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Homogeneous Riemannian manifolds with only one homogeneous geodesic

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Dedicated to Professor L. Tamássy
at the occasion of his 80th birthday

Abstract. In [6], the first author and J. Szenthe proved that each homogeneous Riemannian manifold (M,g) admits at least one homogeneous geodesic, i.e., a geodesic which is an orbit of a one-parameter group of isometries. (For Lie groups this result was proved earlier in [1].) In the present article we show that, for each dimension $n \geq 4$, there is an n-dimensional (solvable) Lie group with a left-invariant metric which admits exactly one homogeneous geodesic through each point, up to a parametrization. (For dimension $n = 3$ such example was found in [5].) Hence the result from [6] cannot be improved, in general.

1. Introduction

Consider a Riemannian homogeneous space $(M,g) = G/H$, i.e., such that G is a connected group of isometries acting transitively on (M,g) and H is the isotropy group at a point $o \in M$. Let $\mathbf{g}$ and $\mathbf{h}$ denote the corresponding Lie algebras. It is well-known [2], [6] that G/H always admits an $\mathrm{ad}(H)$-invariant decomposition $\mathbf{g} = \mathbf{m} + \mathbf{h}$, where $\mathbf{m}$ is a linear subspace

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and which makes $(G/H, g)$ a *reductive space*. Using this decomposition, one can define on (M, g) *the canonical connection* $\widetilde{\nabla}$, which is a metric connection with parallel torsion $\widetilde{T}$ and parallel curvature $\widetilde{R}$. Moreover, the last tensor fields are expressed through the formulas

$$\widetilde{T}(X,Y)_o = -[X,Y]_{\mathbf{m}},\ (\widetilde{R}(X,Y)Z)_o = -[[X,Y]_{\mathbf{h}}, Z], \quad \text{for } X,Y,Z \in \mathbf{m},$$

where the subscripts at the brackets denote the corresponding components, and the elements X, Y are also considered as tangent vectors at o via the natural isomorphism between the subspace $\mathbf{m}$ and the tangent space T_oM. Because the scalar product g_o is defined on T_oM, this natural isomorphism defines an $\mathrm{ad}(H)$-*invariant* scalar product $\langle\, ,\, \rangle$ on $\mathbf{m}$.

We start with the following

Proposition 1.1 ([2], [3], [9]). *Let ∇ denote the Levi–Civita connection of $(G/H, g)$ and $\widetilde{\nabla}$ the canonical connection corresponding to some $\mathrm{ad}(H)$-invariant decomposition $\mathbf{g} = \mathbf{m} + \mathbf{h}$. Denote by D the difference tensor field $\nabla - \widetilde{\nabla}$ between the both connections. Then the tensor field D is determined by the algebraic equation*

$$2g(D_Y X, Z) = g(\widetilde{T}(X,Y), Z) + g(\widetilde{T}(X,Z), Y) + g(\widetilde{T}(Y,Z), X)$$

for all vector fields X, Y, Z. Further, the Riemannian curvature tensor R is given by the algebraic formula

$$R(X,Y) = \widetilde{R}(X,Y) + [D_X, D_Y] + D_{\widetilde{T}(X,Y)}.$$

Hence we see that the Riemannian curvature tensor R can be calculated in a purely algebraic way, using the Lie algebra structure of $\mathbf{g}$, the decomposition $\mathbf{g} = \mathbf{m} + \mathbf{h}$ and the scalar product $\langle\, ,\, \rangle$ on $\mathbf{m}$.

Further, a nonzero vector $X \in \mathbf{g}$ is called a *geodesic vector* if the curve $\gamma(t) = \exp(tX)(o)$ is a geodesic on $(G/H, g)$. The following can be found in [7] or [4]:

Lemma 1.2. *A vector $X \in \mathbf{g} - (0)$ is a geodesic vector if and only if*

$$\langle [X,Y]_{\mathbf{m}}, X_{\mathbf{m}} \rangle = 0 \quad \textit{for all } Y \in \mathbf{m},$$

where $\langle\, ,\, \rangle$ is the $\mathrm{ad}(H)$-invariant scalar product on $\mathbf{m}$ induced by the Riemannian scalar product on T_oM and the subscripts indicate the corresponding projection $\mathbf{g} \to \mathbf{m}$.

Finally, a *homogeneous geodesic* on a homogeneous Riemannian manifold (M,g) is a geodesic which is an orbit of a one-parameter group of isometries. In other words, it is a geodesic determined by a geodesic vector which belongs to the Lie algebra $\mathbf{g}$ of the full isometry group $\mathfrak{I}(M,g)$.

2. The main example

Consider, for each $n \geq 3$, a Lie algebra $\mathbf{g}_n$ of dimension $n+1$ which is given, with respect to a basis $\{X_1, \dots, X_{n+1}\}$, by the multiplication table

$$\begin{aligned} [X_i, X_j] &= 0 \quad \text{for } i,j = 1,\dots,n, \\ [X_{n+1}, X_i] &= a_i X_i + X_{i+1} \quad \text{for } 1 \leq i \leq n-1, \\ [X_{n+1}, X_n] &= a_n X_n, \end{aligned} \tag{1}$$

where $a_1, \dots, a_n$ are arbitrary parameters. Define a scalar product $\langle\, ,\, \rangle$ on $\mathbf{g}_n$ for which the above basis is orthonormal. The family of Lie algebras $(\mathbf{g}_n, \langle\, ,\, \rangle)$ gives rise to an (n-parameter) family of solvable Lie groups G_n with a set of invariant Riemann metrics g. Here we can assume that G_n is always diffeomorphic to the $(n+1)$-dimensional Euclidean space.

Our first aim is to prove that, for a specific choice of the parameters $a_1, \dots, a_n$, all principal Ricci curvatures of (G_n, g) are distinct and hence the group G_n acting on itself by the left translations is the identity component of the full isometry group $\mathfrak{I}(G_n, g)$. This is a nontrivial part of our paper.

The following proposition is a special case of Proposition 1.1 in the situation where $\mathbf{h} = (0)$ and $\mathbf{m} = \mathbf{g}$.

Proposition 2.1 ([2], [3], [9]). *Let ∇ denote the Levi–Civita connection of (G_n, g) and $\widetilde{\nabla}$ the canonical connection on G_n (for which all left-invariant vector fields are parallel). Denote by D the difference tensor field $\nabla - \widetilde{\nabla}$ between the both connections. Then the tensor field D is determined by the algebraic equation*

$$2g(D_Y X, Z) = -g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \tag{2}$$

for all vector fields X, Y, Z. Further, the Riemannian curvature tensor R is given by the algebraic formula

$$R(X,Y) = [D_X, D_Y] - D_{[X,Y]}. \tag{3}$$

Here we used the fact that the torsion and the curvature of the canonical connection $\widetilde{\nabla}$ are given by the formulas $\widetilde{T}(X,Y) = -[X,Y]$, $\widetilde{R} = 0$. It is obvious that all calculations for determining the curvature tensor R are now reduced to those in the Lie algebra $\mathbf{g}_n$. By a routine calculation we obtain the following two propositions:

Proposition 2.2. *The tensor field D is given on $\mathbf{g}_n$, expressed through the vector fields $X_1, \dots, X_{n+1}$, as follows:*

$$\begin{aligned}
&D_{X_i}X_j = 0 \quad \text{for } i,j \le n,\ |i-j| > 1,\\
&D_{X_i}X_j = \frac{1}{2}X_{n+1} \quad \text{for } i,j \le n,\ |i-j| = 1,\\
&D_{X_i}X_i = a_i X_{n+1} \quad \text{for } i = 1,\dots,n; \qquad (4)\\
&D_{X_i}X_{n+1} = -a_i X_i - \frac{1}{2}(X_{i-1} + X_{i+1}) \quad \text{for } 1 < i < n,\\
&D_{X_1}X_{n+1} = -a_1 X_1 - \frac{1}{2}X_2,\\
&D_{X_n}X_{n+1} = -a_n X_n - \frac{1}{2}X_{n-1}; \qquad (5)\\
&D_{X_{n+1}}X_i = \frac{1}{2}(X_{i+1} - X_{i-1}), \quad \text{for } 1 < i < n,\\
&D_{X_{n+1}}X_1 = \frac{1}{2}X_2,\\
&D_{X_{n+1}}X_n = -\frac{1}{2}X_{n-1},\\
&D_{X_{n+1}}X_{n+1} = 0. \qquad (6)
\end{aligned}$$

Proposition 2.3. *The curvature tensor field R is expressed on $\mathbf{g}_n$ through the basic vector fields $X_1, \dots, X_{n+1}$ as follows:*

$$\begin{aligned}
R(X_i, X_{n+1})X_{n+1} = -\frac{1}{4}X_{i-2} - a_i X_{i-1} - \left(\frac{1}{2} + (a_i)^2\right) X_i\\
- a_{i+1}X_{i+1} - \frac{1}{4}X_{i+2},
\end{aligned}$$

$$R(X_1,X_{n+1})X_{n+1}=-\left(\frac{3}{4}+(a_1)^2\right)X_1-a_2X_2-\frac{1}{4}X_3,$$

$$R(X_2,X_{n+1})X_{n+1}=-a_2X_1-\left(\frac{1}{2}+(a_2)^2\right)X_2-a_3X_3-\frac{1}{4}X_4$$

$$(\textit{where one puts } X_4=0 \ \ \textit{for } n=3),$$

$$R(X_{n-1},X_{n+1})X_{n+1}=-\frac{1}{4}X_{n-3}-a_{n-1}X_{n-2}-\left(\frac{1}{2}+(a_{n-1})^2\right)X_{n-1}$$

$$-a_nX_n \qquad (\textit{where one puts } X_0=0 \ \ \textit{for } n=3),$$

$$R(X_n,X_{n+1})X_{n+1}=-\frac{1}{4}X_{n-2}-a_nX_{n-1}+\left(\frac{1}{4}-(a_n)^2\right)X_n; \tag{7}$$

$$R(X_i,X_j)X_k=0 \quad \textit{for } i,j,k\le n,\ |i-k|>1,\ |j-k|>1,$$

$$R(X_i,X_j)X_{n+1}=0 \quad \textit{for } i,j\le n,\ |i-j|>1. \tag{8}$$

Next, we shall calculate the Ricci tensor S.

Proposition 2.4. *We have*

$$S(X_i,X_{n+1})=0 \quad \textit{for } i=1,\dots,n, \tag{9}$$

$$S(X_{n+1},X_{n+1})=-\frac{n-1}{2}-\sum_{j=1}^{n}(a_j)^2. \tag{10}$$

Proof. Because $S(X_{n+1},X_{n+1})=\sum_{i=1}^{n}\langle R(X_i,X_{n+1})X_{n+1},X_i\rangle$, we get (10) directly from (7). Further, for $i\le n$, $S(X_i,X_{n+1})=\sum_{j=1}^{n}\times\langle R(X_j,X_i)X_{n+1},X_j\rangle$, and according to the second formula of (8), the only nontrivial terms can be $\langle R(X_{i-1},X_i)X_{n+1},X_{i-1}\rangle$ and $\langle R(X_{i+1},X_i)X_{n+1},X_{i+1}\rangle$. A direct check shows that they are both equal to zero. □

Proposition 2.5. *We have*

$$S(X_i,X_j)=0 \quad \textit{for } i,j\le n \textit{ and } |i-j|>1. \tag{11}$$

Proof. The proof is routine for $|i-j|>2$. It remains to check that $S(X_i,X_{i+2})=0$ for all $i\le n-2$. But due to the first statement of (8) we have $S(X_i,X_{i+2})=\langle R(X_{i+1},X_i)X_{i+2},X_{i+1}\rangle+\langle R(X_{i+3},X_i)X_{i+2},X_{i+3}\rangle+\langle R(X_{n+1},X_i)X_{i+2},X_{n+1}\rangle$ where the last two terms coincide if $i=n-2$. The middle term can be written in the form $\langle R(X_{i+3},X_{i+2})X_i,X_{i+3}\rangle$, which is zero according to (8), unless $i = n-2$. It remains the sum $\langle R(X_{i+1}, X_i)X_{i+2}, X_{i+1}\rangle + \langle R(X_{n+1}, X_i)X_{i+2}, X_{n+1}\rangle =$ $-\langle D_{X_i}D_{X_{i+1}}X_{i+2}, X_{i+1}\rangle - \langle D_{X_i}D_{X_{n+1}}X_{i+2}, X_{n+1}\rangle -$ $\langle D_{[X_{n+1},X_i]}X_{i+2}, X_{n+1}\rangle = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0.$ □

Proposition 2.6.

$$S(X_i, X_{i+1}) = \frac{1}{2}(a_i - a_{i+1}) - \frac{1}{2}\sum_{j=1}^{n} a_j \quad \textit{for } i = 1, \dots, n-1. \tag{12}$$

Proof.

$$S(X_i, X_{i+1}) = \sum_{j=1, j\neq i, i+1}^{n} \langle D_{X_j}D_{X_i}X_{i+1}, X_j\rangle - \langle D_{X_i}D_{X_{n+1}}X_{i+1}, X_{n+1}\rangle$$

$$- \langle D_{[X_{n+1},X_i]}X_{i+1}, X_{n+1}\rangle = -\frac{1}{2}\left(\sum_{j=1}^{i-1} a_j + \sum_{j=i+2}^{n} a_j\right)$$

$$+ \frac{1}{2}a_i - a_{i+1} - \frac{1}{2}a_i.$$ □

Proposition 2.7. *The diagonal elements of the Ricci tensor are given by the formulas*

$$\begin{aligned} S(X_1, X_1) &= -\frac{1}{2} + sa_1, \\ S(X_i, X_i) &= sa_i \quad \textit{for } \; 2 \le i \le n-1, \\ S(X_n, X_n) &= \frac{1}{2} + sa_n, \\ S(X_{n+1}, X_{n+1}) &= -\frac{n-1}{2} - \sum_{j=1}^{n}(a_j)^2, \end{aligned} \tag{13}$$

where $s = -\sum_{j=1}^{n} a_j$.

Proof. We first recall that $S(X_j, X_j) = \sum_{i=1}^{n+1}\langle R(X_i, X_j)X_j, X_i\rangle$ for all j=1, …, n. A routine calculation using (3), (4) and also the notation $s = -\sum_{j=1}^{n} a_j$ leads to the first three formulas of (13). The last formula is identical with (10). □

Let us denote

$$q = \frac{1}{n(n+1)}, \; k = \left[\frac{2n+1}{3}\right] \; \left(\text{the integral part of } \frac{2n+1}{3}\right). \tag{14}$$

We have $n \geq 3$, so k satisfies the inequalities $2 \leq k \leq n-1$.

Let us fix the parameters $a_1, \dots, a_n$ as follows:

$$a_j = (-2j + \delta_{jk} - \delta_{j,k+1})\; qs \qquad \text{for } j = 1, \dots, n, \tag{15}$$

where s is a parameter and δ_{ij} is the Kronecker's symbol. Then obviously

$$s = -\sum_{j=1}^{n} a_j. \tag{16}$$

From (12), (13) and (15) we obtain

$$\begin{aligned} &S(X_i, X_{i+1}) = ((2 - \delta_{i,k-1} + 2\delta_{ik} - \delta_{i,k+1})q + 1)s/2 \\ &\qquad\qquad\qquad \text{for } i = 1, \dots, n-1, \\ &S(X_1, X_1) = -2qs^2 - 1/2, \\ &S(X_i, X_i) = (-2i + \delta_{ik} - \delta_{i,k+1})\; qs^2 \quad \text{ for } i = 2, \dots, n-1, \\ &S(X_n, X_n) = (-2n - \delta_{n,k+1})\; qs^2 + 1/2, \\ &S(X_{n+1}, X_{n+1}) = -\left(\frac{2}{3}(2n+1) + 6q\right)\; qs^2 - \frac{n-1}{2}. \end{aligned} \tag{17}$$

Now we shall prove the basic

Proposition 2.8. *For the choice of the parameters a_j as in* (15) *and for all sufficiently large values of s, the eigenvalues $\rho_i(s)$ of the Ricci matrix $[S(X_i, X_j)]$ are all distinct.*

Proof. For $s \to +\infty$, all diagonal elements of the matrix are of order s^2 and all other elements are of a lower order. Therefore we define matrices $P(s)$ for all $s \neq 0$ and a constant matrix Q as follows:

$$P(s) = [S(X_i, X_j)] \;/\; qs^2, \quad Q = \lim_{s \to +\infty} P(s). \tag{18}$$

The matrix Q is a *diagonal* matrix. We use (17) to calculate its diagonal elements:

$$Q_{ii} = -2i + \delta_{ik} - \delta_{i,k+1} \qquad \text{for } i = 1, \dots, n, \tag{19}$$

$$Q_{n+1,n+1} = -\frac{2}{3}(2n+1) - 6q = -\left(\frac{4n+2}{3} + \frac{6}{n(n+1)}\right) \notin \mathbb{Z}, \tag{20}$$

where for $n = 3$ we evaluate $Q_{n+1,n+1} = -\frac{31}{6} \notin \mathbb{Z}$ and for $n \geq 4$ we estimate $0 < \frac{6}{n(n+1)} \leq \frac{6}{20} < \frac{1}{3}$, so $Q_{n+1,n+1} \notin \mathbb{Z}$, too.

We see that all elements Q_{ii} $(i = 1, \dots, n+1)$ of the diagonal matrix Q are *distinct*. Consider now the symmetric matrix $\widetilde{P}(t) = P(1/t)$. We see easily that the coefficients of the characteristic polynomial $\widetilde{C}(t, \lambda)$ of $\widetilde{P}(t)$ are polynomials with respect to t and hence they are continuous at $t = 0$. The roots of the equation $\widetilde{C}(0, \lambda) = 0$ are Q_{ii}, and hence all of them are simple roots, which implies $\frac{\partial}{\partial \lambda} \widetilde{C}(0, \lambda)\big|_{\lambda = Q_{ii}} \neq 0$ for all i. Using the implicit function theorem, we see that the roots $\widetilde{\lambda}_i(t)$ of the equation $\widetilde{C}(t, \lambda) = 0$ are continuous functions of t, and because $\widetilde{\lambda}_i(0) = Q_{ii}$, they are all distinct in a neighborhood of $t = 0$. Hence all values $\lambda_i(s) = \widetilde{\lambda}_i(1/s)$ are distinct for each sufficiently large s. But the same is valid for the eigenvalues $\rho_i(s) = qs^2 \lambda_i(s)$ of the Ricci matrix $[S(X_i, X_j)]$. □

Remark. The choice (15) cannot be simplified to the form $a_j = -2jqs$ because, if $2n + 1$ is divisible by 3, then the Q_{ii} are not all distinct and the proof does not work.

We can summarize:

Proposition 2.9. *Consider the space (G_n, g) with the parameters a_j as in (15). Then, for any sufficiently large value of s, each isometry preserving the identity $e \in G_n$ can act on the basis $\{X_1, \dots, X_{n+1}\}$ of $g_n = T_e G_n$ only as a composition of reflections and hence there is only finite number of such isometries. Hence G_n acting on itself by the left translations is the identity component of the full isometry group $\mathfrak{I}(G_n, g)$.*

Corollary 2.10. *For the space (G_n, g) as above, if the parameter s is sufficiently large, then all geodesic vectors are contained in the Lie algebra $\mathbf{g}_n$.*

Now, Lemma 1.2 can be applied in the simplified form and we obtain

Theorem 2.11. *For the space (G_n, g) as above, if the numbers a_j are given by (15) and if the parameter s is sufficiently large, then all geodesic vectors $X \in \mathbf{g}_n$ are multiples of the vector X_{n+1}. Consequently, there is (up to a parametrization) only one homogeneous geodesic through each point.*

Proof. Let us denote $X = \sum_{i=1}^{n+1} x_i X_i$ and let us express the condition of Lemma 1.2 in an explicit form. We can write it as

$$\left\langle \left[\sum_{i=1}^{n+1} x_i X_i, X_j \right], \sum_{k=1}^{n+1} x_k X_k \right\rangle = 0 \quad \text{for } j = 1, \dots, n+1. \tag{21}$$

According to (1), this is reduced to the formulas

$$x_{n+1} \left\langle \left[X_{n+1}, X_j \right], \sum_{k=1}^{n} x_k X_k \right\rangle = 0 \quad \text{for } j = 1, \dots, n, \tag{22}$$

and

$$\left\langle \left[\sum_{i=1}^{n} x_i X_i, X_{n+1} \right], \sum_{k=1}^{n} x_k X_k \right\rangle = 0.$$

The first equations can be expressed in the form

$$\begin{gathered} x_{n+1}(a_j x_j + x_{j+1}) = 0 \quad \text{for } j = 1, \dots, n-1, \\ \text{and} \quad x_{n+1} a_n x_n = 0. \end{gathered} \tag{23}$$

According to our choice, all parameters a_j are negative and hence nonzero. Assuming that x_{n+1} is nonzero, we obtain by induction that all other x_j are zero. Suppose now that $x_{n+1} = 0$. Then using the second equation of (22) we get the following quadratic equation for the components $x_1, \dots, x_n$:

$$\sum_{j=1}^{n} a_j (x_j)^2 + \sum_{k=1}^{n-1} x_k x_{k+1} = 0. \tag{24}$$

Because all a_j are *negative* constant multiples of s, the equation (24) has only trivial solution for any sufficiently large value of s. This concludes the proof of the main result. □

By a simple modification of the previous procedure, we can easily obtain the following

Theorem 2.12. *Let (p, q) denote the prescribed signature of a quadratic form in n variables, $p + q = n$, $p > 0$, $q > 0$. Then there exists a space (G_n, g) and parameters a_j (with sufficiently large absolute values) such that the family of all geodesic vectors $X \in \mathbf{g}_n$ is a disjoint union of* $\operatorname{span}(X_{n+1}) - (0)$ *and of a real hypercone (with deleted vertex) in the orthogonal complement of X_{n+1} whose signature is equal to (p, q).*

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