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Szabó Osserman IP pseudo-Riemannian manifolds

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Dedicated to the 80th birthday of Professor Lajos Tamássy

Abstract. We construct a family of pseudo-Riemannian manifolds so that the skew-symmetric curvature operator, the Jacobi operator, and the Szabó operator have constant eigenvalues on their domains of definition. This provides new and non-trivial examples of Osserman, Szabó, and IP manifolds. We also study when the associated Jordan normal form of these operators is constant.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q). Let R be the Riemann curvature:

$$R(Z_1, Z_2) := \nabla_{Z_1}\nabla_{Z_2} - \nabla_{Z_2}\nabla_{Z_1} - \nabla_{[Z_1,Z_2]},$$

$$R(Z_1, Z_2, Z_3, Z_4) := (R(Z_1, Z_2)Z_3, Z_4).$$

We can use R and ∇R to define several natural operators:

1. The *Jacobi operator* $J(X) : Y \to R(Y, X)X$ is a symmetric operator with $J(X)X = 0$. It plays an important role in the study of geodesic

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sprays. Since $J(cX) = c^2 J(X)$, the natural domains of definition for J are the pseudo-sphere bundles $S^{\pm}(M,g){:=}\{X \in TM : (X,X){=}\pm 1\}$.

2. The *Szabó operator* $\mathfrak{S}(X) : Y \to \nabla_X R(Y,X)X$ is a symmetric operator with $\mathfrak{S}(X)X = 0$. It plays an important role in the study of totally isotropic manifolds (i.e. manifolds where the local isometry group acts transitively on the unit sphere bundles). Since $\mathfrak{S}(cX) = c^3\mathfrak{S}(X)$, the natural domains of definition for $\mathfrak{S}$ are $S^{\pm}(M,g)$.

3. Let $\{X_1, X_2\}$ be an oriented orthonormal basis for a non-degenerate 2 plane π. The *skew-symmetric curvature operator* $\mathcal{R}(\pi) : Y \to R(X_1,X_2)Y$ depends on the orientation of π but not on the particular orthonormal basis chosen. The natural domains of definition for $\mathcal{R}(\cdot)$ are the oriented Grassmannians of timelike, mixed (signature $(1,1)$), and spacelike 2 planes.

It is natural to ask what are the geometric constraints that are imposed by assuming that the eigenvalues (or more generally the Jordan normal form) of one of these 3 natural operators are constant on the appropriate domains of definition.

The *spectrum* $\mathrm{Spec}(T) \subset \mathbb{C}$ of a linear map T is the set of complex eigenvalues of T. Let (M,g) be a pseudo-Riemannian manifold. (M,g) is said to be *spacelike Osserman* if $\mathrm{Spec}(J(\cdot))$ is constant on $S^+(M,g)$, (M,g) is said to be *spacelike Szabó* if $\mathrm{Spec}(\mathfrak{S}(\cdot))$ is constant on $S^+(M,g)$, and (M,g) is said to be *spacelike IP* if $\mathrm{Spec}(\mathcal{R}(\pi))$ is constant on the Grassmannian of oriented spacelike 2 planes in the tangent bundle TM. One defines *timelike Osserman*, *timelike Szabó*, *timelike IP*, and *mixed IP* similarly. The eigenvalue $\{0\}$ plays a distinguished role. We say (M,g) is *nilpotent Osserman* if $\mathrm{Spec}(J(X)) = \{0\}$ for all X, *nilpotent Szabó* and *nilpotent IP* are defined similarly.

The names Osserman, Szabó, and IP are used because the seminal papers for this subject in the Riemannian setting are due to Osserman [13] for the operator $J(\cdot)$, to Szabó [16] for the operator $\mathfrak{S}(\cdot)$, to Ivanov and Petrova [10] and Stanilov and Ivanova [11] for the operator $\mathcal{R}(\cdot)$. The spectral properties of the operators $J(\cdot)$ and $\mathcal{R}(\cdot)$ have been studied extensively; we refer to [4], [5] for a more complete historical discussion and bibliography. By contrast, the operator $\mathfrak{S}(\cdot)$ has received considerably less attention.

Suppose $p \geq 1$ and $q \geq 1$. Then the notions spacelike Osserman (resp. spacelike Szabó) and timelike Osserman (resp. timelike Szabó) are equivalent, so one simply says that (M, g) is *Osserman* (resp. *Szabó*). Similarly, if $p \geq 2$ and if $q \geq 2$, then spacelike, mixed, and timelike IP are equivalent notions so (M, g) is said to be *IP*. See [5] for details. We shall use the words "nilpotent", "Osserman", "Szabó", and "IP" as adjectives. Thus, for example, to say that a manifold is nilpotent Osserman Szabó IP means that it is simultaneously nilpotent Szabó, nilpotent Osserman, and nilpotent IP. We say that (M, g) is *locally symmetric* if $\nabla R = 0$ and *locally homogeneous* if the local isometries of (M, g) act transitively on M. We say that (M, g) is *Ricci flat* if the Ricci tensor vanishes identically.

Let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_p)$ be coordinates on the manifold $M := \mathbb{R}^{2p}$. Let $\psi_{ij}(x) = \psi_{ji}(x)$ be a symmetric 2 tensor ψ on $\mathbb{R}^p$. We define a non-degenerate pseudo-Riemannian metric g_ψ of balanced signature (p, p) on M by setting:

$$ds_\psi^2 := \sum_i dx^i \circ dy^i + \sum_{i,j} \psi_{ij}(x) dx^i \circ dx^j. \tag{1.1}$$

This is closely related to the so called "deformed complete lift" of a metric on $\mathbb{R}^p$ to $T\mathbb{R}^p$; we refer to [3], [12] for further details.

In Section 2, we will prove the following result:

Theorem 1.1. *Let $p \geq 2$ and let $\psi(x)$ be a symmetric 2 tensor. Then (M, g_ψ) is:*

1. *a pseudo-Riemannian manifold of signature (p, p);*
2. *nilpotent Szabó Osserman IP;*
3. *Ricci flat and Einstein;*
4. *neither locally homogeneous nor locally symmetric for generic ψ.*

Nilpotent Osserman manifolds have been constructed previously [1], [3], [4]. We can describe one family which arises from affine geometry as follows. Let ${\Gamma_{ij}}^k(x)$ be the Christoffel symbols of an arbitrary torsion free connection ∇ on $\mathbb{R}^p$. Let R_∇ be the associated curvature operator and let $J_\nabla(X_1) : X_2 \to R_\nabla(X_2, X_1)X_1$ be the associated Jacobi operator on $\mathbb{R}^p$. We say that the connection ∇ is *nilpotent affine Osserman* if for all $X \in T\mathbb{R}^p$, $\mathrm{Spec}(J_\nabla(X)) = \{0\}$, i.e. $J_\nabla(X)^p = 0$.

Following García-Río, Kupeli, and Vázquez-Lorenzo [4] (see page 147), define an associated metric on $M = \mathbb{R}^{2p}$ by setting:

$$ds^2_\nabla = \sum_i dx^i \circ dy^i - 2\sum_{ijk} y_k \Gamma_{ij}{}^k(x) dx^i \circ dx^j. \tag{1.2}$$

Then (M, ds^2_∇) is nilpotent Osserman if and only if ∇ is nilpotent affine Osserman. This metric is quite different in flavor from ours as the coefficients depend on the y variables as well as on the x variables. There does not seem to be any direct connection between the metrics defined in equations (1.1) and (1.2).

Nilpotent IP manifolds have also been constructed previously [9]. However, comparatively little is known about Szabó manifolds – see [6], [7] for some preliminary results in the algebraic setting. In particular, the manifolds (M, g_ψ) are the only known irreducible Szabó manifolds which are not locally symmetric.

The eigenvalue structure does not determine the conjugacy class (i.e. the real Jordan normal form) of a symmetric or skew-symmetric linear operator in the higher signature setting. We will use the words "timelike", "spacelike", and "Jordan" as adjectives. Thus, for example, to say (M, g) is timelike Jordan IP means that the Jordan normal form of the skew-symmetric curvature operator is constant on the Grassmannian of timelike oriented 2-planes. We shall omit the accompanying adjectives 'timelike and spacelike' if both apply. Thus (M, g) is Jordan Osserman means (M, g) is both timelike Jordan Osserman and spacelike Jordan Osserman, i.e. the Jordan normal form of J is constant on $S^+(M, g)$ and on $S^-(M, g)$.

There are no known timelike or spacelike Jordan Szabó manifolds which are not locally symmetric. Section 3 is devoted to the proof of:

Theorem 1.2. *If (M, g_ψ) is not locally symmetric, then (M, g_ψ) is neither spacelike Jordan Szabó nor timelike Jordan Szabó.*

It is useful to consider a subfamily of the metrics defined in equation (1.1). Let f be a real-valued function on $\mathbb{R}^p$. In equation (1.1) we set $\psi = df \circ df$ and define

$$ds^2_f := \sum_i dx^i \circ dy^i + \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} dx^i \circ dx^j. \tag{1.3}$$

We can realize (M, g_f) as a hypersurface in a flat space. Let $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p, \gamma\}$ and $\{\alpha_1^*, \ldots, \alpha_p^*, \beta_1^*, \ldots, \beta_p^*, \gamma^*\}$ be a basis and associated dual basis for vector spaces W and W^*, respectively. We give W the inner product of signature $(p, p+1)$:

$$ds_W^2 := \alpha_1^* \circ \beta_1^* + \cdots + \alpha_p^* \circ \beta_p^* + \gamma^* \circ \gamma^*.$$

Let $F : M \to W$ be the isometric embedding:

$$F(x, y) := \sum_i (x_i \alpha_i + y_i \beta_i) + f(x)\gamma.$$

The normal ν to the hypersurface is given by $\nu = -\frac{\partial f}{\partial x_i}\beta_i + \gamma$, so the second fundamental form of the embedding is

$$L(Z_1, Z_2) = Z_1 Z_2(f). \tag{1.4}$$

We define distributions

$$\mathcal{X} := \operatorname{span}\{\partial_1^x, \ldots, \partial_p^x\} \quad \text{and} \quad \mathcal{Y} := \operatorname{span}\{\partial_1^y, \ldots, \partial_p^y\}.$$

We then have $L(Z_1, Z_2) = 0$ if $Z_1 \in \mathcal{Y}$ or $Z_2 \in \mathcal{Y}$ so the restriction $L_{\mathcal{X}}$ of the second fundamental form L to the distribution $\mathcal{X}$ carries the essential information. If, for example, we set $f(x) = \sum_i \varepsilon_i x_i^2$, then:

$$L_{\mathcal{X}}(\partial_i^x, \partial_j^x) = 2\varepsilon_i \delta_{ij}$$

so there are non-trivial examples where $L_{\mathcal{X}}$ is non-degenerate on M. In Sections 4 and 5, we prove:

Theorem 1.3. *Assume the quadratic form $L_{\mathcal{X}}$ is non-degenerate.*

1. *If $p = 2$ or if $p \geq 3$ and if $L_{\mathcal{X}}$ is definite, then (M, g_f) is:*
   - (a) *nilpotent Jordan Osserman;*
   - (b) *nilpotent spacelike and timelike Jordan IP;*
   - (c) *not mixed Jordan IP.*
2. *If $p \geq 3$ and if $L_{\mathcal{X}}$ is indefinite, then (M, g_f) is:*
   - (a) *neither spacelike Jordan Osserman nor timelike Jordan Osserman;*
   - (b) *nilpotent spacelike and timelike Jordan IP;*
   - (c) *not mixed IP.*

So far, we have discussed the balanced (or neutral) signature $p = q$. There are also results available when $p \neq q$. Give $\mathbb{R}^{(a,b)}$ the canonical flat metric of signature (a, b). We shall prove in Section 6 that:

Theorem 1.4. *Assume the quadratic form $L_{\mathcal{X}}$ of (M, g_f) is positive definite. Let $N := M \times \mathbb{R}^{(a,b)}$ and let g_N be the product metric on N.*

1. *(N, g_N) is a nilpotent Osserman Szabó IP manifold of signature $(p + a, q + b)$.*
2. *For generic f, (N, g_N) is*
   (a) *neither spacelike Jordan Szabó,*
   (b) *nor timelike Jordan Szabó,*
   (c) *nor locally homogeneous,*
   (d) *nor locally symmetric.*
3. *(N, g_N) is not mixed Jordan IP.*
4. *Suppose that $b = 0$. Then (N, g_N) is:*
   (a) *neither timelike Jordan Osserman nor timelike Jordan IP;*
   (b) *spacelike Jordan Osserman and spacelike Jordan IP.*
5. *Suppose that $a = 0$. Then (N, g_N) is:*
   (a) *timelike Jordan Osserman and timelike Jordan IP;*
   (b) *neither spacelike Jordan Osserman nor spacelike Jordan IP.*
6. *Suppose that $a > 0$ and $b > 0$. Then (N, g_N) is:*
   a) *neither timelike Jordan Osserman nor timelike Jordan IP;*
   b) *neither spacelike Jordan Osserman nor spacelike Jordan IP.*

Note. By Theorem 1.3, Jordan Osserman and Jordan IP are different notions. By Theorem 1.4, timelike Jordan Osserman (resp. timelike Jordan IP) and spacelike Jordan Osserman (resp. spacelike Jordan IP) are different notions as well.

The higher order Jacobi operator was first defined by Stanilov and Videv [14] in the Riemannian setting – we consider it here in the pseudo-Riemannian setting. Let (M, g) be a pseudo- Riemannian manifold of signature (r, s) and let $\mathrm{Gr}_{r,s}(M, g)$ be the Grassmannian bundle of non-degenerate subspaces of signature (r, s); we assume $0 \leq r \leq p$, $0 \leq s \leq q$, and $1 \leq r+s \leq m-1$ to ensure $\mathrm{Gr}_{r,s}$ is non-empty and does not consist of a single point; such values are said to be *admissible* and we restrict to such values henceforth. The *higher order Jacobi* operator can be generalized to the pseudo-Riemannian setting by setting:

$$J(\pi) := \sum_{1 \leq i \leq r+s} (e_i, e_i) J(e_i)$$

where $\{e_1, \dots, e_{r+s}\}$ is an orthonormal frame for $\pi \in \mathrm{Gr}_{r,s}(M, g)$. One says (M, g) is *Osserman of type* (r, s) if the eigenvalues are constant on $\mathrm{Gr}_{r,s}(M, g)$; the notion *Jordan Osserman of type* (r, s) is defined similarly. We shall restrict to hypersurface metrics in the interests of notational simplicity, more general results are available. We refer to [8] for the proof of the following result (see also Bonome, Castro, and García-Río [2] for a related result in signature $(2, 2)$):

Theorem 1.5. *Assume the quadratic form L_χ of (M, g_f) is positive definite. Give $N := M \times \mathbb{R}^{(a,b)}$ the product metric g_N of signature $(\bar{p}, \bar{q}) = (p + a, p + b)$. Then:*

1. *(N, g_N) is Osserman of type (r, s) for every admissible (r, s).*
2. *(N, g_N) is Jordan Osserman*
   (a) *of types $(r, 0)$ and $(\bar{p} - r, \bar{q})$ if $a = 0$ and if $0 < r \leq p$;*
   b) *of types $(0, s)$ and $(\bar{p}, \bar{q} - s)$ if $b = 0$ and if $0 < s \leq p$;*
   (c) *of types $(r, 0)$ and $(\bar{p} - r, \bar{q})$ if $a > 0$ and if $a + 2 \leq r \leq \bar{p}$;*
   (d) *of types $(0, s)$ and $(\bar{p}, \bar{q} - s)$ if $b > 0$ and if $b + 2 \leq s \leq \bar{q}$.*
3. *(N, g_N) is not Jordan Osserman for values of (r, s) not listed above in (2).*

2. Nilpotent Jordan Szabó Osserman manifolds

We begin the proof of Theorem 1.1 by determining the curvature tensor of (M, g_ψ). Let $\psi_{ij/k} := \partial_k^x \psi_{ij}$ and let $\psi_{ij/kl} := \partial_k^x \partial_l^x \psi_{ij}$.

Lemma 2.1. *Let Z_ν be vector fields on (M, g_ψ). We have:*

1. $\nabla \partial_i^y = 0$;

2. $R(Z_1, Z_2, Z_3, Z_4) = 0$ *if one of the* $Z_\nu \in \mathcal{Y}$ *for* $1 \le \nu \le 4$;
3. $\nabla R(Z_1, Z_2, Z_3, Z_4; Z_5) = 0$ *if one of the* $Z_\nu \in \mathcal{Y}$ *for* $1 \le \nu \le 5$;
4. $R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = -\frac{1}{2}(\psi_{il/jk} + \psi_{jk/il} - \psi_{ik/jl} - \psi_{jl/ik})$.

PROOF. Let Z_1 and Z_2 be coordinate vector fields. We then have:

$$(\nabla_{Z_1} \partial_i^y, Z_2) = \frac{1}{2}\{\partial_i^y(Z_1, Z_2) + Z_1(\partial_i^y, Z_2) - Z_2(Z_1, \partial_i^y)\} = 0.$$

Assertion (1) now follows. We use it to see

$$R(Z_1, Z_2, \partial_i^y, Z_3) = ((\nabla_{Z_1}\nabla_{Z_2} - \nabla_{Z_2}\nabla_{Z_1} - \nabla_{[Z_1,Z_2]})\partial_i^y, Z_3) = 0. \quad (2.1)$$

This proves assertion (2) if $Z_3 \in \mathcal{Y}$; the curvature symmetries then show that $R(Z_1, Z_2, Z_3, Z_4) = 0$ if any of the remaining vectors belong to $\mathcal{Y}$. Since $\nabla \partial_i^y = 0$, we can covariantly differentiate equation (2.1) and get

$$\nabla_{Z_1} R(Z_2, Z_3, \partial_i^y, Z_4) = 0; \quad (2.2)$$

assertion (3) now follows from equation (2.2) and from the curvature symmetries.

Since $(\nabla_{\partial_i^x} \partial_j^x, \partial_k^y) = 0$ and $(\nabla_{\partial_i^x} \partial_j^x, \partial_k^x) = \frac{1}{2}(\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k})$, we have

$$\nabla_{\partial_i^x} \partial_j^x = \frac{1}{2}\sum_k (\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k})\partial_k^y.$$

We complete the proof of the lemma by computing:

$$\begin{aligned} R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) &= ((\nabla_{\partial_i^x}\nabla_{\partial_j^x} - \nabla_{\partial_j^x}\nabla_{\partial_i^x})\partial_k^x, \partial_l^x) \\ &= \frac{1}{2}(\{\partial_i^x(\psi_{j\nu/k} + \psi_{k\nu/j} - \psi_{jk/\nu}) - \partial_j^x(\psi_{i\nu/k} + \psi_{k\nu/i} - \psi_{ik/\nu})\}\partial_\nu^y, \partial_l^x) \\ &= \frac{1}{2}(\psi_{jl/ki} + \psi_{kl/ji} - \psi_{jk/il} - \psi_{il/jk} - \psi_{kl/ij} + \psi_{ik/jl}). \end{aligned}$$

□

PROOF of Theorem 1.1. 1) It is clear from equation (1.1) that (M, g_ψ) is a pseudo-Riemannian manifold of signature (p, p).

2) We use Lemma 2.1 to see that

$$\mathcal{Y} \subset \ker(J(Z_1)), \quad \mathcal{Y} \subset \ker(R(Z_1, Z_2)), \quad \mathcal{Y} \subset \ker(\mathfrak{S}(Z_1)). \quad (2.3)$$

The distribution $\mathcal{Y}$ is a totally isotropic subspace of T_PM of dimension p, i.e. $(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in \mathcal{Y}$. Consequently, $\mathcal{Y}^\perp = \mathcal{Y}$. Since $(R(Z_1, Z_2)Z_3, \partial_i^y) = 0$ and $(\nabla_{Z_1} R(Z_2, Z_3)Z_4, \partial_i^y) = 0$, we have:

$$\operatorname{range}(J(Z_1)) \subset \mathcal{Y}, \quad \operatorname{range}(R(Z_1, Z_2)) \subset \mathcal{Y}, \quad \operatorname{range}(\mathfrak{S}(Z_1)) \subset \mathcal{Y}. \tag{2.4}$$

We use equations (2.3) and (2.4) to show

$$J(Z_1)^2 = 0, \quad \mathfrak{S}(Z_1)^2 = 0, \quad \text{and} \quad R(Z_1, Z_2)^2 = 0. \tag{2.5}$$

This shows that

$$\operatorname{Spec}(J(Z_1)) = \{0\}, \ \operatorname{Spec}(\mathfrak{S}(Z_1)) = \{0\}, \ \text{and} \ \operatorname{Spec}(R(Z_1, Z_2)) = \{0\}$$

for any vector fields Z_ν. Consequently (M, g_ψ) is nilpotent Jordan Szabó IP.

3) Let ϱ be the Ricci tensor. Since $\varrho(Z, Z) = \operatorname{trace}(J(Z))$ and since $J(Z)^2 = 0$, $\varrho(Z, Z) = 0$. We polarize to see that $\varrho \equiv 0$. Thus (M, g_ψ) is Ricci flat and Einstein.

4) Clearly, (M, g_ψ) is generically neither locally homogeneous nor locally symmetric. $\square$

3. Jordan Szabó manifolds

Theorem 1.2 will follow from the following lemma:

Lemma 3.1. *Let $p \geq 2$ and let $P \in M$. If $\nabla R_\psi(P)$ does not vanish identically, then* $\operatorname{rank}(\mathfrak{S}(\cdot))$ *is constant neither on $S^+(T_PM)$ nor on $S^-(T_PM)$. Thus (M, g_f) is neither spacelike Jordan Szabó nor timelike Jordan Szabó.*

Proof. Suppose $\operatorname{rank}(\mathfrak{S}(\cdot)) = r > 0$ is constant on $S^+(T_PM)$; the timelike case is similar. Let $\mathcal{V}^+$ be a maximal spacelike subspace of T_PM and let $\mathcal{V}^- := (\mathcal{V}^+)^\perp$ be the complementary timelike subspace. Let $\rho^\pm$ be orthogonal projection on $\mathcal{V}^\pm$. If $Z \in S^+(\mathcal{V}^+)$, then we define:

$$\check{\mathfrak{S}}(Z) := \rho^+ \mathfrak{S}(Z) \rho^+.$$

We wish to show that $\operatorname{rank} \check{\mathfrak{S}}(Z) = r$. Let $\{Z_1, \ldots, Z_r\}$ be tangent vectors at P so $\{\mathfrak{S}(Z)Z_1, \ldots, \mathfrak{S}(Z)Z_r\}$ is a basis for $\operatorname{range}(\mathfrak{S}(Z))$.

As $T_PM = \mathcal{Y} + \mathcal{V}^+$, we may decompose $Z_i = V_i^+ + Y_i$, where $V_i^+ \in \mathcal{V}^+$ and $Y_i \in \mathcal{Y}$. Since $\mathcal{Y} \subset \ker \mathfrak{S}(Z)$, $\mathfrak{S}(Z)Z_i = \mathfrak{S}(Z)V_i^+$ and thus $\{\mathfrak{S}(Z)V_1^+, \dots, \mathfrak{S}(Z)V_r^+\}$ is a basis for range$(\mathfrak{S}(Z))$. As $\ker \rho^+ = \mathcal{V}^-$ is timelike, as $\mathcal{Y}$ is totally isotropic, and as range$(\mathfrak{S}(Z)) \subset \mathcal{Y}$, the vectors

$$\{\rho^+\mathfrak{S}(Z)\rho^+V_1^+, \dots, \rho^+\mathfrak{S}(Z)\rho^+V_r^+\}$$

are linearly independent. Consequently, rank$(\check{\mathfrak{S}}(Z)) \geq r$. Since the reverse inequality is immediate, we have as desired that

$$\operatorname{rank}(\check{\mathfrak{S}}(Z)) = r \quad \text{for} \quad Z \in S^{p-1} := S^+(\mathcal{V}^+).$$

Since $\mathfrak{S}$ is self-adjoint and ρ^+ is self-adjoint, $\check{\mathfrak{S}}(Z)$ is a self-adjoint map of $\mathcal{V}^+$. Let E_+ and E_- be the span of the eigenvectors with positive and negative eigenvalues respectively; these are non-trivial as $r > 0$. Since $\check{\mathfrak{S}}$ is a self-adjoint map with constant rank, E_+ and E_- are vector bundles over S^{p-1}. Since $\check{\mathfrak{S}}$ is self-adjoint, and since $\check{\mathfrak{S}}(Z)Z = 0$,

$$E_+(Z) \perp E_-(Z) \quad \text{and} \quad E_\pm(Z) \subset Z^\perp = T_Z S^{p-1}.$$

Let N be the north pole of S^{p-1}. Since $S^{p-1} - \{N\}$ is contractable, there exists a section s_+ to E_+ vanishing only at N. Since $\mathfrak{S}(-Z) = -\mathfrak{S}(Z)$, $E_+(Z) = E_-(-Z)$. Thus $s_-(Z) := s_+(-Z)$ is a section to E_- which only vanishes at $-N$. Since $E_+ \perp E_-$, we have $s_+(Z) \perp s_-(Z)$. Consequently, the vector field

$$s(Z) := s_+(Z) + s_-(Z)$$

is nowhere vanishing on S^{p-1}. Furthermore, we have that $s(Z) = s(-Z)$. This contradicts a result of Szabó [16] and shows that $r = 0$. Hence, $\mathfrak{S}(\cdot)$ vanishes identically on $S^+(T_PM)$. Consequently, $\nabla R = 0$ on T_PM, see for example [7]. □

4. Jordan Osserman manifolds

Let L be the second fundamental form of the hypersurface (M, g_f) and let R be the curvature tensor. We use Lemma 2.1 and equation (1.4) to see

$$R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = -\frac{1}{2}\{\partial_i^x\partial_l^x(\partial_j^x f \cdot \partial_k^x f) + \partial_j^x\partial_k^x(\partial_i^x f \cdot \partial_l^x f)$$

$$
\begin{aligned}
&- \partial_i^x\partial_k^x(\partial_j^x f\cdot\partial_l^x f) - \partial_j^x\partial_l^x(\partial_i^x f\cdot\partial_k^x f)\}\\
&= \partial_i^x\partial_l^x f\cdot\partial_j^x\partial_k^x f - \partial_i^x\partial_k^x f\cdot\partial_j^x\partial_l^x\\
&= L(\partial_i^x,\partial_l^x)L(\partial_j^x,\partial_k^x) - L(\partial_i^x,\partial_k^x)L(\partial_j^x,\partial_l^x).
\end{aligned}
$$

This agrees with the well known formula for the curvature of a hypersurface [5]:

$$R(Z_1,Z_2,Z_3,Z_4) = L(Z_1,Z_4)L(Z_2,Z_3) - L(Z_1,Z_3)L(Z_2,Z_4). \tag{4.1}$$

Assertions (1a) and (2a) of Theorem 1.3 will follow from the following lemma.

Lemma 4.1. *Assume that $L_{\mathcal{X}}$ is non-degenerate on a non-empty connected open subset $\mathcal{O}$ of M.*

1. *If $p = 2$, then $(\mathcal{O}, g_f)$ is Jordan Osserman.*
2. *If $p \geq 3$ and if $L_{\mathcal{X}}$ is definite on $\mathcal{O}$, then $(\mathcal{O}, g_f)$ is Jordan Osserman.*
3. *If $p \geq 3$ and if $L_{\mathcal{X}}$ is indefinite on $\mathcal{O}$, then $(\mathcal{O}, g_f)$ is neither spacelike Jordan Osserman nor timelike Jordan Osserman.*

Proof. We use an argument motivated by results of Stavrov [15]. Let $P\in M$ and suppose $L_{\mathcal{X}}$ is non-degenerate on T_PM. Let $Z\in S^{\pm}(T_PM)$. We decompose $Z = X+Y$ for $X\in\mathcal{X}(P)$ and $Y\in\mathcal{Y}(P)$. Since $(Z,Z)\neq 0$ and since $\mathcal{Y}$ is totally isotropic, $X\neq 0$. By Lemma 2.1, $J(Z) = J(X)$. As $J(X)^2 = 0$, $\operatorname{rank}(J(X))$ determines the Jordan normal form of $J(X)$. Let $0\neq X\in\mathcal{X}(P)$. By equation (4.1), we have

$$(J(X_1)X_2, X_3) = L(X_1,X_1)L(X_2,X_3) - L(X_1,X_2)L(X_1,X_3). \tag{4.2}$$

We have $J(X)\partial_i^y = 0$ and $J(X)X = 0$. Thus

$$\operatorname{rank}(J(X)) \leq p-1.$$

Suppose first that $L(X,X)\neq 0$. We can then choose a basis $\{X_1,\dots,X_p\}$ for $\mathcal{X}(P)$ so $X_1 = X$ and so $L(X_i,X_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i\neq 0$ for $1\leq i\leq p$. We use equation (4.2) to show $\operatorname{rank}(J(X)) = p-1$ by computing:

$$(J(X)X_i,X_j) = \varepsilon_i\varepsilon_1\delta_{ij} \text{ for } i,j\geq 2.$$

Suppose next that $L(X,X)$=0. We can then choose a basis so $L(X_1,X_2)$=1 and so $L(X_1,X_i)$=0 for $i\neq 2$. We show that rank$(J(X))$=1 by computing

$$(J(X)X_1,X_i)=-\delta_{2i}\quad\text{and}\quad(J(X)X_i,X_j)=0\quad\text{for } i\neq 2.$$

Consequently, if $0\neq X\in\mathcal{X}$, then:

$$\operatorname{rank}(J(X))=\begin{cases}p-1 & \text{if } (X,X)\neq 0,\\ 1 & \text{if } (X,X)=0.\end{cases}\tag{4.3}$$

Suppose $L_{\mathcal{X}}$ is definite. Let $X\neq 0$. Then $L(X,X)\neq 0$, so rank$(J(X))=p-1$ by equation (4.3). This shows that $(\mathcal{O},g_f)$ is timelike and spacelike Jordan Osserman. If $p=2$, then $p-1=1$. Equation (4.3) implies rank$(J(X))=1$ and again $(\mathcal{O},g_f)$ is timelike and spacelike Jordan Osserman. Finally, if $L_{\mathcal{X}}$ is indefinite and if $p>2$, then rank$(J(X))=1$ if $L(X,X)=0$ and rank$(J(X))=p-1\neq 1$ if $L(X,X)\neq 0$. Consequently, $(\mathcal{O},g_f)$ is neither spacelike Jordan Osserman nor timelike Jordan Osserman. □

5. Jordan IP manifolds

We complete the proof of Theorem 1.3 by proving

Lemma 5.1. *Let $p\geq 2$. Assume that $L_{\mathcal{X}}$ is non-degenerate on a non-empty connected open subset $\mathcal{O}$ of M. Then $(\mathcal{O},g_f)$ is*

1. *spacelike Jordan IP and timelike Jordan IP;*
2. *not mixed Jordan IP.*

Proof. We adopt arguments of [9] (see Section 5). Let $\{Z_1,Z_2\}$ be an orthonormal basis for a non-degenerate 2 plane in T_PM for $P\in\mathcal{O}$. We expand $Z_\nu=X_\nu+Y_\nu$ and use Lemma 2.1 to see $\mathcal{R}(\pi)=R(X_1,X_2)$. As $\mathcal{R}(\pi)^2=0$, rank$(\mathcal{R}(\pi))$ determines the Jordan normal form. Equation (4.1) implies

$$(\mathcal{R}(\pi)X_3,X_4)=L(X_1,X_4)L(X_2,X_3)-L(X_1,X_3)L(X_2,X_4).\tag{5.1}$$

If π is spacelike or timelike, then π contains no null vectors and thus $\{X_1,X_2\}$ are linearly independent vectors. We extend this set to a basis

$\{X_1, \ldots, X_p\}$ for $\mathcal{X}(P)$. Since $L_{\mathcal{X}}(P)$ is non-degenerate, we can choose a basis $\{\tilde{X}_1, \ldots, \tilde{X}_p\}$ for $\mathcal{X}(P)$ which is dual (with respect to L) to the original basis, i.e. $L(X_i, \tilde{X}_j) = \delta_{ij}$. By equation (5.1),

$$(R(X_1, X_2)\tilde{X}_i, \tilde{X}_j) = \begin{cases} 1 & \text{if } i = 2, j = 1, \\ -1 & \text{if } i = 1, j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It now follows that $\operatorname{rank}(\mathcal{R}(\pi)) = \operatorname{rank}(R(X_1, X_2)) = 2$. Thus $(\mathcal{O}, g_f)$ is spacelike Jordan IP and timelike Jordan IP.

To see that $(\mathcal{O}, g_f)$ is not mixed IP, we consider the following 2 planes:

$$\pi_1 := \operatorname{span}\{\partial_1^y, \partial_1^x\} \quad \text{and} \quad \pi_2(\varepsilon) := \operatorname{span}\{\varepsilon^{-1}\partial_1^y + \varepsilon\partial_1^x, -\varepsilon^{-1}\partial_2^y + \varepsilon\partial_2^x\},$$

respectively, where ε is a real parameter. The matrices giving the induced inner products on π_1 and $\pi_2(\varepsilon)$ are given by:

$$A_1 := \begin{pmatrix} 0 & 1 \\ 1 & \varrho_{11} \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 2 + \varepsilon 2\varrho_{11} & \varepsilon 2\varrho_{12} \\ \varepsilon 2\varrho_{12} & -2 + \varepsilon 2\varrho_{22} \end{pmatrix},$$

where $\varrho_{ij} := (\partial_i^x, \partial_j^x)$. Since $\det(A_1) = -1$ and $\det(A_2) = -4 + O(\varepsilon^2)$, π_1 and $\pi_2(\varepsilon)$ are mixed 2 planes for ε small. Since $\mathcal{R}(\pi_1) = 0$ and $\mathcal{R}(\pi_2(\varepsilon)) = c(\varepsilon)R(\partial_1^x, \partial_2^x) \neq 0$, $\mathcal{R}(\pi_1)$ and $\mathcal{R}(\pi_2(\varepsilon))$ are not Jordan equivalent and hence $(\mathcal{O}, g_f)$ is not mixed Jordan IP. $\square$

6. Manifolds of signature $p \neq q$

Proof of Theorem 1.4. Let N be the isometric product of $\mathbb{R}^{(a,b)}$ with (M, g_f) this has signature $(p + a, q + b)$. Let R^N and R^M be the curvature tensors on N and M respectively. Let U_ν be tangent vectors on N. We decompose $U_\nu = W_\nu + Z_\nu$, where W_ν is tangent to $\mathbb{R}^{(u,v)}$ and Z_ν is tangent to M. Since

$$R^N(U_1, U_2)U_3 = R^M(Z_1, Z_2)Z_3, \qquad \text{and}$$

$$\nabla^N_{U_1} R^N(U_2, U_3) = \nabla^M_{Z_1} R^M(Z_2, Z_3),$$

$J^N(U) = J^M(Z)$, $\mathfrak{S}^N(U) = \mathfrak{S}^M(Z)$, and $R^N(U_1, U_2) = R^M(Z_1, Z_2)$. We use (2.5) to see that (N, g_N) is nilpotent Osserman Szabó IP; this proves assertion (1); assertions (2) and (3) follow from the corresponding assertions for (M, g_f).

Suppose that $b = 0$. Let $0 \neq U \in TN$ be spacelike. Expand $U = W+Z$ and $Z = X + Y$. If $X = 0$, then $Z \in \mathcal{Y}$ so $(U, U) = (W, W) \leq 0$, which is false. Thus $X \neq 0$ and by equation (4.3)

$$\operatorname{rank}(J(U)) = \operatorname{rank}(J(X)) = p - 1.$$

Thus (N, g_N) is spacelike Jordan Osserman. One shows similarly that (N, g_N) is spacelike Jordan IP. This proves assertion (4b); assertion (5a) follows similarly.

Suppose $b > 0$. We can choose $0 \neq W \in T(\mathbb{R}^{(a,b)})$ spacelike. Then we have that $\operatorname{rank}(J(W)) = 0$. We can choose $0 \neq Z \in TM$ spacelike so $\operatorname{rank}(J(Z)) = p-1$. Thus (N, g_N) is not timelike Jordan Osserman. Similarly, we may show that (N, g_N) is not timelike IP. This proves assertions (5b) and (6b); the proof of assertion (6a) is similar. $\square$

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