u_n \mid n \in \mathbb{N}\}.$$

Then there exist $x_0, x^* \in I$ such that $g(x_0) = u_0$ and $g(x^*) = u$. We may assume that $x_0 < x^*$, the other case can be handled similarly.

Let

$$H := \{t \in I \mid x_0 \leq t \leq x^* \text{ and } g(t) = u\}.$$

Then H is a *closed* set, and since $x^* \in H$, H is *not empty*. Denote

$$x := \inf H,$$

then we clearly have that $x_0 < x$. Since g is continuous, $g(x) = u$ and if $x_0 \leq t < x$ then $g(t) \neq u$. The function g takes each value between u_0 and u on the closed interval $[x_0, x]$, therefore there *exists* a sequence $x_n \in [x_0, x[$ such that $g(x_n) = u_n$ $(n \in \mathbb{N})$. We show that $x_n \to x$ as $n \to \infty$. If this were not so then there would be a subsequence (x_{n_k}) $(n_1 < n_2 < n_3 < \dots)$ converging to $\bar{x} \neq x$, and from this we would have $\bar{x} < x$. Since g is continuous, $g(x_{n_k}) \to g(\bar{x})$ $(k \to \infty)$ and $g(x_{n_k}) = u_{n_k} \to u = g(x)$ $(k \to \infty)$. These would imply $g(\bar{x}) = g(x)$, contradicting the definition of x. Therefore $x_n \to x$ as $n \to \infty$ indeed.

The previous and equation (3) imply

$$\begin{aligned} f(M(x_n, x)) &= \mu \frac{f(x_n)g(x) - f(x)g(x_n)}{g(x) - g(x_n)} \\ &= \mu \frac{F(u_n)u - F(u)u_n}{u - u_n} = \mu \left(-u \frac{F(u_n) - F(u)}{u_n - u} + F(u) \right). \end{aligned}$$

Since f is continuous and M is a mean, we have that

$$\lim_{n \to \infty} f(M(x_n, x)) = f(x) = F(g(x)) = F(u).$$

Thus, by the previous equation, the limit

$$\lim_{n \to \infty} \frac{F(u_n) - F(u)}{u_n - u} = F'(u)$$

exists, too, and

$$F(u) = \mu(-uF'(u) + F(u))$$

holds for all $u \in J$. Since $\mu \neq 0, 1$, from this we obtain

$$\left(\ln F(u) - \frac{\mu - 1}{\mu} \ln u \right)' = 0$$

for all $u \in J$, that is, there exists a constant $q > 0$ satisfying

$$F(u) = qu^{\frac{\mu-1}{\mu}} \qquad (u \in J).$$

By (7), this implies the assertion of the theorem. □

In view of Theorem 2, the investigation of the functional equation (3) reduces to the discussion of a functional equation involving only one unknown function. The following results deal with this reduced problem.

Corollary 1. *Let $M : I^2 \to I$ be a mean on I and let $\mu \neq 0, 1$ be a real constant. If $f, g : I \to \mathbb{R}_+$ are continuous solutions of the functional equation* (3) *then $f : I \to \mathbb{R}_+$ is a continuous solution of the functional equation*

$$f(M(x,y))\Big(f(x)^{\frac{\mu}{1-\mu}} - f(y)^{\frac{\mu}{1-\mu}}\Big) = \mu\Big(f(x)^{\frac{\mu}{1-\mu}+1} - f(y)^{\frac{\mu}{1-\mu}+1}\Big) \qquad (x, y \in I). \tag{8}$$

Theorem 3. *Let $M : I^2 \to I$ be a strict mean and $\mu \neq 0, 1$. If $f : I \to \mathbb{R}_+$ is a continuous solution of the functional equation* (8) *then either there exists $c > 0$ such that*

$$f(x) = c \quad \text{if } x \in I, \tag{9}$$

or, f is injective on I and

$$f(M(f^{-1}(u), f^{-1}(v))) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(u, v) \tag{10}$$

for all $u \neq v$, $u, v \in f(I) =: J \subset \mathbb{R}_+$. Here $S_{a,b} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the Stolarsky mean on $\mathbb{R}_+$ of parameters a, b defined in the case $ab(a-b) \neq 0$ by

$$S_{a,b}(u,v) := \begin{cases} \left(\dfrac{b(u^a - v^a)}{a(u^b - v^b)}\right)^{\frac{1}{a-b}} & \text{if } u \neq v \\ u & \text{if } u = v \end{cases} \qquad (u, v \in \mathbb{R}_+). \tag{11}$$

(cf. [11], [12]*).*

Proof. Under these conditions (9) always solves (8), thus we can suppose that f is a nonconstant continuous solution of (8). Then if $f(x) \neq$

$f(y)$ $x,y \in I$, (8) yields

$$f(M(x,y)) = \mu \frac{f(x)^{\frac{\mu}{1-\mu}+1} - f(y)^{\frac{\mu}{1-\mu}+1}}{f(x)^{\frac{\mu}{1-\mu}} - f(y)^{\frac{\mu}{1-\mu}}} = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(f(x), f(y)). \quad (12)$$

Since the Stolarsky means defined by (11) are strict, it follows from Theorem 1 that f is injective. Thus, for all $u,v \in f(I) =: J$ with $u \neq v$, (12) implies (10). □

In the sequel we shall need the following lemma.

Lemma 1. *Let $K \subset \mathbb{R}_+$ be a nonvoid open interval. The Stolarsky mean $S_{a,b}$ of parameters a, b ($ab(a-b) \neq 0$) is a quasi-arithmetic mean on K if and only if either $a = 2b$ or $2a = b$ or $a + b = 0$.*

Proof. Let φ be a continuous strictly monotonic function such that

$$S_{a,b}(u,v) = \varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right) =: A_\varphi(u,v) \qquad (u,v \in K).$$

Then for all $t > 0$ and $u,v \in K$ with $tu, tv \in K$, $S_{a,b}(tu,tv) = tS_{a,b}(u,v)$ holds, which yields

$$A_\varphi(tu,tv) = tA_\varphi(u,v).$$

Thus, the quasi-arithmetic mean A_φ is homogeneous on the interval K. Now, by [10, Corollary 2],

$$A_\varphi(u,v) = H_p(u,v) \qquad (u,v \in K)$$

for some parameter p, where

$$H_p(u,v) := \begin{cases} \left(\dfrac{u^p+v^p}{2}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{uv} & \text{if } p = 0 \end{cases} \qquad (u,v \in K)$$

is the pth Hölder (or power) mean. On the other hand, by the comparison theorem of Stolarsky means (see [9] and also [6]) $S_{a,b} = H_p = S_{2p,p}$ if and only if $a = 2b$ or $2a = b$ or $a + b = 0$. □

4. The generalization of the Matkowski–Sutô problem

Let $\mathcal{CM}(I)$ denote the class of strictly monotonic and continuous functions defined on I. A mean $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I if there exist a constant $0 < \lambda < 1$ and $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)) =: A_\varphi(x,y;\lambda) \tag{13}$$

for all $x, y \in I$. If $\varphi(x) = \mathrm{id}(x) := x$ then we write

$$A_\varphi(x,y;\lambda) = A(x,y;\lambda) = \lambda x + (1-\lambda)y \qquad (x,y \in I).$$

The number $0 < \lambda < 1$ is called a *weight* and the function $\varphi \in \mathcal{CM}(I)$ is called a *generator function.*

A possible generalization of the Matkowski–Sutô problem can be formulated in the following way: *Determine the constants* $0 < \lambda < 1$ *and* $\mu \neq 0, 1$ *and the functions* $\varphi, \psi \in \mathcal{CM}(I)$ *such that*

$$\lambda x + (1-\lambda)y = \mu A_\varphi(x,y;\lambda) + (1-\mu)A_\psi(x,y;\lambda) \tag{14}$$

holds for all $x, y \in I$. The case $\lambda = \mu = \frac{1}{2}$ is the original Matkowski–Sutô problem whose solution with analitycity assumptions was determined to Sutô [13], [14]. Matkowski [7] found the same set of solutions under twice continuous differentiability assumptions. The unnatural regularity assumptions were step by step eliminated in a sequence of papers [1], [2], [4] by the authors and Gy. Maksa. Finally, the complete solution was found in [3].

Now we consider the equation (14) under the following assumptions: *φ and ψ are continuously differentiable and $\varphi'(x) \neq 0$, $\psi'(x) \neq 0$ if $x \in I$.*

Theorem 4. *Let $0 < \lambda < 1$ and $\mu \neq 0, 1$. If $\varphi, \psi \in \mathcal{CM}(I)$ solve (14) and φ and ψ are continuously differentiable and $\varphi'(x) > 0$, $\psi'(x) > 0$ for $x \in I$ then, with the notations*

$$J := \varphi(I), \qquad f := \varphi' \circ \varphi^{-1}, \qquad g := \psi' \circ \varphi^{-1}, \tag{15}$$

the continuous functions $f, g : J \to \mathbb{R}_+$ satisfy the functional equation

$$f(\lambda u + (1-\lambda)v)(g(v) - g(u)) = \mu(f(u)g(v) - f(v)g(u)) \tag{16}$$

for all $u, v \in J$.

Proof. Differentiate (14) with respect to x and y. (The assumptions of the theorem make this possible.) Then

$$\lambda = \mu\frac{\lambda\varphi'(x)}{\varphi'(A_\varphi(x,y;\lambda))} + (1-\mu)\frac{\lambda\psi'(x)}{\psi'(A_\psi(x,y;\lambda))}$$

and

$$1-\lambda = \mu\frac{(1-\lambda)\varphi'(y)}{\varphi'(A_\varphi(x,y;\lambda))} + (1-\mu)\frac{(1-\lambda)\psi'(y)}{\psi'(A_\psi(x,y;\lambda))}$$

for all $x, y \in I$. Multiplying the first equation by $(1-\lambda)\psi'(y)$ and the second one by $\lambda\psi'(x)$, then subtracting the equations, we obtain

$$\psi'(y) - \psi'(x) = \frac{\mu(\varphi'(x)\psi'(y) - \varphi'(y)\psi'(x))}{\varphi'(A_\varphi(x,y;\lambda))}$$

for all $x, y \in I$. Let $u = \varphi(x)$ and $v = \varphi(y)$ $(u, v \in J)$ be arbitrary. Then, with the notations (15), we get that (16) holds. □

In what follows, without loss of generality $\mu > 0$ $(\mu \neq 1)$ can be assumed. Since in (14) either μ or $\mu - 1$ is positive, interchanging φ and ψ if necessary, we may examine only the case when $\mu > 0$ $(\mu \neq 1)$. In this case, the roles of φ and ψ are not interchangeable.

Theorem 5. *Let $0 < \lambda < 1$ and $\mu > 0$, $(\mu \neq 1)$. If the continuous functions $f, g : J \to \mathbb{R}_+$ satisfy the functional equation* (16) *then the following cases are possible:*

(i) *If $\lambda \neq \frac{1}{2}$ then there exists $c \in \mathbb{R}_+$ such that $f(u) = c$ for all $u \in J$;*

(ii) *If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then there exists $c \in \mathbb{R}_+$ such that $f(u) = c$ for all $u \in J$;*

(iii) *If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exist $a, b \in \mathbb{R}$ such that*

$$f(u) = au + b > 0 \quad \text{if } u \in J;$$

(iv) *If $\lambda = \frac{1}{2}$ and $\mu = 2$ then there exist $a, b \in \mathbb{R}$ such that*

$$f(u) = \frac{1}{au+b} > 0 \quad \text{if } u \in J.$$

Proof. With the strict mean $M(u,v) := \lambda u + (1-\lambda)v$ the functional equation (16) has the form (3). Then, by Theorem 2, there exists $c > 0$ such that $f(u)^\mu g(u)^{1-\mu} = c$ for all $u \in J$. This implies that the continuous function $f : J \to \mathbb{R}_+$ satisfies the functional equation

$$f(\lambda u + (1-\lambda)v)\left(f(u)^{\frac{\mu}{1-\mu}} - f(v)^{\frac{\mu}{1-\mu}}\right) = \mu\left(f(u)^{\frac{\mu}{1-\mu}+1} - f(v)^{\frac{\mu}{1-\mu}+1}\right)$$

for all $u, v \in J$. By Theorem 3, then either $f(u) = c$ $(u \in J)$ for some $c \in \mathbb{R}_+$, or f is injective on J, and thus, for all $x = f(u)$, $y = f(v)$ $(x, y \in f(J) =: K \subset \mathbb{R}_+)$ $x \neq y$

$$f(\lambda f^{-1}(x) + (1-\lambda)f^{-1}(y)) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(x,y) \tag{17}$$

holds. The following cases are possible:

(i) If $\lambda \neq \frac{1}{2}$ then f cannot be injective. Indeed, if it were so, then from the symmetry of the right-hand side of (17),

$$f(\lambda f^{-1}(x) + (1-\lambda)f^{-1}(y)) = f(\lambda f^{-1}(y) + (1-\lambda)f^{-1}(x))$$

would follow, i.e., $x = y$, which is a contradiction. Thus in this case f is constant on J.

(ii) If $\lambda = \frac{1}{2}$ and f is injective then from (17) we have

$$f\left(\frac{f^{-1}(x) + f^{-1}(y)}{2}\right) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(x,y) \tag{18}$$

for all $x, y \in K \subset \mathbb{R}_+$. By Lemma 1, (18) holds if and only if either $\mu = \frac{1}{2}$ or $\mu = 2$. Thus if $\mu \notin \{\frac{1}{2}, 2\}$, f is constant on J.

(iii) If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then we may suppose that f is injective. From (18) we have

$$f\left(\frac{f^{-1}(x) + f^{-1}(y)}{2}\right) = \frac{x+y}{2} \qquad (x \neq y;\ x, y \in K),$$

which implies the existence of $a, b \in \mathbb{R}$ with $a \neq 0$ such that

$$f(u) = au + b > 0 \quad \text{if } u \in J.$$

If $a = 0$ then $f(u) = b > 0$ gives the constant solutions.

(iv) If $\lambda = \frac{1}{2}$ and $\mu = 2$ one can easily see that there exist $a, b \in \mathbb{R}_+$ for which

$$f(u) = \frac{1}{au + b} > 0 \quad \text{if } u \in J$$

because in this case the function $\frac{1}{f}$ satisfies the Jensen equation.

Thus, the proof of the theorem is complete. $\square$

Returning to the generalized Matkowski–Sutô problem, we need the following definitions.

Definition 1. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in I$$

then we say that φ is equivalent to ψ on I and denote it by $\varphi(x) \sim \psi(x)$ if $x \in I$ or in short $\varphi \sim \psi$ on I. If $(\varphi, \psi) \in \mathcal{CM}(I)^2$, $(\Phi, \Psi) \in \mathcal{CM}(I)^2$ and $\varphi \sim \Phi$ and $\psi \sim \Psi$ on I then we say that the pair (φ, ψ) is equivalent to the pair (Φ, Ψ) on I and this fact is denoted by $(\varphi, \psi) \sim (\Phi, \Psi)$ on I.

The statement of the following lemma is well know, therefore its proof will be omitted.

Lemma 2. *Let $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$. Then $A_\varphi(x, y; \lambda) = A_\psi(x, y; \lambda)$ for all $x, y \in I$ if and only if $\varphi \sim \psi$ on I.*

Definition 2. Let $I \subset \mathbb{R}$ be a nonvoid open interval. Define the one parameter family of functions $\chi_p : I \to \mathbb{R}$ $(p \in \mathbb{R})$ as follows

$$\chi_p(x) = \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I).$$

We also define the following sets

$$P_+(I) := \{p \in \mathbb{R} \mid I + p \subset \mathbb{R}_+\}$$
$$P_-(I) := \{p \in \mathbb{R} \mid -I + p \subset \mathbb{R}_+\}.$$

With the help of these notations, set

$$\gamma_p^+(x) := \sqrt{x + p} \qquad \text{if } p \in P_+(I) \qquad (x \in I)$$
$$\gamma_p^-(x) := \sqrt{-x + p} \quad \text{if } p \in P_-(I) \qquad (x \in I).$$

Finally, we introduce the following notations

$$\mathcal{S}(I) := \{(\chi_p, \chi_{-p}) \mid p \in \mathbb{R}\},$$

$$\mathcal{T}(I) := \{(\chi_0, \chi_0)\} \cup \{(\gamma_p^+, \log\circ\gamma_p^+) \mid p \in P_+(I)\} \\ \cup \{(\gamma_p^-, \log\circ\gamma_p^-) \mid p \in P_-(I)\}.$$

Theorem 6. *Let $0 < \lambda < 1$ and $\mu > 0$, $(\mu \neq 1)$. If $\varphi, \psi \in \mathcal{CM}(I)$ solve the generalized Matkowski–Sutô problem* (14) *and φ, ψ are continuously differentiable with nonvanishing derivatives on I then the following cases are possible:*

(i) *If $\lambda \neq \frac{1}{2}$ then $(\varphi, \psi) \sim (\chi_0, \chi_0)$ on I;*

(ii) *If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then $(\varphi, \psi) \sim (\chi_0, \chi_0)$ on I;*

(iii) *If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exists $(s_1, s_2) \in \mathcal{S}(I)$ such that $(\varphi, \psi) \sim (s_1, s_2)$ on I;*

(iv) *If $\lambda = \frac{1}{2}$ and $\mu = 2$ then then there exists $(t_1, t_2) \in \mathcal{T}(I)$ such that $(\varphi, \psi) \sim (t_1, t_2)$ on I.*

The pairs (φ, ψ) given in the cases (i), (ii), (iii), *and* (iv) *are solutions of equation* (14).

Proof. By Lemma 2, it is enough solve the functional equation (14) up to the equivalence of the functions φ and ψ. Thus we may assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ if $x \in I$. Then, with the notations (15), Theorem 4 implies that (16) holds. Now using Theorem 5, we have that the cases (i)–(iv) are possible for $f := \varphi' \circ \varphi^{-1}$. Due to the definition of f, we obtain the differential equation for the function φ:

$$\varphi'(x) = f(\varphi(x)) \qquad (x \in I). \tag{19}$$

In the cases (i) and (ii) (19) yields $\varphi(x) = cx + b$ $(x \in I)$, that is, $\varphi \sim \chi_0$ on I. Furthermore, equation (14) results that $\psi \sim \chi_0$ on I.

In the case (iii), (19) reduces to the differential equation

$$\varphi'(x) = a\varphi(x) + b > 0 \qquad (x \in I),$$

whence we get that there exists $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ on I. Now applying (14), we simply obtain that $\psi \sim \chi_{-p}$, i.e., $(\varphi, \psi) \sim (s_1, s_2)$ for some pair $(s_1, s_2) \in \mathcal{S}(I)$.

Finally, in the case (iv), (19) simplifies to the differential equation

$$\varphi'(x) = \frac{1}{a\varphi(x)+b} > 0 \quad (x \in I)$$

from which we deduce that either $\varphi \sim \chi_0$ on I, or there exists $p \in P_+(I)$ such that $\varphi \sim \gamma_p^+$ on I or there exists $p \in P_-(I)$ such that $\varphi \sim \gamma_p^-$ on I. Now applying (14), we get $(\varphi, \psi) \sim (t_1, t_2)$ for some pair $(t_1, t_2) \in \mathcal{T}(I)$. $\square$

The result of Theorem 6 can be restated in a different but equivalent form, too. For, we introduce the following notations.

If $I \subset \mathbb{R}$ is a nonvoid open interval then we define the following means: If $p \in \mathbb{R}$ then set

$$T_p(x,y) := \begin{cases} \dfrac{x+y}{2} & \text{if } p = 0 \\ \dfrac{1}{p}\log\left(\dfrac{e^{px}+e^{py}}{2}\right) & \text{if } p \neq 0 \end{cases} \qquad (x, y \in I). \tag{20}$$

For $p \in P_+(I)$ define

$$\begin{aligned} \Gamma_p^+(x,y) &:= \left(\frac{\sqrt{x+p}+\sqrt{y+p}}{2}\right)^2 - p \qquad \text{and} \\ G_p^+(x,y) &:= \sqrt{(x+p)(y+p)} - p \qquad (x, y \in I) \end{aligned} \tag{21}$$

Finally, for $p \in P_-(I)$ define

$$\begin{aligned} \Gamma_p^-(x,y) &:= -\left(\frac{\sqrt{-x+p}+\sqrt{-y+p}}{2}\right)^2 + p \qquad \text{and} \\ G_p^-(x,y) &:= -\sqrt{(-x+p)(-y+p)} + p \qquad (x, y \in I). \end{aligned} \tag{22}$$

One can easily check that the expressions given by (20), (21) and (22) are the quasi-arithmetic means on I generated by the functions χ_p in (20), γ_p^+ and $\log\gamma_p^+$ in (21) and γ_p^- and $\log\gamma_p^-$ in (22), respectively.

Theorem 7. *Let $0 < \lambda < 1$ and $\mu > 0$ $(\mu \neq 1)$. Suppose that the quasi-arithmetic means $M, N : I^2 \to I$ weighted by λ satisfy the generalized Matkowski–Sutô equation*

$$\lambda x + (1-\lambda)y = \mu M(x,y) + (1-\mu)N(x,y) \tag{4}$$

for all $x, y \in I$. *If the generator functions of* M *and* N *are continuously differentiable with nonvanishing derivatives on* I *then the following cases are possible:*

(i) *If* $\lambda \neq \frac{1}{2}$ *then*

$$M(x,y) = N(x,y) = A(x,y;\lambda) \qquad (x, y \in I);$$

(ii) *If* $\lambda = \frac{1}{2}$ *and* $\mu \notin \{\frac{1}{2}, 2\}$ *then*

$$M(x,y) = N(x,y) = A(x,y;1/2) = A(x,y) \qquad (x, y \in I);$$

(iii) *If* $\lambda = \frac{1}{2}$ *and* $\mu = \frac{1}{2}$ *then*

$$M(x,y) = T_p(x,y) \quad \text{and} \quad N(x,y) = T_{-p}(x,y) \qquad (x, y \in I)$$

for some $p \in \mathbb{R}$;

(iv) *If* $\lambda = \frac{1}{2}$ *and* $\mu = 2$ *then either*

$$M(x,y) = N(x,y) = A(x,y) \quad (x, y \in I),$$

or

$$M(x,y) = \Gamma_p^+(x,y) \quad \text{and} \quad N(x,y) = G_p^+(x,y) \qquad (x, y \in I)$$

for some $p \in P_+(I)$, *or*

$$M(x,y) = \Gamma_p^-(x,y) \quad \text{and} \quad N(x,y) = G_p^-(x,y) \qquad (x, y \in I)$$

for some $p \in P_-(I)$.

Conversely, all the means M *and* N *listed in the above cases are solutions of* (4).

Proof. Theorem 6 immediately implies the statements. □

References

[1] Z. Daróczy and Gy. Maksa, On a problem of Matkowski, *Colloq. Math.* **82** (1999), 117–123.

[2] Z. Daróczy, Gy. Maksa and Zs. Páles, Extension theorems for the Matkowski–Sutô problem, *Demonstratio Math.* **33** (2000), 547–556.

[3] Z. Daróczy and Zs. Páles, Gauss-type composition of means and the solution of the Matkowski–Sutô problem, *Publ. Math. Debrecen* **61** (2002), 157–218.

[4] Z. Daróczy and Zs. Páles, On means that are both quasi-arithmetic and conjugate arithmetic, *Acta Math. Hungar.* **90** (2001), 271–282.

[5] Z. Daróczy and Zs. Páles, The Matkowski–Sutô problem for weighted quasi-arithmetic means, *Acta Math. Hungar.* (*submitted*).

[6] E. Leach and M. Sholander, Extended mean values II, *J. Math. Anal. Appl.* **92** (1983), 207–223.

[7] J. Matkowski, Invariant and complementary quasi-arithmetic means, *Aequationes Math.* **57** (1999), 87–107.

[8] D. Głazowska, W. Jarczyk and J. Matkowski, Arithmetic means as a linear combination of two quasi-arithmetic means, *Publ. Math. Debrecen* **61** (2002), 455–467.

[9] Zs. Páles, Inequalities for differences of powers, *J. Math. Anal. Appl.* **131** (1988), 271–281.

[10] Zs. Páles, Nonconvex functions and separation by power means, *Math. Ineq. Appl.* **3** (2000), 169–176.

[11] K. B. Stolarsky, Generalizations of the logarithmic mean, *Math. Mag.* **48** (1975), 87–92.

[12] K. B. Stolarsky, The power and generalized logarithmic means, *Amer. Math. Monthly* **87** (1980), 545–548.

[13] O. Sutô, Studies on some functional equations I, *Tôhoku Math. J.* **6** (1914), 1–15.

[14] O. Sutô, Studies on some functional equations II, *Tôhoku Math. J.* **6** (1914), 82–101.

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(Received May 7, 2002)