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The smallest univoque number is not isolated

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Dedicated to the 80th birthday of Professor Lajos Tamássy

Abstract. KOMORNIK and LORETI [9] showed that there exists a smallest univoque number $q' \approx 1.787$. Later ALLOUCHE and COSNARD [1] proved that this number is transcendental. The aim of this note is to construct a (decreasing) sequence of algebraic univoque numbers converging to q'.

1. Introduction

Given a real number $1 \leq q \leq 2$, there exists at least one sequence (c_i) of zeroes and ones satisfying the equality

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \dots \tag{1}$$

One such sequence, denoted by (γ_i), can be obtained by the so-called *greedy* algorithm of RÉNYI [13]: proceeding by induction, we choose $c_i = 1$ whenever possible. Among all expansions for a given q, this is lexicographically the largest.

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If $q=2$, then this is the unique possible expansion: $c_i=1$ for all i. Erdős, Horváth and Joó [5] discovered that there exist also smaller numbers q having this curious uniqueness property; following Daróczy and Kátai [3] we call them *univoque* numbers. Subsequently, they were characterized algebraically in [6] (see also [10] for an extension of this result):

Theorem 1. *A number $1\le q\le 2$ is univoque if and only if there exists an expansion (γ_i) of 1 satisfying the following two conditions (in the lexicographic sense):*

$$\gamma_{i+1}\gamma_{i+2}\cdots<\gamma_1\gamma_2\ldots \quad \textit{whenever} \quad \gamma_i=0 \tag{2}$$

and

$$\overline{\gamma_{i+1}\gamma_{i+2}\cdots}<\gamma_1\gamma_2\ldots \quad \textit{whenever} \quad \gamma_i=1. \tag{3}$$

Here and in the sequel we use the notation $\overline{c}:=1-c$.

Among several interesting properties of the set $\mathcal{U}$ of univoque numbers, for which we refer to the papers [1], [2], [3], [4], [5], [8] and [9], we recall from [9] that there exists a smallest univoque number $q'\approx 1.787$, and the corresponding expansion is given by the truncated Thue–Morse sequence

$$(\tau_i)_{i=1}^{\infty}=1101\ 0011\ldots$$

The purpose of this note is to investigate the following two questions:

- One may wonder whether q' is an isolated univoque number or not. In the first case one could look for the second smallest univoque number, and so on.

- Allouche and Cosnard proved in [1] that q' is transcendental. It is than natural to look for the smallest *algebraic* univoque number if it exists.

Both problems are solved by the following

Theorem 2. *There exists a (decreasing) sequence of algebraic univoque numbers converging to q'. In particular, q' is not an isolated point of $\mathcal{U}$.*

2. Proof of Theorem 2

For the purpose of the present paper, it is advantageous to adopt the following definition of the Thue–Morse sequence (τ_i): if

$$i = \varepsilon_k 2^k + \cdots + \varepsilon_0$$

is the dyadic expansion of some nonnegative integer i, then we define

$$\tau_i := \begin{cases} 1 & \text{if } \varepsilon_k + \cdots + \varepsilon_0 \text{ is odd,} \\ 0 & \text{if } \varepsilon_k + \cdots + \varepsilon_0 \text{ is even.} \end{cases} \tag{4}$$

In particular, $\tau_0 = 0$. See [9] for its equivalence with another usual definition.

Our main tool is the following strenghtening of a property of the Thue–Morse sequence τ_1, $\tau_2, \dots$, established in [9].

Lemma 3. *Let $1 \le i < 2^{N+1}$ for some nonnegative integer N.*

(a) *If $\tau_i = 0$, then $\tau_{i+1} \dots \tau_{i+2^N} < \tau_1 \dots \tau_{2^N}$ in the lexicographic sense.*

(b) *If $\tau_i = 1$, then $\overline{\tau_{i+1} \dots \tau_{i+2^N}} < \tau_1 \dots \tau_{2^N}$ in the lexicographic sense.*

Remark. In fact, part (a) remains valid even if $\tau_i = 1$, except the case where $N = 0$ and $i = 1$, while part (b) remains always valid even if $\tau_i = 0$. An analogous property was established recently by Glendinning and Sidorov [7].

Proof. Consider first the case $\tau_i = 0$. Then $\varepsilon_k + \cdots + \varepsilon_0$ is even and therefore $\varepsilon_k + \cdots + \varepsilon_0 \ge 2$ because $i \ge 1$ by assumption. Hence we may write $i = 2^n + 2^m + j$ with $2^n > 2^m > j \ge 0$. We claim that

$$\tau_{i+1} \dots \tau_{i+2^N} < \tau_{j+1} \dots \tau_{j+2^N}. \tag{5}$$

We distinguish two cases. If $n \geq m+2$, then using (4) we have

$$\tau_{i+k} = \tau_{j+k} \quad \text{for} \quad 1 \leq k < 2^m - j$$

but

$$\tau_{i+2^m-j} = \tau_{2^n+2^{m+1}} = 0 < 1 = \tau_{2^m} = \tau_{j+2^m-j}.$$

Since

$$2^m - j \leq 2^m \leq 2^{N-1} < 2^N,$$

this proves (5).

If $n = m+1$, then using (4) we obtain by a similar reasoning that

$$\tau_{i+k} = \tau_{j+k} \quad \text{for} \quad 1 \leq k < 2^{m+1} - j$$

but

$$\tau_{i+2^{m+1}-j} = \tau_{2^{m+2}+2^m} = 0 < 1 = \tau_{2^{m+1}} = \tau_{j+2^{m+1}-j}.$$

Since

$$2^{m+1} - j \leq 2^{m+1} = 2^n \leq 2^N,$$

(5) follows again.

Since $\tau_j = \tau_i = 0$, we may iterate (5) until we obtain $j = 0$, thereby proving the desired inequality.

Now consider the case $\tau_i = 1$ and write $i = 2^m + j$ with $2^m > j \geq 0$. We claim that

$$\overline{\tau_{i+1} \ldots \tau_{i+2^N}} < \tau_{j+1} \ldots \tau_{j+2^N}. \tag{6}$$

Indeed, using (4) we have

$$\overline{\tau_{i+k}} = \tau_{j+k} \quad \text{for} \quad 1 \leq k < 2^m - j$$

but

$$\overline{\tau_{i+2^m-j}} = \overline{\tau_{2^{m+1}}} = 0 < 1 = \tau_{2^m} = \tau_{j+2^m-j}.$$

Since

$$2^m - j \leq 2^m \leq 2^N,$$

this proves (6).

If $j = 0$, then we are done. If $j > 0$, then we complete the proof by combining (5) and (6). $\square$

Now fix a nonnegative integer N and introduce the following sequence:

$$c_i := \begin{cases} \tau_i & \text{if } 1 \le i < 2^{N+1}, \\ c_{i-2^N} & \text{if } i \ge 2^{N+1}. \end{cases} \tag{7}$$

This sequence was used for different purposes in a recent work of GLENDINNING and SIDOROV [7]. Observe that the sequence (c_n) is periodic with period 2^N beginning with c_{2^N}. Let us write down the first 16 elements of the Thue–Morse sequence and of the sequences (c_n) for $N = 0, 1, 2$:

$$\begin{array}{ll} (\tau_i): & 1101\ 0011\ 0010\ 1101\dots \\ N=0: & 1111\ 1111\ 1111\ 1111\dots \\ N=1: & 1101\ 0101\ 0101\ 0101\dots \\ N=2: & 1101\ 0011\ 0011\ 0011\dots \end{array}$$

Let us note for further reference that

$$\tau_i = \tau_{i-2^N} \quad \text{for} \quad 2^{N+1} \le i < 2^{N+1} + 2^N. \tag{8}$$

Indeed, this follows easily from (4).

It is clear that the equation

$$1 = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \dots \tag{9}$$

defines an algebraic number $1 < q_N \le 2$ satisfying $q_N \to q'$ as $N \to \infty$.

PROOF of Theorem 2. Thanks to Theorem 1, it suffices to verify that the sequence (c_n) is admissible in the following sense:

$$c_{i+1} \dots c_{i+2^N} < c_1 \dots c_{2^N} \quad \text{whenever} \quad c_i = 0 \tag{10}$$

and

$$\overline{c_{i+1} \dots c_{i+2^N}} < c_1 \dots c_{2^N} \quad \text{whenever} \quad c_i = 1. \tag{11}$$

For $1 \le i < 2^{N+1}$ both relations follow from the similar properties of the Thue–Morse sequence established in the preceding lemma because the first $2^{N+1} + 2^N - 1$ of the two sequences coincide by equation (8).

For $i \geq 2^{N+1}$ the relations (10) and (11) now follow by induction because the sequences $c_{i+1} \dots c_{i+2^N}$ and $c_{i+1-2^N} \dots c_i$ coincide, and also $c_i = c_{i-2^N}$, so that $c_i = 0$ implies $c_{i-2^N} = 0$ and $c_i = 1$ implies $c_{i-2^N} = 1$. □

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