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On weakly conformally symmetric Ricci-recurrent spaces

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Dedicated to Professor Lajos Tamássy on his 80th birthday

Abstract. The present paper is concerned with n-dimensional $(n > 3)$ Riemannian spaces are weakly conformally symmetric. It is proved that such a space satisfying the second Bianchi identity can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is weakly symmetric. Finally we study weakly conformally symmetric Ricci-recurrent spaces.

1. Introduction

The notion of weakly symmetric and weakly projective symmetric spaces was introduced by Tamássy and Binh [1]. A non-flat Riemannian space V_n $(n > 2)$ is called a weakly symmetric space if the curvature tensor R_{hijk} satisfies the condition:

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + d_i R_{hljk} + e_j R_{hilk} + f_k R_{hijl} \tag{1.1}$$

where a, b, d, e, f are 1-forms (non-zero simultaneously) and the comma ' , ' denotes covariant differentiation with respect to the metric tensor of the space.

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The 1-forms a, b, d, e, f are called the associated 1-forms of the space, and an n-dimensional space of this kind is denoted by $(WS)_n$. Such a space have been studied by M. Prvanović [2], T. Q. Binh [3], U. C. De and S. Bandhyopadhyay [4] and others. In [4] De and Bandhyaopadhyay proved that the associated 1-forms d and f are identical with b and e respectively. Hence $(WS)_n$ is characterized by the condition

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + b_i R_{hljk} + e_j R_{hilk} + e_k R_{hijl}. \tag{1.2}$$

In the same paper [4] De and Bandhyopadhyay proved the existence of a $(WS)_n$ by considering a metric. In a subsequent paper [5] De and Bandhyopadhyay introduce the notion of weakly conformally symmetric spaces.

An n-dimensional $(n > 3)$ non-conformally flat Riemannian space is called weakly *conformally symmetric* if its conformal curvature tensor C_{hijk} defined by

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}) \end{aligned} \tag{1.3}$$

satisfies the condition

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lijk} + d_i C_{hljk} + e_j C_{hilk} + f_k C_{hijl} \tag{1.4}$$

where a, b, d, e, f are 1-forms (non-zero simultaneously). Such a space is denoted by $(WCS)_n$. Since the conformal curvature tensor C_{hijk} satisfies the skew-symmetric property for the indices, h, i and j, k, like curvature tensor R_{hijk} also, the defining relation (1.4) can be expressed in the following form:

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lijk} + b_i C_{hljk} + e_j C_{hilk} + e_k C_{hijl}. \tag{1.5}$$

It is well known that the conformal curvature tensor satisfies the conditions

$$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0, \tag{1.6}$$

$$C^r_{rjk} = C^r_{irk} = C^r_{ijr} = 0, \tag{1.7}$$

$$C_{hijk} = -C_{hikj} = -C_{ihjk} = C_{jkhi}. \tag{1.8}$$

An n-dimensional Riemannian space is said to be Ricci-recurrent [6] if its Ricci tensor is non-zero and satisfies the conditions

$$R_{ij,k} = \lambda_k R_{ij}, \tag{1.9}$$

where λ_k is a non-zero 1-form.

If the 1-forms $a = b = e = o$ in (1.5), then the space is called conformally symmetric [7]. If the 1-forms are all equal, the space is called conformally quasi-recurrent space introduced by M. Prvanović [8]. Confomally symmetric Ricci-recurrent spaces have been studied by W. Roter [9].

2. Semi-symmetric metric connection

In general, the tensor C^h_{ijk} does not satisfy the second Bianchi identity

$$C^h_{ijk,l} + C^h_{ikl,j} + C^h_{ilj,k} = 0. \tag{2.1}$$

In this section we suppose that condition (2.1) holds in the investigated $(WCS)_n$.

Transvecting (1.5) with g^{sh} we obtain

$$C^s_{ijk,l} = a_l C^s_{ijk} + b^s C_{lijk} + b_i C^s_{ljk} + e_j C^s_{ilk} + e_k C^S_{ijl}. \tag{2.2}$$

Using (1.6) and (1.8) from (2.1) and (2.2) it follows that

$$A_l C^h_{ijk} + A_j C^h_{ikl} + A_k C^h_{ilj} = 0, \tag{2.3}$$

where $A_i = a_i - 2e_i$.

Contracting h and l in (2.3) and using (1.7) we get

$$A_h C^h_{ijk} = 0. \tag{2.4}$$

Transvecting (2.3) with A^l and using (2.4) we obtain

$$A_l A^l C^h_{ijk} = 0. \tag{2.5}$$

Hence $A_l A^l = 0$, since by assumption $C^h_{ijk} \neq 0$. Thus we can state the following

Theorem 1. *The vector $A_i = a_i - 2e_i$ in a $(WCS)_n$ is a null vector.*

Now we shall prove

Theorem 2. *A weakly conformally symmetric space can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is weakly conformally symmetric.*

Proof. A connection of the form

$$\tilde{\Gamma}^h_{jk} = \begin{Bmatrix} i \\ j \quad k \end{Bmatrix} + \delta^h_j S_k - g_{jk} S^h,$$

where S_i is a vector field, is a semi-symmetric metric connection. In fact, denoting the operator of covariant differention with respect to this connection by $\tilde{\nabla}$, we have

$$\tilde{\nabla}_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \tilde{\Gamma}^r_{kl} g_{rj} - \tilde{\Gamma}^r_{kj} g_{ir} = 0,$$

while the torsion tensor has the form

$$T^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj} = S_k \delta^i_j - S_j \delta^i_k.$$

We shall denote the tensors determined with respect to $\tilde{\Gamma}$ by a tilde above. For example, $\tilde{R}^h_{ijk}$ is the curvature tensor of this connection, $\tilde{R}_{ij}$ is the Ricci tensor, R is the scalar curvature, while

$$\begin{aligned} \tilde{C}^h_{ijk} = \tilde{R}^h_{ijk} &- \frac{1}{n-2}(g_{ij}\tilde{R}^h_k - g_{ik}\tilde{R}^h_j + \delta^h_k \tilde{R}_{ij} - \delta^h_j \tilde{R}_{ik}) \\ &+ \frac{\tilde{R}}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \end{aligned}$$

It is known that [10]

$$\tilde{C}^h_{ijk} = C^h_{ijk}. \tag{2.6}$$

Applying the operator $\tilde{\nabla}$ to (2.6) and using (2.2), we get

$$\begin{aligned} \tilde{\nabla}\tilde{C}^h_{ijk} = a_s C^h_{ijk} &+ b^h C_{sijk} + b_i C^h_{sjk} + e_j C^h_{isk} + e_k C^h_{ijs} - S^h C_{sijk} \\ &- S_i C^h_{sjk} - S_j C^h_{isk} - S_k C^h_{ijs} + \delta^h_s S_r C^r_{ijk} + g_{is} S^r C^h_{rjk} \\ &+ g_{js} S^r C^h_{irk} + g_{ks} S^r C^h_{ijr}. \end{aligned} \tag{2.7}$$

Therefore, if $S_i = A_i$, i.e., if

$$\tilde{\Gamma}^h_{jk} = \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} + \delta^h_j A_k - g_{jk} A^h,$$

and if we take into account (2.4) and (2.6), we find,

$$\begin{aligned}\tilde{\nabla}_s C^h_{ijk} = a_s C^h_{ijk} + (b^h - A^h) C_{sijk} + (b_i - A_i) C^h_{sjk} \\ + (e_j - A_j) C^h_{isk} + (e_k - A_k) C^h_{ijs}\end{aligned} \tag{2.8}$$

which implies that the conformal curvature tensor is weakly conformally symmetric with respect to the connection $\tilde{\nabla}$. This completes the proof of the theorem. □

If, in particular, $A_i = b_i = e_i$, then (2.8) reduces to $\tilde{\nabla}_s C^h_{ijk} = a_s C^h_{ijk}$. Hence we can state the following

Corollary. *A weakly conformally symmetric space can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is recurrent if the associated vector S_i of the semi-symmetric connection is equal to the associated vectors b_i and e_i of a weakly conformally symmetric space.*

3. Weakly conformally symmetric Ricci-recurrent space

A Ricci-recurrent space is defined by (1.6). Transvecting (1.6) with g^{ij}, we obtain

$$R_{,k} = \lambda_k R. \tag{3.1}$$

Hence if $R \neq 0$, the recurrence vector λ_k is gradient. W. Roter [9] proved that in a Ricci-recurrent space the following relations

$$R_{ri} R^r_{jlm} + R_{rj} R^r_{ilm} = 0 \tag{3.2}$$

and

$$R_{ri} R^r_j = \frac{1}{2} R R_{ij} \tag{3.3}$$

hold.

In view of the known result $R^s_{j,s} = \frac{1}{2}R_{,j}$ (1.6) implies that

$$\lambda_s R^s_j = \frac{1}{2}R\lambda_j. \tag{3.4}$$

Transvecting (1.5) with g^{sh} we obtain

$$C^s_{ijk,l} = a_l C^s_{ijk} + b^s C_{lijk} + b_i C^s_{ljk} + e_j C^s_{ilk} + e_k C^s_{ijl}.$$

Since $C^s_{sjk} = C^s_{jsk} = C^s_{jks} = 0$, the above equation reduces to

$$C^s_{ijk,s} = T_s C^s_{ijk}, \tag{3.5}$$

where $T_s = a_s + b_s$.

It is known that in a Riemannian space [11, p. 91]

$$C^s_{ijk,s} = \frac{n-3}{n-2}\left[\left(R_{ij,k} - \frac{1}{2(n-1)}R_{,k}\, g_{ij}\right) - \left(R_{ik,j} - \frac{1}{2(n-1)}R_{,j}\, g_{ik}\right)\right]$$

which in view of (1.6) and (3.4) yields,

$$\begin{aligned} T_s C^s_{ijk} = \frac{n-3}{n-2}\bigg[&\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right)\lambda_k \\ &- \left(R_{ik} - \frac{1}{2(n-1)}Rg_{ik}\right)\lambda_j\bigg]. \end{aligned} \tag{3.6}$$

Since $T_s T^i C^s_{ijk} = 0$, equation (3.6) implies

$$\left(T_s R^s_j - \frac{1}{2(n-1)}RT_j\right)\lambda_k = \left(T_s R^s_{k,} - \frac{1}{2(n-1)}RT_k\right)\lambda_j. \tag{3.7}$$

Transvecting (3.7) with $R^k_{p,}$ using (3.3) and (3.4), and then comparing it with (3.7) we get

$$T_s R^s_j = \frac{1}{2}RT_j. \tag{3.8}$$

Transvecting (1.5) with T^h and by virtue of (3.6) and (3.8) we get

$$\begin{aligned} &\frac{n-3}{n-2}\left[\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right)\lambda_k - \left(R_{ik} - \frac{2}{2(n-1)}Rg_{ik}\right)\lambda_j\right] \\ &\qquad = T_s R^s_{ijk} - \frac{1}{n-2}(T_k R_{ij} - T_j R_{ik}) \end{aligned} \tag{3.9}$$

$$+\frac{3-n}{2(n-1)(n-2)}R(T_k g_{ij}-T_j R_{ik}).$$

Now

$$\begin{aligned} T_s R^s_{ijk} R^i_p &= T^s R^i_p R_{sijk} = -T^s R^i_p R_{isjk} \\ &= -T^s R_{ip} R^i_{sjk} = T^s R_{is} R^i_{pjk}, \quad \text{by (3.2)} \\ &= \frac{1}{2} R T_i T^i_{pjk} \quad \text{by (3.8)}. \end{aligned}$$

Transvecting (3.9) with R^i_p and using the above relation and (3.3) we obtain

$$\frac{n-3}{n-1}(\lambda_k R_{jp}-\lambda_j R_{kp}) = T_s R^s_{pjk} - \frac{2}{n-1}(T_k R_{jp} - T_j R_{kp})$$

which, by contraction with g^{pj} yields

$$\frac{n-3}{n-1}(R\lambda_k - \lambda_j R^j_k) = T_s R^s_k - \frac{2}{n-1}(T_k R - T_j R^j_k).$$

Now using (3.4) and (3.8) we obtain $T_k = \lambda_k$.

Thus we can state the following

Theorem 3. *In a weakly conformally symmetric Ricci-recurrent space with $R \neq 0$, the relation $\lambda_k = a_k + b_k$ holds.*

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