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# Submersions on nilmanifolds and their geodesics

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*Dedicated to Professor Lajos Tamássy on his 80th birthday*

**Abstract.** We describe the geodesics of two-step nilpotent Lie groups $N$ with respect to left invariant Riemannian metrics $\langle .,.\rangle$ using the Riemannian submersion structure of the fiber bundle $\pi : N \to N/\mathcal{Z}$, where $\mathcal{Z}$ denotes the center of $N$. We characterize two-step nilmanifolds $(N, \langle .,.\rangle)$ which have the property that the projections of geodesics of $N$ onto the factor space $N/\mathcal{Z}$ are Euclidean lines or circles.

## 1. Introduction

Geodesics of two-step nilpotent Lie groups with respect to left invariant Riemannian metrics are investigated by many authors in the last 20 years. Their equations are determined for different classes of nilmanifolds (cf. [5], [6], [1], [3], [7]) and applied to the spectral geometry of Riemannian manifolds (e.g. [3], [9], [8], [2]). We intoduce a natural Riemannian submersion structure on a two-step nilpotent Lie group equipped with a left invariant Riemannian metric and investigate the fundamantal equations of this submersion (cf. [11]). The O'Neill's differential equations of geodesics of a Riemannian submersion (cf. [12], [10]) give the equations of geodesics of two-step nilmanifolds in a form which is obtained by A. Kaplan for

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Heisenberg type groups ([5]) and by P. Eberlein for the general case ([3]). Our paper is devoted to the characterization of modified H-type Riemannian manifolds, introduced in [7], using the properties of their geodesics in the Riemannian submersion representation of two-step nilpotent Riemannian nilmanifolds.

*Two-step nilpotent Lie groups.* A connected Riemannian manifold which admits a transitive nilpotent group $\mathcal{N}$ of isometries is called a *nilmanifold*. The action of $\mathcal{N}$ is neccesarily simply transitive (cf. [13], Theorem 2, pp. 341–342), thus the manifold may be identified with the group $\mathcal{N}$ endowed with a left invariant metric. If the manifold is simply connected then the exponential map $\exp : \mathfrak{n} \to N$, where $\mathfrak{n}$ is the Lie algebra of $N$, is a diffeomorphism.

A Lie algebra $\mathfrak{n}$ is said to be *two-step nilpotent* if $[\mathfrak{n},\mathfrak{n}] \neq \{0\}$ but $[[\mathfrak{n},\mathfrak{n}],\mathfrak{n}] = \{0\}$. If the Lie algebra $\mathfrak{n}$ of the simply connected Lie group $N$ is two-step nilpotent, then from the Campbell–Hausdorff formula follows that the multiplication can be expressed in the form:

$$\exp(X)\cdot\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y]\right) \quad \text{for all} \quad X, Y \in \mathfrak{n}.$$

Let $N$ be a simply connected two-step nilpotent Lie group and $\mathfrak{n}$ be its Lie algebra. A left invariant Riemannian metric $g$ on $N$ is determined by an inner product $\langle .\,,.\rangle$ on $\mathfrak{n}$. This two-step nilpotent Riemannian nilmanifold on $N$ will be denoted by $(N, \langle .\,,.\rangle)$. We denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{a} = \mathfrak{z}^\perp$ the orthogonal complement of $\mathfrak{z}$. Then we have the orthogonal direct sum decomposition $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$. We denote by $\mathfrak{so}(\mathfrak{a})$ and $\mathfrak{so}(\mathfrak{z})$ the Lie algebras of skew-symmetric transformations of the Euclidean vector subspaces $(\mathfrak{a}, \langle .\,,.\rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle .\,,.\rangle_{\mathfrak{z}})$ of $(\mathfrak{n}, \langle .\,,.\rangle)$. For each element $Z$ of $\mathfrak{z}$ we obtain a skew-symmetric transformation $j(Z) : \mathfrak{a} \to \mathfrak{a}$ defined by

$$\langle j(Z)X, Y\rangle = \langle [X,Y], Z\rangle \quad \text{for all } X, Y \in \mathfrak{a}. \tag{1}$$

Then $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{a})$ is a linear map.

The geometry of $(N, \langle . , . \rangle)$ can be expressed in terms of the maps $j(Z)$ with $Z \in \mathfrak{z}$. Namely, if there are given inner product spaces $(\mathfrak{a}, \langle . , . \rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle . , . \rangle_{\mathfrak{z}})$ and a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{a})$, we obtain a simply connected Riemannian nilmanifold $(N, \langle . , . \rangle)$ by letting $(\mathfrak{n}, \langle . , . \rangle)$ be the orthogonal direct sum of inner product spaces $(\mathfrak{a}, \langle . , . \rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle . , . \rangle_{\mathfrak{z}})$, defining a Lie bracket on $\mathfrak{n}$ by (1) together with the conditions:

$$[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}, \quad \text{and} \quad [\mathfrak{n}, \mathfrak{z}] = 0,$$

and then letting $N$ be the associated simply connected Lie group with the left invariant Riemannian metric $g$ defined by the inner product $\langle . , . \rangle$. Since $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$ and $[\mathfrak{n}, \mathfrak{z}] = 0$ the Riemannian manifold $N$ is a two-step nilmanifold. If $\mathfrak{z}$ is equal to the commutator of $\mathfrak{n}$, i.e. $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, then $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{a})$ is an injective linear map. Among two-step nilpotent Lie groups with left invariant metric the Heisenberg-type Lie groups are of particular significance. These spaces were introduced and studied seriously by A. Kaplan in [6]. A two-step nilpotent Lie group $(N, \langle . , . \rangle)$ with left invariant metric is said to be an *Heisenberg-type* (H-type) *Lie group* if $[j(Z)]^2 = -\langle Z, Z \rangle \, \mathrm{id}_{\mathfrak{a}}$ for any $Z \in \mathfrak{z}$.

In [7] J. Lauret introduced the following generalization of the notion of H-type Lie groups with left invariant metric by weakening the *H-type condition.*

*Definition 1.* A two-step nilpotent Lie group $(N, \langle . , . \rangle)$ with left invariant metric is said to be a modified Heisenberg-type (modified H-type) Lie group if $[j(Z)]^2 = \lambda(Z) \, \mathrm{id}_{\mathfrak{a}}$ for any $Z \in \mathfrak{z}$ with some function $\lambda(Z) < 0$.

The modified H-type Lie groups with left invariant metric are classified in [7], up to isometry, proving that they are given by pairs $(N, \langle . , . \rangle_S)$ of an H-type nilmanifold $(N, \langle . , . \rangle)$ and of a scalar product $\langle . , . \rangle_S$ of the form

$$\langle X + A, Y + B \rangle_S = \langle X, Y \rangle + \langle SA, B \rangle \quad \text{for all } X, Y \in \mathfrak{a}, \ A, B \in \mathfrak{z},$$

where $S$ is a symmetric positive definite transformation on $(\mathfrak{z}, \langle . , . \rangle)$.

*Riemannian submersions.* Let $M$ and $B$ be Riemannian manifolds. A *Riemannian submersion* $\pi : M \to B$ is a smooth mapping of $M$ onto $B$, which satisfies the following axioms:

(i) $\pi$ has maximal rank,

(ii) $\pi_*$ preserves lengths of horizontal tangent vectors to $M$.

The submanifolds $\pi^{-1}(b)$ are called fibers. A vector field on $M$ (respectively a tangent vector to $M$) is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers.

For a submersion $\pi : M \to B$, let $\mathfrak{H}$ and $\mathfrak{V}$ denote the projections of the tangent spaces of $M$ onto the subspaces of horizontal and vertical vectors. The character of a submersion can be described by the *fundamental tensors* of the submersion: $T$ is determined by the *second fundamental form* of the fibers $\pi^{-1}(b)$ and $A$ is the *integrability tensor* of the horizontal distribution $\mathfrak{H}$ on $M$. They are expressed by

$$T_E F = \mathfrak{H}\nabla_{\mathfrak{V}E}(\mathfrak{V}F) + \mathfrak{V}\nabla_{\mathfrak{V}E}(\mathfrak{H}F),$$

$$A_E F = \mathfrak{V}\nabla_{\mathfrak{H}E}(\mathfrak{H}F) + \mathfrak{H}\nabla_{\mathfrak{H}E}(\mathfrak{V}F)$$

for arbitrary vector fields $E$ and $F$, where $\nabla$ is the covariant derivative of $M$. These tensors have the following properties:

$T_E$ and $A_E$ are *skew-symmetric* linear operators and they *reverse* the *horizontal* and *vertical* subspaces of the tangent spaces of $M$,

$T$ is *vertical* (i.e. $T_E = T_{\mathfrak{V}E}$) and $A$ is *horizontal* (i.e. $A_E = A_{\mathfrak{H}E}$),

$T$ is *symmetric*: $T_V W = T_W V$ for *vertical* $V$ and $W$,

$A$ is *skew-symmetric*: $A_X Y = -A_Y X$ for *horizontal* $X$ and $Y$,

$A_X Y = \frac{1}{2}\mathfrak{V}[X,Y]$ for *horizontal* $X$ and $Y$.

The relation between the fundamental tensors $T$ and $A$ and the covariant derivative $\nabla$ of $M$ is given by the equations:

$$\nabla_V W = T_V W + \mathfrak{V}\nabla_V W, \qquad \nabla_V X = \mathfrak{H}\nabla_V X + T_V X, \tag{2}$$

$$\nabla_X V = A_X V + \mathfrak{V}\nabla_X V, \qquad \nabla_X Y = \mathfrak{H}\nabla_X Y + A_X Y \tag{3}$$

for horizontal vector fields $X$ and $Y$ and for vertical vector fields $V$ and $W$.

## 2. The submersion $\pi : N \to N/\mathcal{Z}$

Let $N$ be a simply connected two-step nilpotent Lie group and let $\mathfrak{n}$ be its Lie algebra. Then we can identify the vector space $\mathfrak{n}$ with $N$ and the product space $\mathfrak{n} \times \mathfrak{n}$ with the tangent bundle $TN$ via the diffeomorphism $\exp : \mathfrak{n} \to N$. Let $\langle .\,, . \rangle$ be an inner product in $\mathfrak{n} \cong T_e N$ which defines a left invariant Riemannian metric $g_p(X, Y) = \langle (\lambda_p^{-1})_* X, (\lambda_p^{-1})_* Y \rangle$ on $N$, where $\lambda$ denotes the left translation map. Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{n}$ and $\mathcal{Z}$ be the corresponding Lie subgroup. The horizontal distribution $T_x^h N$ is left invariant and orthogonal to $(\lambda_x)_* T_e \mathcal{Z}$. Using the orthogonal direct sum decomposition $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$, where $\mathfrak{a} = \mathfrak{z}^\perp$, we have the identities

$$[X \oplus U, Y \oplus V] = 0 \oplus [X, Y],$$

$$(X \oplus U) \circ (Y \oplus V) = (X + Y) \oplus \Big( U + V + \frac{1}{2}[X, Y] \Big)$$

for all $X, Y \in \mathfrak{a}$ and $U, V \in \mathfrak{z}$. Hence the tangent map of the left multiplication map $\lambda_{X \oplus U}$ satisfies

$$(\lambda_{X \oplus U})_*|_{0 \oplus 0}(Y \oplus V) = Y \oplus \Big( V + \frac{1}{2}[X, Y] \Big). \tag{4}$$

It follows that a left invariant vector field $(\lambda_{X \oplus U})_*|_{0 \oplus 0}(Y \oplus V)$ can be written as the map $X \oplus U \mapsto Y \oplus (V + \frac{1}{2}[X, Y])$. The tangent space $T_{X \oplus U} N$ is the orthogonal direct sum

$$T_{X \oplus U} N = \Big\{ Y \oplus \frac{1}{2}[X, Y];\; Y \in \mathfrak{a} \Big\} \oplus \{ 0 \oplus Z;\; Z \in \mathfrak{z} \}$$

of the horizontal subspace $T^{(h)}_{X \oplus U} N = \{ Y \oplus \frac{1}{2}[X, Y];\; Y \in \mathfrak{a} \}$ and of the vertical subspace $T^{(v)}_{X \oplus U} N = \{ 0 \oplus Z;\; Z \in \mathfrak{z} \}$. The horizontal subspace $T^{(h)}_{X \oplus U} N$ is independent of the center $\mathcal{Z}$ of $N$ and hence the horizontal distribution determines a connection $\tau$ in the principal fiber bundle $\pi : N \to N/\mathcal{Z}$. Now, we want to describe the corresponding parallel translation of the fibers along the curves of the base space $N/\mathcal{Z}$. Using the identification $\exp : \mathfrak{n} \to N$ we see that the cosets of the factor space $N/\mathcal{Z}$ concide with the cosets of the factor space $\mathfrak{n}/\mathfrak{z}$ with respect to the additive structure of $\mathfrak{n}$. Hence we can identify the points of the base space $N/\mathcal{Z}$ with the vectors of the space $\mathfrak{a}$.

**Lemma 2.** *Let $X(t)$ be a differentiable curve in the factor space $N/\mathcal{Z}$. We denote by $\tau_{t_0, t} : \pi^{-1}(X(t_0) \oplus 0) \to \pi^{-1}(X(t) \oplus 0)$ the map which is determined by the horizontal lifts to $N$ of the base curve $X(t)$. Then we* *have*

$$\tau_{t_0,\,t}(X(t_0)\oplus Z) = X(t)\oplus\left(Z+\frac{1}{2}\int_{t_0}^{t}[X(u),X'(u)]du\right). \tag{5}$$

Proof. The shape of the horizontal distribution means that a curve $X(t)\oplus Z(t)$ is a horizontal lift of the curve $X(t)$ if and only if its tangent vector satisfies

$$X'(t)\oplus Z'(t) = X'(t)\oplus\frac{1}{2}[X(t),X'(t)].$$

This is equivalent to the equation $Z'(t)=\frac{1}{2}[X(t),X'(t)]$ from which follows the assertion. $\square$

Now we show that we can introduce a unique Euclidean metric $\bar{g}$ on the factor space $N/\mathcal{Z}$ such that the principal fiber bundle $\pi : N \to N/\mathcal{Z}$ will be a Riemannian submersion with respect to the Riemannian metrics $g$ and $\bar{g}$. Indeed, the horizontal distribution is independent from $U\in\mathfrak{z}$, hence the left invariant Riemannian scalar product $g_p(X,Y)=\langle(\lambda_p^{-1})_*X,(\lambda_p^{-1})_*Y\rangle$, restricted to the horizontal distribution $T^{(h)}_{X\oplus U}N$, can be projected to the tangent bundle $T(N/\mathcal{Z})$. This means that $\pi : N\to N/\mathcal{Z}$ is a Riemannian submersion.

Since $\{t(X\oplus 0);\ t\in\mathbb{R}\}$ is a 1-parameter subgroup in $\exp(\mathfrak{a})$ for any $X\in\mathfrak{a}$, one has $\exp(\mathfrak{a})=\mathfrak{a}$. The factor space $N/\mathcal{Z}$ can be identified with $\mathfrak{a}$ via the map $N/\mathcal{Z}\to\mathfrak{a}$ defined by

$$(X\oplus 0)\circ\mathcal{Z}=\{(X\oplus 0)\circ(0\oplus V)=X\oplus V;\ V\in\mathfrak{z}\}\mapsto X.$$

Let $\bar{g}_X$ denote the Riemannian metric in $\mathfrak{a}$, at $X\in\mathfrak{a}$, which corresponds to the Riemannian metric of the factor space $N/\mathcal{Z}$. Let $Y_1,Y_2\in\mathfrak{a}$. The lifts of $Y_i$ to $T^{(h)}_{X\oplus 0}N$ are $\{Y_i\oplus\frac{1}{2}[X,Y_i]\}$, $i=1,2$, from which follows $(\lambda^{-1}_{X\oplus 0})_*(Y_i\oplus\frac{1}{2}[X,Y_i])=Y_i\oplus 0$. Thus we get

$$\begin{aligned}\bar{g}_X(Y_1,Y_2)&=\Big\langle(\lambda^{-1}_{X\oplus 0})_*\Big(Y_1\oplus\frac{1}{2}[X,Y_1]\Big),(\lambda^{-1}_{X\oplus 0})_*\Big(Y_2\oplus\frac{1}{2}[X,Y_2]\Big)\Big\rangle\\&=\langle Y_1\oplus 0,Y_2\oplus 0\rangle=\langle Y_1,Y_2\rangle_{\mathfrak{a}}.\end{aligned}$$

It follows, that the Riemannian scalar product on the factor space $N/\mathcal{Z}$ is constant and hence $N/\mathcal{Z}$ is an Euclidean space.

The covariant derivative $\nabla_X Y$ of the Lie group $(N, \langle . , . \rangle)$ with left invariant metric has the following form:

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad \nabla_X Z = \nabla_Z X = -\frac{1}{2} j(Z)X, \quad \nabla_Z Z^* = 0, \tag{6}$$

where $X, Y \in \mathfrak{a}$ and $Z, Z^* \in \mathfrak{z}$ regarded as left invariant vector fields on $N$ (cf. [3], p. 630).

**Lemma 3.** *The fundamental tensors $T|_e$ and $A|_e$ of the Riemannian submersion $\pi : N \to N/\mathcal{Z}$ at the identity element $e \in N$ satisfy*

$$T|_e = 0 \quad \textit{and} \quad (A|_e)_{X+U}(Y + V) = -\frac{1}{2} j(V)X \oplus \frac{1}{2}[X, Y]$$

*for all $X, Y \in T_e^{(h)}N \cong \mathfrak{a}$ and $U, V \in T_e^{(v)}N \cong \mathfrak{z}$.*

Proof. We extend the vectors $X, Y \in T_e^{(h)}N \cong \mathfrak{a}$ and $U, V \in T_e^{(v)}N \cong \mathfrak{z}$ to left invariant vector fields on $N$, which will be denoted by the same letter. Then we obtain from the relation (2) between the fundamental tensors $T$ and the covariant derivative $\nabla$ of $(N, \langle . , . \rangle)$ and from the equations (6) that

$$T_V W = -\mathfrak{H} \nabla_V W = 0, \quad T_V Y = -\mathfrak{V} \nabla_V Y = \frac{1}{2} \mathfrak{V} j(V)Y = 0$$

for all left invariant vector fields $Y \in \mathfrak{a}$ and $V, W \in \mathfrak{z}$. Since the tensor $T$ is vertical, it follows $T_{X+V}(Y + W) = T_V(Y + W) = T_V(Y) + T_V(W) = 0$ for all $X, Y \in \mathfrak{a}$ and $V, W \in \mathfrak{z}$.

The identity $A_{X+U}(Y + V) = A_X(Y + V)$ holds for all $X, Y \in \mathfrak{a}$ and $U, V \in \mathfrak{z}$ since $A$ is horizontal. Similarly as in the previous case we obtain the equations

$$A_X V = -\mathfrak{H} \nabla_X V = \frac{1}{2} j(V)X, \quad A_X Y = -\mathfrak{V} \nabla_X Y = -\frac{1}{2}[X, Y]$$

from the relations (2) and (6) for left invariant horizontal vector fields $X, Y$ and vertical vector field $V$, which proves the assertion. $\square$

Since the tensorfield $T$ is left invariant, it follows from the last lemma that $T$ vanishes identically. The tensorfield $A$ is also left invariant, hence one can express $(A|_{X \oplus U})_{Y_1 \oplus Z_1}(Y_2 \oplus Z_2)$ at an arbitrary point $X \oplus U$ using the relation (4):

$$(A|_{X\oplus U})_{Y_1\oplus Z_1}(Y_2\oplus Z_2)$$
$$= (\lambda_{X\oplus U})_*(A|_{0\oplus 0})_{(\lambda_{X\oplus U})_*^{-1}(Y_1\oplus Z_1)}(\lambda_{X\oplus U})_*^{-1}(Y_2\oplus Z_2).$$

An easy computation gives:

**Lemma 4.** *The Riemannian submersion $\pi : N \to N/\mathcal{Z}$ has totally geodesic fibers, i.e. the tensorfield $T$ vanishes identically. The tensorfield $A$ has the following expression*

$$(A|_{X\oplus U})_{Y_1\oplus Z_1}(Y_2\oplus Z_2) = \Big(-\frac{1}{2}j(Z_2)Y_1 + \frac{1}{4}j([X,Y_2])Y_1\Big)$$
$$\oplus\Big(\frac{1}{2}[Y_1,Y_2] - \frac{1}{4}[X,j(Z_2)Y_1] + \frac{1}{8}[X,j([X,Y_2])Y_1]\Big)$$

*for all $X, Y_1, Y_2 \in \mathfrak{a}$ and $U, Z_1, Z_2 \in \mathfrak{z}$.*

### 3. Characterization of the geodesics

We can apply the results of O'Neill on the differential equations of geodesics of the total space $N$ of $\pi : N \to N/\mathcal{Z}$ (cf. Theorem 1 in [12], p. 364) and we obtain a Riemannian submersion setting of the equations of geodesics of two-step nilmanifolds (cf. [5], [3]). Since the tensorfield $T$ vanishes identically, a differentiable curve $\alpha(s)$ is a geodesic of $N$ if and only if it satisfies the differential equations

$$\alpha''_* = -2A_{\mathfrak{H}\alpha'}\mathfrak{V}\alpha' \quad \text{and} \quad \mathfrak{V}(\mathfrak{V}\alpha')' = 0, \tag{7}$$

where $\alpha''_*$ denotes the horizontal lift to $\alpha$ of the acceleration $(\pi\circ\alpha)''$ of the projected curve $\pi\circ\alpha$ in the Euclidean space $N/\mathcal{Z}$. We identify the manifold $N$ with its Lie algebra $\mathfrak{n}$ and the tangent bundle $TN$ with the product vector space $\mathfrak{n}\times\mathfrak{n}$ as before. We write the geodesic $\alpha(s)$ in the form $\alpha(s) = X(s)\oplus U(s)$, where $X(s)\in\mathfrak{a}, U(s)\in\mathfrak{z}$ for any $s\in\mathbb{R}$. Then $\alpha'(s) = X'(s)\oplus U'(s)$ and we have

$$\mathfrak{H}\alpha'(s) = X'(s)\oplus\frac{1}{2}[X(s),X'(s)],$$

$$\mathfrak{V}\alpha'(s) = 0 \oplus U'(s) - \frac{1}{2}[X(s), X'(s)].$$

Moreover, the horizontal lift of $(\pi \circ \alpha)''$ has the expression

$$\alpha''_* = X''(s) \oplus \frac{1}{2}[X(s), X''(s)].$$

Now, using the shape of the tensor $A$ given in Lemma 4, the equations (7) give the system

$$\begin{aligned} X''(s) \oplus \frac{1}{2}[X(s), X''(s)] &\Big( j(U'(s) - \frac{1}{2}[X(s), X'(s)] \Big) X'(s) \\ &\oplus \frac{1}{2}\Big[X, j(U'(s) - \frac{1}{2}[X(s), X'(s)])X'(s)\Big], \\ U''(s) - \frac{1}{2}[X(s), X''(s)] &= 0. \end{aligned}$$

The second equation means that $U'(s) - \frac{1}{2}[X(s), X'(s)]$ is constant. We denote this constant vector by $W_0$.

The first equation can be written in the equivalent form

$$(X''(s) - (j(W_0)X'(s)) \oplus \frac{1}{2}[X(s), (X''(s) - j(W_0)X'(s))] = 0 \oplus 0.$$

Hence we obtain the system of equations of geodesics in the form:

$$X''(s) = j(W_0)X'(s), \tag{8}$$

$$U'(s) - \frac{1}{2}[X(s), X'(s)] = W_0, \tag{9}$$

where $W_0 \in \mathfrak{z}$ is a constant vector. These equations are proved by A. Kaplan ([5] p. 133) for H-type nilmanifolds and are generalized by P. Eberlein ([3], (3.1) Proposition, p. 625) for arbitrary two-step nilmanifolds. The equation (9) means that the vertical component of the tangent vector field $\mathfrak{V}\alpha'(s) = 0 \oplus (U'(s) - \frac{1}{2}[X(s), X'(s)])$ along the geodesic $\alpha(s)$ of $N$ is represented by a constant vector $0 \oplus W_0$, where we identify $N$ with its Lie algebra $\mathfrak{n}$ and the tangent bundle $TN$ with the product vector space $\mathfrak{n} \times \mathfrak{n}$. The constant $W_0$ can be an arbitrary vector contained in $\mathfrak{z}$, which is determined by the vertical part of the initial value of the tangent vector of the geodesic.

Since the fibers of the submersion $\pi : N \to N/\mathcal{Z}$ are totally geodesic submanifolds of $N$ the maps $\tau_{s_0,s} : \pi^{-1}(\pi(\alpha(s_0))) \to \pi^{-1}(\pi(\alpha(s)))$ from $\pi^{-1}(\pi(\alpha(s_0)))$ onto $\pi^{-1}(\pi(\alpha(s)))$, where $s \in \mathbb{R}$, are isometries. It follows that the map

$$\psi : \mathbb{R} \times \pi^{-1}(\pi(\alpha(s_0))) \to \bigcup_{s\in\mathbb{R}} \pi^{-1}(\pi(\alpha(s))),$$

defined by

$$\psi(s,z) = \tau_{s_0,s} z; \quad \text{where} \quad s \in \mathbb{R},\ z \in \pi^{-1}(\pi(\alpha(s_0))),$$

is an isometry from the Euclidean product $\mathbb{R} \times \pi^{-1}(\pi(\alpha(s_0)))$ of the Euclidean spaces $\mathbb{R}$ and $\pi^{-1}(\pi(\alpha(s_0))) \subset N$ onto the Riemannian submanifold $\bigcup_{s\in\mathbb{R}} \pi^{-1}(\pi(\alpha(s))) \subset N$.

We can give now an interpretation of the equation (9).

**Lemma 5.** *If the curve $X(s) \oplus U(s)$ is a geodesic of the Riemannian two-step nilmanifold $N$ then it is the image*

$$\{\psi(s, sW_0);\ s \in \mathbb{R}\} = \{\tau_{s_0,s} sW_0;\ s \in \mathbb{R}\}$$

*in the submanifold $\bigcup_{s\in\mathbb{R}} \pi^{-1}(X(s) \oplus 0) \subset N$ of a line $\{(s, sW_0);\ s \in \mathbb{R}\}$ of the Euclidean space $\mathbb{R} \times \pi^{-1}(\pi^{-1}(X(s_0) \oplus 0))$.*

Proof. Using the identification $\exp : \mathfrak{n} \to N$ we obtain from Lemma 2 that the point $\psi(s, sW_0) = \tau_{s_0,s} sW_0 \in \pi^{-1}(X(s) \oplus 0)$ of the image of the line $\{(s, sW_0);\ s \in \mathbb{R}\}$ has the form

$$X(s) \oplus U(s) = X(s) \oplus \Big(sW_0 + \frac{1}{2}\int_{s_0}^{s} [X(u), X'(u)]du\Big).$$

The tangent vector of this curve is expressed by

$$X'(s) \oplus U'(s) = X'(s) \oplus \Big(W_0 + \frac{1}{2}[X(s), X'(s)]\Big),$$

which is equivalent to the equation (9). □

Now, we investigate the equation (8). As we have seen, the constant vector $W_0$ is and arbitrary element of $\mathfrak{z}$ which determinates the vertical part of the initial value of the tangent vector of the geodesic $\alpha(s)$. The projection $\pi \circ \alpha(s)$ of the geodesic $\alpha(s)$ is describeded by the vector field $X(s)$ satisfying the equation (8), where $X(s) \in \mathfrak{a}$ for all $s \in \mathbb{R}$.

**Proposition 6.** *The projection $\pi \circ \alpha(s)$ of any geodesic $\alpha(s)$ has constant curvatures in the Euclidean space $N/\mathcal{Z}$.*

Proof. Let $X(s)$ be an $\mathfrak{a}$-valued vector field cooresponding to the projection $\pi\circ\alpha(s)$ of a geodesic $\alpha(s)$. We denote by $X^{(n)}(s) = j(W_0)^{n-1}X'(s)$ the $n$-th derivative of $X(s)$ if $n \geq 1$, but we use also $X'$ and $X''$ for the first and second derivative. Then according to the equation (8) we have $\frac{1}{2}\langle X^{(n)}(s), X^{(n)}(s)\rangle' = \langle X^{(n+1)}(s), X^{(n)}(s)\rangle = \langle j(W_0)X^{(n)}(s), X^{(n)}(s)\rangle = 0$, since the operator $j(W_0)$ is skew-symmetric. Hence we can parametrize the geodesic $\alpha$ by the arc-length $t$ of $\pi \circ \alpha$. We denote $\mathbf{e}_1(t) = X'(t)$. Then the first curvature $\kappa_1$ of $\pi \circ \alpha(t)$ is the constant $\langle X''t), X''(t)\rangle^{\frac{1}{2}} = \langle \mathbf{e}_1'(t), \mathbf{e}_1'(t)\rangle^{\frac{1}{2}}$. If $\kappa_1 \neq 0$ we define $\mathbf{e}_2(t)$ by $\mathbf{e}_1'(t) = \kappa_1\mathbf{e}_2(t)$. In this way we define the Frenet frame $\kappa_1, \kappa_2, \dots, \kappa_n$ recursively by $\mathbf{e}_i'(t) = -\kappa_{i-1}\mathbf{e}_{i-1}(t) + \kappa_i\mathbf{e}_{i+1}(t)$, $i = 2, \dots, n-1$, and $\mathbf{e}_n'(t) = -\kappa_{n-1}\mathbf{e}_{n-1}(t)$. By induction we assume that $\kappa_{i-1}$ is constant and $\mathbf{e}_i(t)$ is a linear combination of $X'(t), \dots, X^{(i)}(t)$ with constant coefficients and hence the length of $\mathbf{e}_i'(t) = j(W_0)\mathbf{e}_i(t)$ is constant. Using $\mathbf{e}_i'(t) = -\kappa_{i-1}\mathbf{e}_{i-1}(t) + \kappa_i\mathbf{e}_{i+1}(t)$ we obtain that $\kappa_i$ is constant and $\mathbf{e}_{i+1}(t)$ is a linear combination of $X'(t), \dots, X^{(i+1)}(t)$ with constant coefficients. □

**Corollary 7.** *The projection $\pi \circ \alpha(s)$ of a geodesic $\alpha = (X(t), U(t))$ into the Euclidean space $N/\mathcal{Z}$ is an Euclidean line if and only if $j(U(t_0))X(t_0) = 0$ is satisfied in a point $t_0$.*

## 4. Geodesics of modified H-type groups

Now we characterize the two-step nilmanifolds, the projection of geodesics of which are points, Euclidean lines or circles.

**Proposition 8.** *Let $(N, \langle .,. \rangle)$ be a two-step nilpotent Lie group. Then the projections of geodesics are points, lines or circles in the Euclidean space $N/\mathcal{Z}$ if and only if $j(U)^2 = -q(U)\,\mathrm{id}_{\mathfrak{a}}$, where $q(U)$ is a positive semidefinite quadratic form on $\mathfrak{z}$.*

Proof. Let $(N, \langle .\,,.\rangle)$ be a two-step nilpotent Lie group. Denote $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$ the corresponding Lie algebra. The projected curves are contained in two-dimensional subspaces of $N/\mathcal{Z}$ if and only if $\kappa_2 = 0$ for all these curves. It is the case if and only if $X''(t) = \mathbf{e}_1'(t) = j(W_0)\mathbf{e}_i(t) = \kappa_1 \mathbf{e}_2(t)$ and $X'''(t) = \mathbf{e}_1''(t) = j(W_0)^2 \mathbf{e}_1(t) = \kappa_1 \mathbf{e}_2'(t) = -\kappa_1^2 \mathbf{e}_1(t)$ is satisfied. It follows that for any $U \in \mathfrak{z}$ and $X \in \mathfrak{a}$ we have $j(U)^2 X = \lambda X$ with a suitable coefficient $\lambda$. Since the linear operator $j$ is skew-symmetric, we obtain that $\langle j(U)^2 X, X\rangle = -\langle j(U)X, j(U)X\rangle$, or equivalently $\lambda = -\frac{|j(U)X|^2}{|X|^2}$. Since all vectors $X \in \mathfrak{a}$ are eigenvectors of $j(U)^2$ the operator $j^2(U)$ has only one eigenvalue for any $U \in \mathfrak{z}$. Hence $\lambda = -\frac{||j(U)X||^2}{||X||^2}$ is independent from $X$. Then one obtains $\lambda = -\frac{|j(U)X|^2}{|X|^2} = -q(U)$, where $q(U)$ is a positive semidefinite quadratic form on $\mathfrak{z}$. $\square$

The following result follows from the previous proposition and from the fact that one may always split off an abelian factor from a two-step nilpotent Lie algebra.

**Theorem 9.** *Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra and let $N$ be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $\mathcal{Z}$ denote the center of $N$. All projections of the geodesics of $N$ onto the Euclidean space $N/\mathcal{Z}$ are planar curves if and only if $N$ is direct sum of a modified H-type group with the Euclidean de Rahm factor of $N$. In this case the projections of geodesics are points, lines or circles.*

Proof. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with center $\mathfrak{z}$. Then $\mathfrak{n} = \mathfrak{n}^* \oplus \xi$ and $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}] \oplus \xi$, where the ideals $\mathfrak{n}^*$ (nonabelian factor) and $\xi$ (abelian factor) of $\mathfrak{n}$ are uniquely determined and $\mathfrak{n}^*$ is a two-step nilpotent Lie algebra such that the commutator subalgebra $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{n}^*, \mathfrak{n}^*]$ is the center of $\mathfrak{n}^*$.

Let $\langle .\,,.\rangle$ denote an inner product on $\mathfrak{n}$ and also the corresponding left invariant metric on $N$. If $\xi$ has dimension $p \geq 0$, then the kernel of the linear map $j$ defined by (1) has dimension $p$. Hence the Euclidean de Rahm factor of $\{N, \langle .\,,.\rangle\}$ has the dimension of the abelian factor $\mathfrak{n}^*$ of $\mathfrak{n}$, too. (See for further details [4].)

In Lemma 6 we have seen that the projected curves are planar curves in the Euclidean space $N/\mathcal{Z}$ if and only if $j(U)^2 = -q(U)\,\mathrm{id}_{\mathfrak{a}}$, where $q(U)$ is a positive semidefinite quadratic form on $\mathfrak{z}$. If $q(U)$ is a positive definite quadratic form on $\mathfrak{z}$, then we obtain the class of modified H-type groups (cf. [7]). If there exists $U \neq 0 \in \mathfrak{z}$, which satisfies $q(U) = 0$, then $j$ is not injective and in this case $U$ is orthogonal to $[\mathfrak{n}, \mathfrak{n}]$. It follows from this fact that $N$ is direct product of a modified H-type group with the Euclidean de Rahm factor of $N$.

Conversely if $N$ is direct product of a modified H-type group with the Euclidean de Rahm factor of $N$, then it follows from the definition of modified H-type groups that $j(U)^2 = -q(U)\,\mathrm{id}_{\mathfrak{a}}$, where $q(U)$ is a positive semidefinite quadratic form on $\mathfrak{z}$. This proves that the canonical projection onto the quotient group $N/\mathcal{Z}$ maps every geodesic in $N$ to a planar curve in $N/\mathcal{Z}$. $\square$

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