Publ. Math. Debrecen
62/3-4 (2003), 447–460

The Weyl tensors

By DEMETER KRUPKA (Brno)

Dedicated to Professor L. Tamássy on his 80th birthday

Abstract. The projections of tensor spaces of types $(1,3)$, $(2,2)$, and $(1,4)$ over a real, n-dimensional vector space onto their complementary subspaces of Weyl (i.e. traceless), and Kronecker tensors are considered. The corresponding trace decomposition formulas providing a basis for an algebraic classification of these tensor, are discussed.

1. Introduction

In this paper, $\mathbb{R}$ denotes the field of real numbers, and E is a real, n-dimensional vector space. The space of tensors of type (r,s) over E is denoted by $T^r_s E$. In a (fixed) basis e_i of E, we usually denote a tensor $U \in T^r_s E$ by its components, i.e., we write $U = U^{j_1 j_2 \dots j_r}_{i_1 i_2 \dots i_s}$.

Let r and s be any positive integers. We define a *Weyl tensor* to be any traceless tensor $W \in T^r_s E$. We define a *Kronecker tensor* as a tensor $V \in T^r_s E$, $V = V^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$, of the form

$$V^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \delta^{i_1}_{j_1} V^{(1) i_2 i_3 \dots i_r}_{(1) j_2 j_3 \dots j_s} + \delta^{i_1}_{j_2} V^{(1) i_2 i_3 \dots i_r}_{(2) j_1 j_3 \dots j_s} + \dots + \delta^{i_1}_{j_s} V^{(1) i_2 i_3 \dots i_r}_{(s) j_1 j_3 \dots j_{s-1}}$$
$$+ \delta^{i_2}_{j_1} V^{(2) i_1 i_3 \dots i_r}_{(1) j_2 j_3 \dots j_s} + \delta^{i_2}_{j_2} V^{(2) i_1 i_3 \dots i_r}_{(2) j_1 j_3 \dots j_s} + \dots + \delta^{i_2}_{j_s} V^{(2) i_1 i_3 \dots i_r}_{(s) j_1 j_3 \dots j_{s-1}}$$

Mathematics Subject Classification: 15A72, 53A45, 53B20.
Key words and phrases: tensor, trace, duality, curvature tensor, Weyl tensor.
Supported by grants 201/00/0724 and 201/03/0512 of the Czech Grant Agency (GACR).

$$+\dots$$
$$+\delta^{i_r}_{j_1}V^{(r)i_2i_3\dots i_{r-1}}_{(1)j_2j_3\dots j_s}+\delta^{i_r}_{j_2}V^{(r)i_2i_3\dots i_{r-1}}_{(2)j_1j_3\dots j_s}+\dots+\delta^{i_r}_{j_s}V^{(r)i_2i_3\dots i_{r-1r}}_{(s)j_1j_3\dots j_{s-1}}$$

for some tensors $V^{(p)}_{(q)}\in T^{r-1}_{s-1}E$, $V^{(p)}_{(q)}=V^{(p)i_1i_2\dots i_{r-1}}_{(q)j_1j_2\dots j_{s-1}}$; the Kronecker tensors are also called δ-*generated.* According to the trace decomposition theorem, every tensor $U\in T^r_sE$ can be uniquely decomposed into the *Weyl* (i.e., traceless), and *Kronecker* components. Examples show, however, that for lower dimensions, the tensors $V^{(p)}_{(q)}$ may not be unique. For the general trace decomposition theory we refer to Krupka [8], [9], and Mikes [10].

The *natural* trace operation is defined on the basis of *duality* of vector spaces, i.e., only in mixed tensor spaces, and does not require any metric tensor structure on the underlying vector space. In contradistinction to the metric tensor case (see e.g. Weyl [14], Hamermesh [5], Chapter 10, and Welsh [13]), the trace operation is invariant with respect to the general linear group. In this setting, the trace decomposition problem appears as a problem of the theory of systems of linear equations, or of natural projectors in mixed tensor spaces [7], rather than a problem of the group representation theory.

The aim of this note is to review recent results on the trace decomposition of tensors of type $(1,3)$, $(2,2)$, and $(1,4)$ (see [6], [8], [9]). Tensors of these types appear in differential geometry of Riemann, and Finsler spaces (curvature tensors and their covariant derivatives), and their trace decompositions represent a basis for an (algebraic) *classification* of the underlying geometric structures. In particular, it turns out that the Weyl components in these cases coincide with the classical Weyl tensors, characterizing various geometric properties of smooth manifolds endowed with linear connections (see e.g., Bokan [1], Chern, Chen, Lam [2], Eisenhart [3], Gromoll, Klingenberg, Meyer [4], Tamássy, Binh [11], Thomas [12], Weyl [15]).

2. The Weyl tensors of type $(1,3)$

Consider the tensor space T^1_3E. In general, a tensor $U\in T^1_3E$ has three traces, the $(1,1)$-, $(1,2)$-, and $(1,3)$-traces. the *Ricci tensor* of U is defined to be the $(1,2)$-trace U^s_{ksm}. In the following theorem, we give the basic trace decomposition formula for tensors of type $(1,3)$, which defines the Weyl component of these tensors. The proof, based on an explicit solution of the trace decomposition equations, can be found in [8]. E denotes a real vector space of dimension $n \geq 3$.

Theorem 1. *Let* $\dim E \geq 3$, *and let* $U \in T^1_3 E$, $U = U^i_{k\ell m}$. *There exists a unique Weyl tensor* $W \in T^1_3 E$, $W = W^i_{k\ell m}$, *and unique tensors* $P, Q, R \in T^0_2 E$, $P = P_{\ell m}$, $Q = Q_{km}$, $R = R_{k\ell}$ *such that*

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_k P_{\ell m} + \delta^i_\ell Q_{km} + \delta^i_m R_{k\ell}.$$

These tensors are given by

$$\begin{aligned} P_{k\ell} = \frac{1}{(n^2-1)(n^2-4)} &(n(n^2-3)U^t_{tk\ell} - (n^2-2)U^t_{kt\ell} \\ &+ nU^t_{k\ell t} - 2U^t_{t\ell k} + nU^t_{\ell tk} - (n^2-2)U^t_{\ell kt}), \end{aligned}$$

$$\begin{aligned} Q_{k\ell} = \frac{1}{(n^2-1)(n^2-4)} &(-(n^2-2)U^t_{tk\ell} + n(n^2-3)U^t_{kt\ell} \\ &- (n^2-2)U^t_{k\ell t} + nU^t_{t\ell k} - 2U^t_{\ell tk} + nU^t_{\ell kt}), \end{aligned}$$

$$\begin{aligned} R_{k\ell} = \frac{1}{(n^2-1)(n^2-4)} &(nU^t_{tk\ell} - (n^2-2)U^t_{kt\ell} + n(n^2-3)U^t_{k\ell t} \\ &- (n^2-2)U^t_{t\ell k} + nU^t_{\ell tk} - 2U^t_{\ell kt}), \end{aligned}$$

and

$$W^i_{k\ell m} = U^i_{k\ell m} - \delta^i_k P_{\ell m} - \delta^i_\ell Q_{km} - \delta^i_m R_{k\ell}.$$

In a series of corollaries, we now compute the Weyl components of tensors $U = U^i_{k\ell m}$, satisfying certain symmetry conditions. The proofs are easy applications of Theorem 1.

Corollary 1. *Let* $n \geq 3$, *and let* $U \in T^1_3 E$, $U = U^i_{k\ell m}$. *Assume that*

$$U^i_{k\ell m} + U^i_{km\ell} = 0. \tag{1}$$

Then there exists a unique Weyl tensor $W \in T^1_3 E$, $W = W^i_{k\ell m}$, *and unique tensors* $P, Q \in T^0_2 E$, $P = P_{\ell m}$, $Q = Q_{km}$, *such that*

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_k P_{\ell m} + \delta^i_\ell Q_{km} - \delta^i_m Q_{k\ell}. \tag{2}$$

These tensors are given by

$$P_{k\ell} = \frac{1}{(n+1)(n-2)}((n-1)U^t_{tk\ell} + U^t_{\ell tk} - U^t_{kt\ell}),$$

$$Q_{k\ell} = \frac{1}{(n^2-2)(n-2)}(-(n-1)U^t_{tk\ell} + (n^2-n-1)U^t_{kt\ell} - U^t_{\ell tk}),$$

and

$$\begin{aligned} W^i_{k\ell m} = U^i_{k\ell m} &- \frac{1}{(n+1)(n-2)}\delta^i_k((n-1)U^t_{t\ell m} + U^t_{mt\ell} - U^t_{\ell tm}) \\ &- \frac{1}{(n^2-1)(n-2)}\delta^i_\ell(-(n-1)U^t_{tkm} + (n^2-n-1)U^t_{ktm} - U^t_{mtk}) \\ &+ \frac{1}{(n^2-1)(n-2)}\delta^i_m(-(n-1)U^t_{tk\ell} + (n^2-n-1)U^t_{kt\ell} - U^t_{\ell tk}). \end{aligned}$$

Proof. The proof is straightforward. We substitute from (1) to (2), and then we use Theorem 1 and the identities $n^3-3n+2 = (n+2)(n-1)^2$, $n^3+n^2-3n-2 = (n+2)(n^2-n-1)$, and $n^2+n-2 = (n+2)(n-1)$. □

Remark 1. Corollary 1 describes, in particular, the trace decomposition of the curvature tensors of linear connections on smooth manifolds; these tensors satisfy property (1). Note that the δ-components in (2) depends on both the $(1,1)$-trace, and $(1,2)$-trace of U.

Corollary 2. *Let $n \geq 3$, and assume that $U = U^i_{k\ell m}$ satisfies*

$$U^i_{k\ell m} + U^i_{km\ell} = 0, \quad U^i_{k\ell m} + U^i_{mk\ell} + U^i_{\ell mk} = 0. \tag{3}$$

Then there exists a unique Weyl tensor $W \in T^1_3 E$, $W = W^i_{k\ell m}$, and unique tensors $P, Q \in T^0_2 E$, $P = P_{\ell m}$, $Q = Q_{km}$, such that

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_k P_{\ell m} + \delta^i_\ell Q_{km} - \delta^i_m Q_{k\ell}.$$

These tensors are given by

$$\begin{gathered} P_{k\ell} = \frac{1}{n+1}(U^t_{kt\ell} - U^t_{\ell tk}), \\ Q_{k\ell} = \frac{1}{2(n-1)}(U^t_{kt\ell} + U^t_{\ell tk}) + \frac{1}{2(n+1)}(U^t_{kt\ell} - U^t_{\ell tk}), \end{gathered} \tag{4}$$

and

$$W^i_{k\ell m} = U^i_{k\ell m} - \frac{1}{n+1}\delta^i_k(U^t_{\ell tm} - U^t_{mt\ell}) - \frac{1}{n^2-1}\delta^i_\ell(nU^t_{ktm} + U^t_{mtk}) + \frac{1}{n^2-1}\delta^i_m(nU^t_{kt\ell} + U^t_{\ell tk}). \tag{5}$$

Proof. Using identities (3), we find $U^t_{tk\ell} = -U^t_{k\ell t} - U^t_{\ell tk} = U^t_{kt\ell} - U^t_{\ell tk}$. Formulas (4), (5) now follow from Theorem 1. □

Remark 2. The tensor $W = W^i_{k\ell m}$ (5) is the *Weyl projective curvature tensor*. It follows from Corollary 2 that under the symmetry assumptions (3), the tensor $U = U^i_{jk\ell}$ is completely determined by its $(1,2)$-trace $U^t_{kt\ell}$, i.e., by the *Ricci tensor*, and by the Weyl projective curvature tensor. Note, however, that the Weyl and the Ricci tensors are not sufficient to determine the trace decomposition (2) in Corollary 1.

Remark 3. Identities (3) are satisfied by the curvature tensor of a connection without torsion (see e.g. Gromoll, Klingenberg, Meyer [4]), and by the first Chern connection in Finsler geometry (see Chern, Chen, Lam [2], Chapter 8).

Corollary 3. *Let $n \geq 3$, and assume that $U = U^i_{k\ell m}$ satisfies*

$$U^i_{k\ell m} + U^i_{km\ell} = 0, \quad U^t_{\ell tk} = U^t_{kt\ell}.$$

Then there exists a unique Weyl tensor $W \in T^1_3 E$, $W = W^i_{k\ell m}$, and unique tensors $P, Q \in T^0_2 E$, $P = P_{\ell m}$, $Q = Q_{km}$, such that

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_k P_{\ell m} + \delta^i_\ell Q_{km} - \delta^i_m Q_{k\ell}.$$

These tensors are given by

$$P_{k\ell} = \frac{n-1}{(n+1)(n-2)}U^t_{tk\ell}, \quad Q_{k\ell} = -\frac{1}{(n+1)(n-2)}U^t_{tk\ell} + \frac{1}{n-1}U^t_{kt\ell},$$

and

$$W^i_{k\ell m} = U^i_{k\ell m} - \frac{n-1}{(n+1)(n-2)}\delta^i_k U^t_{t\ell m} + \delta^i_\ell\left(\frac{1}{(n+1)(n-2)}U^t_{tkm} - \frac{1}{n-1}U^t_{ktm}\right)$$

$$- \delta^i_m \left(\frac{1}{(n+1)(n-2)} U^t_{tk\ell} - \frac{1}{n-1} U^t_{kt\ell} \right).$$

PROOF. We use Theorem 1, and the identity $n^2-n-2=(n-2)(n+1)$. □

Corollary 4. *Let $n \geq 3$, and assume that $U = U^i_{k\ell m}$ satisfies*

$$U^i_{k\ell m} + U^i_{km\ell} = 0, \qquad U^i_{k\ell m} + U^i_{mk\ell} + U^i_{\ell mk} = 0, \quad U^t_{tk\ell} = 0. \tag{6}$$

Then there exists a unique Weyl tensor $W \in T^1_3 E$, $W = W^i_{k\ell m}$, and a unique tensor $Q \in T^0_2 E$, $Q = Q_{km}$, such that

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_\ell Q_{km} - \delta^i_m Q_{k\ell}.$$

These tensors are given by

$$Q_{k\ell} = \frac{1}{n-1} U^t_{\ell tk}, \quad W^i_{k\ell m} = U^i_{k\ell m} - \frac{1}{n-1}(\delta^i_\ell U^t_{mtk} - \delta^i_m U^t_{\ell tk}).$$

PROOF. This follows from Corollary 2. □

Remark 4. Conditions (6) are equivalent with

$$U^i_{k\ell m} + U^i_{km\ell} = 0, \quad U^i_{k\ell m} + U^i_{mk\ell} + U^i_{\ell mk} = 0, \quad U^t_{kt\ell} = U^t_{\ell tk}.$$

The third of these conditions means that the Ricci tensor is *symmetric.*

We now discuss a symmetry condition which appears in Finsler geometry.

Corollary 5. *Let $\dim E \geq 3$, and let $U \in T^1_3 E$, $U = U^i_{k\ell m}$. Assume that*

$$U^i_{k\ell m} - U^i_{\ell km} = 0.$$

Then there exists a unique Weyl tensor $W \in T^1_3 E$, $W = W^i_{k\ell m}$, and unique tensors $P, R \in T^0_2 E$, $P = P_{\ell m}$, $R = R_{k\ell}$, such that

$$U^i_{k\ell m} = W^i_{k\ell m} + \delta^i_k P_{\ell m} + \delta^i_\ell P_{km} + \delta^i_m R_{k\ell}.$$

These tensors are given by

$$P_{k\ell} = \frac{1}{(n^2-1)(n+4)}((n^2+n-1)U^t_{kt\ell} + U^t_{\ell tk} - (n+1)U^t_{k\ell t}),$$

$$R_{k\ell} = \frac{1}{(n-1)(n+2)}(-U^t_{kt\ell} - U^t_{\ell tk} + (n+1)U^t_{k\ell t}),$$

and

$$\begin{aligned} W^i_{k\ell m} = U^i_{k\ell m} - &\frac{1}{(n^2-1)(n+4)} \\ &\times (\delta^i_k((n^2+n-1)U^t_{ktm} + U^t_{mtk} - (n+1)U^t_{kmt}) \\ &- \delta^i_\ell((n^2+n-1)U^t_{ktm} + U^t_{mtk} - (n+1)U^t_{kmt})) \\ &+ \frac{1}{(n-1)(n+2)}\delta^i_m(U^t_{kt\ell} + U^t_{\ell tk} - (n+1)U^t_{k\ell t}). \end{aligned}$$

PROOF. We use Theorem 1, and the identities $n^3 - n^2 - 3n + 2 = (n-2)(n^2+n-1)$, $n^2 - n - 2 = (n-2)(n+1)$, and $n^3 - 3n - 2 = (n+1)^2(n-2)$. □

3. The Weyl tensors of type $(2,2)$

Consider the tensor space T^2_2E. A tensor $U \in T^2_2E$, $U = U^{ij}_{k\ell}$, has four traces, the $(1,1)$-, $(1,2)$-, $(2,1)$-, and $(2,2)$-traces. In the following Theorem 2 we recall the trace decomposition formula for tensors of type $(2,2)$; this formula defines the Weyl component of a tensor $U \in T^2_2E$. The proof of Theorem 2 can be found in [9]. As before, E denotes a real, n-dimensional vector space such that $n \geq 3$.

Theorem 2. *Let* $\dim E \geq 3$, *and let* $U \in T^2_2E$, $U = U^{ij}_{k\ell}$. *There exists a unique Weyl tensor* $W \in T^2_2E$, $W = W^{ij}_{k\ell}$, *unique traceless tensors* $P, Q, R, S \in T^1_1E$, $P = P^i_k$, $Q = Q^i_k$, $R = R^i_k$, $S = S^i_k$, *and unique numbers* $G, H \in \mathbb{R}$ *such that*

$$U^{ij}_{k\ell} = W^{ij}_{k\ell} + \delta^i_k P^j_\ell + \delta^i_\ell Q^j_k + \delta^j_k R^i_\ell + \delta^j_\ell S^i_k + \delta^i_k\delta^j_\ell G + \delta^i_\ell\delta^j_k H.$$

These tensors are given by

$$P^i_j = \frac{1}{n(n^2-4)}((n^2-2)U^{si}_{sj} - nU^{si}_{js} - nU^{is}_{sj} + 2U^{is}_{js} - n\delta^i_j U^{st}_{st} + 2\delta^i_j U^{st}_{ts}),$$

$$Q^i_j = \frac{1}{n(n^2-4)}(-nU^{si}_{sj} + (n^2-2)U^{si}_{js} + 2U^{is}_{sj} - nU^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),$$

$$R^i_j = \frac{1}{n(n^2-4)}(-nU^{si}_{sj} + 2U^{si}_{js} + (n^2-2)U^{is}_{sj} - nU^{is}_{js} + 2\delta^i_j U^{st}_{st} - n\delta^i_j U^{st}_{ts}),$$

$$S^i_j = \frac{1}{n(n^2-4)}(2U^{si}_{sj} - nU^{si}_{js} - nU^{is}_{sj} + (n^2-2)U^{is}_{js} + 2\delta^i_j U^{st}_{ts} - n\delta^i_j U^{st}_{st}),$$

$$G = \frac{1}{n(n^2-1)}(nU^{st}_{st} - U^{st}_{ts}),$$

$$H = \frac{1}{n(n^2-1)}(-U^{st}_{st} + nU^{st}_{ts}),$$

and

$$W^{ij}_{k\ell} = U^{ij}_{k\ell} - \delta^i_k P^j_\ell - \delta^i_\ell Q^j_k - \delta^j_k R^i_\ell - \delta^j_\ell S^i_k - \delta^i_k \delta^j_\ell G - \delta^i_\ell \delta^j_k H.$$

Before going on to special cases, we fix some notation. We denote by E a real, n-dimensional vector space, endowed with a tensor g of type $(0,2)$. g is supposed to be *symmetric* and *regular*, but is not necessarily positive definite. As an application of Theorem 2, we discuss the trace decomposition of tensors of type $(0,4)$ on E. Since g allows us to raise and to lower indices, the trace operation can be applied to the corresponding tensors of type $(2,2)$. In a basis e_i of E, we write $g = g_{ij} e^i \otimes e^j$, where e^i is the dual basis of this one. Then $g_{ij} = g_{ji}$ and $\det(g_{ij}) \neq 0$. As usual, we denote by g^{ij} the components of the inverse matrix to g_{ij}; then $g_{ij} g^{jk} = \delta^k_i$, $g^{ij} = g^{ji}$.

We consider tensors $V \in T^0_4 E$, $V = V_{mjk\ell}$, satisfying

$$V_{ijk\ell} = -V_{ij\ell k}, \quad V_{ijk\ell} + V_{i\ell jk} + V_{ik\ell j} = 0, \quad V_{ijk\ell} = -V_{jik\ell}. \tag{1}$$

Then V also satisfies

$$V_{ijk\ell} = V_{k\ell ij}.$$

We define a tensor $U \in T^1_3 E$, $U = U^i_{jk\ell}$, by

$$U^i_{jk\ell} = g^{is} V_{jsk\ell}.$$

U obviously satisfies

$$U^i_{jk\ell} + U^i_{j\ell k} = 0, \quad U^i_{jk\ell} + U^i_{\ell jk} + U^i_{k\ell j} = 0.$$

The *Ricci tensor* of U,

$$M_{j\ell} = U^s_{js\ell} = g^{st} V_{jst\ell}, \tag{2}$$

is symmetric, $M_{j\ell} = g^{st} V_{jst\ell} = g^{st} V_{\ell tsj} = M_{\ell j}$. Therefore, the $(1,1)$-trace of U satisfies

$$U^s_{sk\ell} = -U^s_{\ell sk} - U^s_{k\ell s} = -U^s_{\ell sk} + U^s_{ks\ell} = 0.$$

The scalar

$$M = g^{ij} M_{ij} = g^{ij} g^{k\ell} V_{ik\ell j} = g^{ij} U^s_{isj} \tag{3}$$

is the *scalar curvature* of V.

The following is a modification of Corollary 4, Section 2 (see Remark 4, Section 2).

Corollary 1. *Every tensor $V \in T^0_4 E$, $V = V_{mjk\ell}$, satisfying properties* (1) *admits a unique decomposition*

$$V_{rjk\ell} = W_{rjk\ell} + g_{kr} Q_{j\ell} - g_{\ell r} Q_{jk},$$

such that $U^i_{jk\ell} = g^{ri} W_{rjk\ell}$ is a Weyl tensor. The tensors $Q_{k\ell}$ and $W_{rjk\ell}$ are determined by

$$Q_{k\ell} = \frac{1}{n-1} M_{k\ell}, \quad W_{ijk\ell} = V_{ijk\ell} - \frac{1}{n-1}(g_{jk} M_{i\ell} - g_{j\ell} M_{ik}).$$

Proof. It follows from Corollary 4, Section 2, that $U^i_{jk\ell} = W^i_{jk\ell} + \delta^i_k Q_{j\ell} - \delta^i_\ell Q_{jk}$, where

$$Q_{k\ell} = \frac{1}{n-1} U^s_{\ell sk} = \frac{1}{n-1} M_{k\ell},$$

$$W^i_{jk\ell} = g^{is} V_{jsk\ell} - \frac{1}{n-1} \delta^i_k M_{j\ell} + \frac{1}{n-1} \delta^i_\ell M_{jk}.$$

Then

$$W_{ijk\ell} = g_{js} W^s_{ik\ell} = V_{ijk\ell} - \frac{1}{n-1} g_{jk} M_{i\ell} + \frac{1}{n-1} g_{j\ell} M_{ik}. \qquad \square$$

Remark 1. If $U = g^{is} V_{jsk\ell}$ is the curvature tensor of a Levi–Civita connection, then $W = W^i_{jk\ell}$ is the well-known *Weyl projective curvature tensor* of U.

Corollary 2. *Every tensor $V = T^0_4 E$, $V = V_{mjk\ell}$, satisfying properties* (1), *admits a unique decomposition*

$$\begin{aligned} V_{rsk\ell} = g_{ir}g_{js}W^{ij}_{k\ell} + g_{kr}g_{is}P^i_\ell - g_{\ell r}g_{is}P^i_k \\ - g_{ir}g_{ks}P^i_\ell + g_{ir}g_{\ell s}P^i_k - (g_{\ell r}g_{ks} - g_{kr}g_{\ell s})G, \end{aligned}$$

such that the tensors $W^{ij}_{k\ell}$, P^i_ℓ are Weyl tensors. The tensors G, P^i_ℓ, $V^{ij}_{k\ell}$ are given by

$$G = -\frac{1}{n(n-1)}M, \quad P^i_j = -\frac{1}{n-2}g^{iq}M_{qj} + \frac{1}{n(n-2)}\delta^i_j M,$$

and

$$W^{ij}_{k\ell} = g^{ip}g^{jq}V_{pqk\ell} - \delta^i_k P^j_\ell + \delta^i_\ell P^j_k + \delta^j_k P^i_\ell - \delta^j_\ell P^i_k + (\delta^i_\ell\delta^j_k - \delta^i_k\delta^j_\ell)G.$$

Proof. We introduce a tensor $T \in T^2_2 E$, $T = T^{ij}_{k\ell}$, by

$$T^{ij}_{k\ell} = g^{ip}U^j_{pk\ell} = g^{ip}g^{jq}V_{pqk\ell}. \tag{4}$$

The trace decomposition of T is of the form

$$T^{ij}_{k\ell} = W^{ij}_{k\ell} + \delta^i_k P^j_\ell + \delta^i_\ell Q^j_k + \delta^j_k R^i_\ell + \delta^j_\ell S^i_k + \delta^i_k\delta^j_\ell G + \delta^i_\ell\delta^j_k H, \tag{5}$$

where the tensors $W^{ij}_{k\ell}$, P^j_ℓ, Q^j_k, R^i_ℓ, S^i_k, G, and H are uniquely determined by Theorem 2. But by (4), (2), and (3),

$$\begin{aligned} &T^{sj}_{s\ell} = -g^{jq}M_{q\ell}, \quad T^{sj}_{ks} = g^{jq}M_{qk}, \quad T^{is}_{s\ell} = g^{ip}M_{p\ell}, \\ &T^{is}_{ks} = -g^{ip}M_{pk}, \quad T^{st}_{st} = -M, \quad T^{st}_{ts} = M. \end{aligned}$$

In particular, $T^{sj}_{s\ell} = -T^{sj}_{\ell s} = -T^{js}_{s\ell} = T^{js}_{\ell s}$, $T^{st}_{st} = -T^{st}_{ts}$, and we easily get

$$\begin{aligned} &P^i_j = S^i_j = -Q^i_j = -R^i_j = -\frac{1}{n-2}g^{iq}M_{qj} + \frac{1}{n(n-2)}\delta^i_j M, \\ &G = -H = -\frac{1}{n(n-1)}M. \end{aligned} \tag{6}$$

Consequently,

$$\begin{aligned} W^{ij}_{k\ell} &= T^{ij}_{k\ell} - \delta^i_k P^j_\ell - \delta^i_\ell Q^j_k - \delta^j_k R^i_\ell - \delta^j_\ell S^i_k - \delta^i_k\delta^j_\ell G - \delta^i_\ell\delta^j_k H \\ &= g^{ip}g^{jq}V_{pqk\ell} - \delta^i_k P^j_\ell + \delta^i_\ell P^j_k + \delta^j_k P^i_\ell - \delta^j_\ell P^i_k + (\delta^i_\ell\delta^j_k - \delta^i_k\delta^j_\ell)G. \end{aligned} \tag{7}$$

Formulas (6), (7) completely determine the trace decomposition (5). We have

$$T^{ij}_{k\ell} = W^{ij}_{k\ell} + \delta^i_k P^j_\ell - \delta^i_\ell P^j_k - \delta^j_k P^i_\ell + \delta^j_\ell P^i_k - (\delta^i_\ell \delta^j_k - \delta^i_k \delta^j_\ell) G.$$

To conclude the proof, it is now sufficient to compute the induced decomposition of the tensor $V_{rsk\ell} = g_{ir} g_{js} T^{ij}_{k\ell}$. □

Remark 2. We have more possibilities of raising the subscripts of $V = V_{mjk\ell}$ in (4). However, the components of tensors which arise in this way are expressible as linear combinations of (4).

Remark 3. Denote in Corollary 2,

$$C_{ijk\ell} = g_{ir} g_{js} W^{rs}_{k\ell}.$$

Since

$$\begin{aligned} g_{is} P^s_j &= -\frac{1}{n-2} g_{is} g^{sq} M_{qj} + \frac{1}{n(n-2)} g_{is} \delta^s_j M \\ &= -\frac{1}{n-2} M_{ij} + \frac{1}{n(n-2)} g_{ij} M, \end{aligned}$$

we get by a routine calculation

$$\begin{aligned} C_{ijk\ell} = W_{rsk\ell} &+ \frac{1}{n-2}(g_{kr} W_{s\ell} - g_{\ell r} W_{sk} - g_{ks} W_{r\ell} + g_{\ell s} W_{rk}) \\ &+ \frac{1}{(n-1)(n-2)} g_{\ell r} g_{ks} W - \frac{1}{(n-1)(n-2)} g_{kr} g_{\ell s} W. \end{aligned}$$

This is the well-known expression for the *Weyl conformal curvature tensor* of a metric tensor $g = g_{ij}$.

4. The Weyl tensors of type $(1, 4)$

In this section, we consider the tensor space $T^1_4 E$. In Theorem 3 we give the trace decomposition formula for tensors $U \in T^1_4 E$, $U = U^i_{jk\ell m}$, which defines the Weyl component of U. The formula has been derived by Kovár with the help of *Maple* (see [6]). For the sake of uniqueness of the trace decomposition in this case, we have to assume that $\dim E = n \geq 4$.

Since the trace decomposition formula in this case is very long, we first introduce some abbreviations. We set

$$A = (n^2-9)(n^2-4)(n^2-1),$$

$$B = n^6 - 11n^4 + 22n^2 - 6,$$

$$C = n^4 - 9n^2 + 10, \quad D = n^4 - 6n^2 + 3,$$

$$E = n^2 + 1, \quad F = n^2 - 4, \quad G = 2n^2 - 3.$$

Theorem 3. *Let* $\dim E \geq 4$, *and let* $U \in T^1_4 E$, $U = U^i_{jk\ell m}$. *There exists a unique Weyl tensor* $W \in T^1_4 E$, $W = W^i_{jk\ell m}$ *and unique tensors* $P, Q, R, S \in T^0_3 E$, $P = P_{k\ell m}$, $Q = Q_{k\ell m}$, $R = R_{k\ell m}$, $S = S_{k\ell m}$, *such that*

$$U^i_{jk\ell m} = W^i_{jk\ell m} + \delta^i_j P_{k\ell m} + \delta^i_k Q_{j\ell m} + \delta^i_\ell R_{jkm} + \delta^i_m S_{jk\ell}.$$

These tensors are given by

$$\begin{aligned}
P_{k\ell m} = {} & \frac{1}{A}\Big(\frac{B}{n}U^s_{sk\ell m} + \frac{3E}{n}(U^s_{s\ell mk} + U^s_{smk\ell}) - 2F(U^s_{s\ell km} + U^s_{sm\ell k} + U^s_{skm\ell})\Big)\\
& + \frac{1}{A}\Big(-CU^s_{ks\ell m} + \frac{D}{n}(U^s_{\ell skm} + U^s_{ms\ell k}) - E(U^s_{\ell smk} + U^s_{msk\ell}) + \frac{2G}{n}U^s_{ksm\ell}\Big)\\
& + \frac{1}{A}\Big(-CU^s_{\ell ksm} + \frac{D}{n}(U^s_{k\ell sm} + U^s_{mks\ell}) - E(U^s_{m\ell sk} + U^s_{kms\ell}) + \frac{2G}{n}U^s_{\ell msk}\Big)\\
& + \frac{1}{A}\Big(-CU^s_{mk\ell s} + \frac{D}{n}(U^s_{\ell kms} + U^s_{km\ell s}) - E(U^s_{k\ell ms} + U^s_{\ell mks}) + \frac{2G}{n}U^s_{m\ell ks}\Big),\\
Q_{k\ell m} = {} & \frac{1}{A}\Big(-CU^s_{sk\ell m} + \frac{D}{n}(U^s_{s\ell km} + U^s_{sm\ell k}) - E(U^s_{s\ell mk} - U^s_{smk\ell}) + \frac{2G}{n}U^s_{skm\ell}\Big)\\
& + \frac{1}{A}\Big(\frac{B}{n}U^s_{ks\ell m} + \frac{3E}{n}(U^s_{\ell smk} + U^s_{msk\ell}) - 2F(U^s_{\ell skm} + U^s_{ms\ell k} + U^s_{ksm\ell})\Big)\\
& + \frac{1}{A}\Big(-CU^s_{k\ell sm} + \frac{D}{n}(U^s_{\ell ksm} + U^s_{kms\ell}) - E(U^s_{\ell msk} + U^s_{mks\ell}) + \frac{2G}{n}U^s_{m\ell sk}\Big)\\
& + \frac{1}{A}\Big(-CU^s_{km\ell s} + \frac{D}{n}(U^s_{k\ell ms} + U^s_{mk\ell s}) - E(U^s_{\ell kms} + U^s_{m\ell ks}) + \frac{2G}{n}U^s_{\ell mks}\Big),\\
R_{k\ell m} = {} & \frac{1}{A}\Big(-CU^s_{s\ell km} + \frac{D}{n}(U^s_{sk\ell m} + U^s_{s\ell mk}) - E(U^s_{sm\ell k} + U^s_{skm\ell}) + \frac{2G}{n}U^s_{smk\ell}\Big)\\
& + \frac{1}{A}\Big(-CU^s_{ks\ell m} + \frac{D}{n}(U^s_{\ell skm} + U^s_{ksm\ell}) - E(U^s_{\ell smk} + U^s_{msk\ell}) + \frac{2G}{n}U^s_{ms\ell k}\Big)\\
& + \frac{1}{A}\Big(\frac{B}{n}U^s_{k\ell sm} + \frac{3E}{n}(U^s_{\ell msk} + U^s_{mks\ell}) - 2F(U^s_{\ell ksm} + U^s_{m\ell sk} + U^s_{kms\ell})\Big)
\end{aligned}$$

$$+\frac{1}{A}\Big(-CU^s_{k\ell ms}-E(U^s_{\ell mks}+U^s_{mk\ell s})+\frac{D}{n}(U^s_{m\ell ks}+U^s_{km\ell s})+\frac{2G}{n}U^s_{\ell kms}\Big),$$

$$S_{k\ell m}=\frac{1}{A}\Big(-CU^s_{s\ell mk}+\frac{D}{n}(U^s_{s\ell km}+U^s_{skm\ell})-E(U^s_{sk\ell m}+U^s_{smk\ell})+\frac{2G}{n}U^s_{sm\ell k}\Big)$$
$$+\frac{1}{A}\Big(-CU^s_{ksm\ell}+\frac{D}{n}(U^s_{ks\ell m}+U^s_{\ell smk})-E(U^s_{\ell skm}+U^s_{ms\ell k})+\frac{2G}{n}U^s_{msk\ell}\Big)$$
$$+\frac{1}{A}\Big(-CU^s_{k\ell sm}+\frac{D}{n}(U^s_{m\ell sk}+U^s_{kms\ell})-E(U^s_{\ell msk}+U^s_{mks\ell})+\frac{2G}{n}U^s_{\ell ksm}\Big)$$
$$+\frac{1}{A}\Big(\frac{B}{n}U^s_{k\ell ms}+\frac{3E}{n}(U^s_{\ell mks}+U^s_{mk\ell s})-2F(U^s_{\ell kms}+U^s_{m\ell ks}+U^s_{km\ell s})\Big),$$

and

$$W^i_{jk\ell m}=U^i_{jk\ell m}-\delta^i_jP_{k\ell m}-\delta^i_kQ_{j\ell m}-\delta^i_\ell R_{jkm}-\delta^i_mS_{jk\ell}.$$

PROOF. To prove Theorem 3, we solve the trace decomposition equations

$$\begin{gathered}U^i_{jk\ell m}=W^i_{jk\ell m}+\delta^i_jP_{k\ell m}+\delta^i_kQ_{j\ell m}+\delta^i_\ell R_{jkm}+\delta^i_mS_{jk\ell},\\ W^s_{sk\ell m}=0,\quad W^s_{js\ell m}=0,\quad W^i_{jksm}=0,\quad W^i_{jk\ell s}=0.\end{gathered}\tag{1}$$

□

Remark 1. If $n\le 3$, then the trace decomposition equations (1) have more solutions. For $n=3$, all solutions are described in [6].

Remark 2. Theorem 3 provides a basis for an algebraic (as well as geometric) classification of the tensors, arising as the covariant derivative of curvature tensors.

References

[1] N. BOKAN, On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection, *Rendiconti del Circ. Mat. di Palermo* Ser. II **39** (1990), 331–380.

[2] S. S. CHERN, W. H. CHEN and K. S. LAM, Lectures on Differential Geometry, *World Scientific*, 2000.

[3] L. P. EISENHART, Riemannian Geometry, *Princenton University Press*, *Princenton*, 1964.

[4] D. GROMOLL, W. KLINGENBERG and W. MEYER, Riemannsche Geometrie im Grossen, *Springer*, *Berlin*, 1968.

[5] M. HAMERMESH, Group Theory and Its Applications to Physical Problems, *Addison-Wesley*, *Massachusetts*, 1964.

[6] M. Kovár, Decompositions of tensors by the trace operation, Thesis, *Mathematical Institute*, *Silesian Univ. Opava*, 1996 (in *Czech*).

[7] D. Krupka, Natural projectors in tensor spaces, *Beiträge zur Algebra und Geometrie* **43** (2002), 217–231.

[8] D. Krupka, The trace decompositions of tensors of type $(1,2)$ and $(1,3)$, in: New Developments in Differential Geometry, Proc. Colloq. on Diff. Geom., July 1994, Debrecen, (J. Szenthe and L. Tamássy, eds.), *Kluwer Academic Publishers*, *Dordrecht*, 1996, 243–253.

[9] D. Krupka, The trace decomposition problem, *Beiträge zur Algebra und Geometrie* **36** (1995), 303–315.

[10] J. Mikes, On the general trace decomposition problem, in: Differential Gemetry and its Applications, Proc. Conf., Brno (Czech Republic), August 1995, (J. Janyska, I. Kolár, J. Slovák, eds.), *Masaryk University*, *Brno*, 1996, 45–50.

[11] L. Tamássy and T. Q. Binh, On weakly symmetric Riemannian manifolds, Coll. Math. Soc. János Bolyai 56, Budapest, Hungary, (J. Szenthe and L. Tamássy, eds.), *Elsevier Science Publishers*, 1992, 663–670.

[12] T. Y. Thomas, The Differential Invariants of Generalized Spaces, *Cambridge, At the University Press*, 1934.

[13] T. A. Welsh, Young Tableaux and Modules of Groups and Lie Algebras, Ph.D. thesis, *University of Southampton*, 1992.

[14] H. Weyl, The Classical Groups, *Princeton Univ. Press*, *Princeton*, 1946.

[15] H. Weyl, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung, *Götinger Nachrichten*, 1921, 99–112.

DEMETER KRUPKA
INSTITUTE OF THEORETICAL PHYSICS AND ASTROPHYSICS
FACULTY OF SCIENCE, MASARYK UNIVERSITY
KOTLARSKA 2, 611 37 BRNO
CZECH REPUBLIC

E-mail: krupka@physics.muni.cz

(Received November 8, 2002; revised March 24, 2003)