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Variational metric structures

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Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday

Abstract. Relations between Lagrangian structures, metric structures, and semispray connections on a manifold are investigated. Generalized Finsler structures (called *quasifinslerian*) are studied, coming from *integrable time, position and velocity dependent metrics.* For every quasifinslerian metric one has a naturally associated semispray connection, called *canonical connection*, and a *global* Lagrangian, called kinetic energy. One obtains the most general form of metrical connections and related equations for geodesics, which at the same time are variational. As expected, canonical connections generalize the Levi–Civita connection and the connections appearing in Finsler geometry. Relations between quasifinslerian and Lagrange spaces, as well as between metrizability of semispray connections and the existence of variational integrators for second-order ordinary differential equations are also discussed.

1. Introduction

Metric structures on a manifold, such as (pseudo)-Riemannian or Finsler structures, are known to be closely connected with the calculus of

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variations. Indeed, geodesic curves in Riemannian or Finsler geometry are *extremals* of a variational functional defined by a Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \tag{1.1}$$

where, in the case of a Finsler metric, the g_{ij}'s are functions on the tangent space TM of a manifold M, satisfying the following *integrability* and *homogeneity* conditions, respectively:

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}, \qquad \frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \tag{1.2}$$

This fact motivated studies on *metric properties of Lagrangian structures*, as well as studies on *variationality properties of connections appearing in Riemannian and Finsler geometry*. In these directions important results were achieved by M. Anastasiei [1], I. Anderson and G. Thompson [3], M. Crampin, W. Sarlet, E. Martínez, G. B. Byrnes and G. E. Prince [5], J. Grifone [8], J. Grifone and Z. Muzsnay [9], J. Klein [12], [13], D. Krupka and A. Sattarov [16], O. Krupková [17], [19], R. Miron [24], [25], R. Miron and G. Atanasiu [27], A. Rapcsák [30], J. Szilasi and Z. Muzsnay [35], L. Tamássy [36]–[39], and others (see also P. L. Antonelli and R. Miron [2], J. Grifone and Z. Muzsnay [10], O. Krupková [20]).

In this paper we are interested in metric structures connected with systems of second order ordinary differential equations

$$\frac{d^2 c^i}{dt^2} = \Gamma^i \left(t, c(t), \frac{dc}{dt} \right), \quad i = 1, 2, \dots, m, \tag{1.3}$$

for smooth curves $c : R \to M$ (where M is a smooth manifold of dimension m), defined on an open neighborhood of zero in R. Geometrically, such systems of ODE's have the meaning of equations for geodesics of certain non-linear connections, called *semispray connections*, defined to be sections $\Gamma : R \times TM \to R \times T^2M$, where $T^2M \subset TTM$ denotes the manifold of 2-jets of curves defined on a neighborhood of zero into M. In this sense semispray connections can be viewed as a generalization of connections appearing in Riemannian or Finsler geometry, as well as of sprays on tangent bundles. They are closely related with Lagrangian structures, since any regular Lagrangian on $R \times TM$ gives rise to a semispray connection

$\Gamma : R \times TM \to R \times T^2M$. Hence, it is interesting to investigate relations between Lagrangian structures, metric structures and connections from a general point of view. The aim of this paper is to review some recent results in this direction with an emphasis on unifying aspects applying to Riemannian and Finsler geometry, calculus of variations, and applications in physics. The exposition closely follows that of refs. [16]–[19].

We consider a general concept of a *metric* on a manifold M, as a regular symmetric fibered morphism $g : R \times TM \to T^0_2 M$ over id_M, where $T^0_2 M$ denotes the bundle of all tensors of type $(0,2)$ over M. Obviously, such a metric depends on time, positions and velocities. A usual (Riemannian or pseudo-Riemannian) metric, as well as Finsler metrics represent particular cases. Requiring 'homogeneity' one obtains a class of metrics which is studied in a generalized Finsler geometry. Another interesting class, generalizing Finsler spaces, appears when 'integrability' is required. We call a manifold M endowed with an integrable metric g a *quasifinslerian manifold* (*semi-finslerian* in [19]). In this case, as discovered in [19], every metric g has a *unique* associated semispray connection, called *canonical connection.* This connection is *variational* in the sense that there exists a canonical *global* Lagrangian λ such that the (covariant) equations for geodesics of the canonical connection coincide with the Euler–Lagrange equations of λ. Thus, every quasifinslerian manifold is a Lagrange space in the sense of Miron. On the other hand, different Lagrange spaces may give rise to the same quasifinslerian structure. We provide a characterization of equivalent Lagrange spaces and their associated semispray connections with respect to the canonical connection.

Another remarkable feature concerning relation between quasifinslerian structures and Lagrange spaces concerns *global aspects.* Namely, in the calculus of variations one often has to consider not a 'true' Lagrange space, given by a *global* Lagrangian, but rather a class of *local* Lagrangians giving rise to a 'global dynamics'. However, also in this case there is a *global quasifinslerian structure*, meaning that such a class of local Lagrangians defines a global metric and the corresponding global canonical connection [19]. From the point of view of physics this means that to every Lagrangian system (in this general sense) there corresponds a unique global canonical quasifinslerian structure, having the physical meaning of a *free particle* (or a *kinetic energy*) for the arising quasifinslerian manifold. At the same time the semispray connection of the Lagrangian system is uniquely decomposed into the canonical connection, and a soldering form, having the meaning of a *force* (giving a *potential energy* term in the Lagrangian).

Existence of a canonical connection naturally leads to a classification of semispray connections on a quasifinslerian manifold [19]. This, however, is closely related with the question of *metrizability of connections* as well as with the problem of *variational integrators* for second-order differential equations (see e.g. [3], [5], [6], [10], [19], [31], and many others). Applying the new point of view it turns out that metrizability means almost (but not exactly) the same as variationality: metrizable semispray connections are variational, and variational semispray connections express as a sum of a metrizable (canonical) connection and a potential soldering form.

2. Notations and preliminaries

Throughout the paper, manifolds and mappings are smooth, and summation over repeated indices is understood. We denote by T the tangent functor, J^r the r-jet prolongation functor, id the identity mapping, * the pull-back, d the exterior derivative, i_ξ the contraction and ∂_ξ Lie derivative by a vector field ξ.

Let $\pi : Y \to X$ be a fibered manifold, $\dim X = 1$, $\dim Y = m+1$, and $\pi_1 : J^1Y \to X$ and $\pi_{1,0} : J^1Y \to Y$ natural projections. A mapping $\gamma : X \to Y$ defined on an open subset of X is called a *section* of π if $\pi \circ \gamma = \mathrm{id}$. The first prolongation of γ is a section $J^1\gamma$ of π_1. A vector field ξ on J^1Y is said to be π_1-*vertical* (resp. $\pi_{1,0}$-*vertical*) if $T\pi_1.\xi = 0$ (resp. $T\pi_{1,0}.\xi = 0$). A q-form η on J^1Y is called π_1-*horizontal* (resp. $\pi_{1,0}$-*horizontal*), if $i_\xi\eta =$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y. η is called *contact*, if $J^1\gamma^*\eta = 0$ for every section γ of π. A contact $\pi_{1,0}$-horizontal q-form η is called 1-*contact*, if for every π_1-vertical vector field ξ on J^1Y, the form $i_\xi\eta$ is π_1-horizontal; η is called k-*contact*, $2 \le k \le q$, if $i_\xi\eta$ is $(k-1)$-contact. For every q-form on Y there is a unique decomposition

$$\pi_{1,0}^*\eta = \eta_0 + \eta_1 + \cdots + \eta_q, \tag{2.1}$$

where η_0 is a π_1-horizontal form, and η_i, $1 \le i \le q$, are i-contact forms on

J^1Y. We set $h\eta = \eta_0$, $p_i\eta = \eta_i$, and call it the *horizontal* and *i-contact part* of η, respectively. In particular, for a 1-form ρ and a 2-form α we have

$$\pi_{1,0}^*\rho = h\rho + p_1\rho, \qquad \pi_{1,0}^*\alpha = p_1\alpha + p_2\alpha. \tag{2.2}$$

The ideal of contact forms on J^1Y is generated by local contact forms ω^i and $d\omega^i$, $1 \leq i \leq m$, where (in fibered coordinates denoted by $(t, x^i, \dot{x}^i)$)

$$\omega^i = dx^i - \dot{x}^i dt. \tag{2.3}$$

For more details on jet manifolds, calculus of horizontal and contact forms, semispray connections and Lagrangian structures, used throughout the paper, we refer e.g. to D. Krupka [14], [15], O. Krupková [20], L. Mangiarotti and M. Modugno [21], D. J. Saunders [31], [32] and A. Vondra [41].

3. Second order ordinary differential equations on manifolds

First of all, let us recall a geometric model for a system of second order ordinary differential equations of type (1.3). To do this, it is better to work with *graphs* of curves into M, rather than the curves themselves. This means that we shall consider a fibered manifold $\pi : R \times M \to R$, where M is an m-dimensional smooth manifold, and π is the first canonical projection. A curve $c : R \to M$, defined in a neighborhood of $0 \in R$, will then be represented by its graph $\gamma : R \ni t \to \gamma(t) = (t, c(t)) \in R \times M$, which is a *section* of the fibered manifold π. For the first and second jet prolongation of π, i.e. the fibered manifolds $\pi_1 : J^1(R \times M) \to R$ and $\pi_2 : J^2(R \times M) \to R$, respectively, it holds $J^1(R \times M) \approx R \times TM$ and $J^2(R \times M) \approx R \times T^2M$, where $T^2M \subset TTM$ denotes the manifold of 2-velocities, i.e. 2-jets of curves into M.

On $R \times M$ *we use charts adapted to the product structure*, i.e. coordinates of the form (t, x^i), $1 \leq i \leq m$, where t is a global coordinate on R and (x^i) are coordinates on M. In this case the transformation formulas simply read

$$\bar{t} = t, \quad \bar{x}^i = \bar{x}^i(x^1, \dots, x^m), \qquad 1 \leq i \leq m, \tag{3.1}$$

i.e., the "time" and "space" coordinates transform independently. Associated coordinates on $R \times TM$ and $R \times T^2M$ are denoted by $(t, x^i, \dot{x}^i)$ and $(t, x^i, \dot{x}^i, \ddot{x}^i)$, respectively, and the transformation rules are given by

$$\dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^k} \dot{x}^k, \quad \ddot{\bar{x}}^i = \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} \dot{x}^j \dot{x}^k + \frac{\partial \bar{x}^i}{\partial x^k} \ddot{x}^k. \tag{3.2}$$

In what follows, solely charts of this kind are considered.

Any section γ of the fibered manifold π (graph of a curve into M) can be prolonged to a section $J^1\gamma$ of the fibered manifold $R \times TM$, and $J^2\gamma$ of $R \times T^2M$. If $\gamma(t) = (t, c(t))$ then $J^1\gamma(t) = (t, c(t), \dot{c}(t))$ and $J^2\gamma(t) = (t, c(t), \dot{c}(t), \ddot{c}(t))$.

A *semispray connection* on π is a (local) section Γ of the fibered manifold $\pi_{2,1} : R \times T^2M \to R \times TM$. Semispray connections are a kind of the so-called jet connections, or Ehresmann connections, representing a generalization of the classical concept of a linear connection on a manifold (therefore, they are also called "nonlinear connections"). In coordinates the definition $\pi_{2,1} \circ \Gamma = \mathrm{id}_{R \times TM}$ of a semispray connection Γ reads

$$t \circ \Gamma = t, \quad x^i \circ \Gamma = x^i, \quad \dot{x}^i \circ \Gamma = \dot{x}^i, \quad \ddot{x}^i \circ \Gamma = \Gamma^i. \tag{3.3}$$

The functions $\Gamma^i(t, x^k, \dot{x}^k)$ are called *components* of Γ. Note that under coordinate transformations they transform like the coordinates $\ddot{x}^i$.

A (local) section γ of π is called a *geodesic* (or a *path*) of a semispray connection Γ if it satisfies the equation

$$\Gamma \circ J^1\gamma = J^2\gamma. \tag{3.4}$$

In coordinates this turns out to be a system of $m = \dim M$ *second order ordinary differential equations*

$$\frac{d^2c^i}{dt^2} = \Gamma^i \left(t, c^k(t), \frac{dc^k}{dt} \right), \qquad 1 \le i \le m, \tag{3.5}$$

for (the components of) curves $R \ni t \to c(t) \in M$.

There is a number of geometric objects which can be identified with a semispray connection Γ. Namely, it is a *distribution* H_Γ on $R \times TM$, spanned by the (global) vector field

$$D_\Gamma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial \dot{x}^i}, \tag{3.6}$$

or (equivalently) annihilated by $2m$ (local) one-forms

$$\omega^i = dx^i - \dot{x}^i dt, \quad \dot{\omega}^i_\Gamma = d\dot{x}^i - \Gamma^i dt. \tag{3.7}$$

Note that $T(R \times TM) = H_\Gamma \oplus V\pi_1$ (meaning that H_Γ is a horizontal distribution, i.e. complementary to the bundle $V\pi_1$ of π_1-vertical vectors). Moreover, integral sections of the distribution H_Γ coincide with geodesics of Γ. Next, it is the *horizontal form* h_Γ of Γ, which is a projectable (onto identity) vector-valued one-form on $R \times TM$, defined by

$$h_\Gamma = D_\Gamma \otimes dt, \tag{3.8}$$

or the "complementary" vector-valued one form, the *vertical form* of Γ,

$$v_\Gamma = I - h_\Gamma = \frac{\partial}{\partial x^i} \otimes \omega^i + \frac{\partial}{\partial \dot{x}^i} \otimes \dot{\omega}^i_\Gamma, \tag{3.9}$$

where I denotes the identity vector-valued one-form. We can see that $H_\Gamma = \operatorname{Im} h_\Gamma = \ker v_\Gamma$.

The *covariant differential* of a vector valued p-form η with respect to Γ is a vector-valued $(p+1)$-form defined by

$$d_\Gamma \eta = [h_\Gamma, \eta], \tag{3.10}$$

where $[\,,\,]$ is the Frölicher–Nijenhuis bracket (see [33, p. 81]).

There is an important relation between semispray connections and $\pi_{1,0}$-vertical-valued π_1-horizontal one-forms on $R \times TM$, which are called *soldering forms*: namely, soldering forms have the meaning of 'deformations' or 'differences' of connections. More precisely, if Γ is a semispray connection and s is a soldering form then the section Γ' given by $h_{\Gamma'} = h_\Gamma + s$ is another semispray connection. In coordinates,

$$s = s^i \frac{\partial}{\partial \dot{x}^i} \otimes dt, \quad s^i = \Gamma'^i - \Gamma^i. \tag{3.11}$$

A semispray connection $\Gamma : R \times TM \to R \times T^2M$ is called *time-independent* or *autonomous* if there exists a section $\tilde{\Gamma}$ of the fibered manifold $T^2M \to TM$ such that the following diagram is commutative:

$$\begin{array}{ccc} R \times T^2M & \xrightarrow{\ \mathrm{p}_2\ } & T^2M \\ \Big\uparrow \Gamma & & \tilde{\Gamma} \Big\uparrow \\ R \times TM & \xrightarrow{\ \mathrm{p}_2\ } & TM \end{array} \tag{3.12}$$

(p_2 denotes the second canonical projection). This means that the horizontal distribution H_Γ is projectable onto TM (i.e. $T\,\mathrm{p}_2\,.D_\Gamma$ is a vector field on TM), or, equivalently, the components Γ^i of Γ do not depend on t.

4. Dynamical forms

A 2-form E on $R \times T^2M$ is called a *dynamical form* if it is 1-contact and $\pi_{2,0}$-horizontal. A section γ of π is called a *path* of E if

$$E \circ J^2\gamma = 0. \tag{4.1}$$

In coordinates one has $E = E_i(t, x^k, \dot{x}^k, \ddot{x}^k)\, dx^i \wedge dt$ and $\gamma(t) = (t, c^k(t))$, $1 \leq k \leq m$, hence equation (4.1) represents a *system of m second order ordinary differential equations*,

$$E_i\left(t, c^k(t), \frac{dc^k}{dt}, \frac{d^2c^k}{dt^2}\right) = 0, \quad 1 \leq i \leq m. \tag{4.2}$$

Comparing these equations with equations for the geodesics of a semispray connection (3.5), it turns out that equations for paths of dynamical forms correspond to *general* systems of m SODE's, while equations for gedesics of semispray connections cover only those equations which are *explicitly solved with respect to the second derivatives.*

Throughout this paper we restrict ourselves to dynamical forms with components *affine in the second derivatives*, i.e., of the form

$$E_i = A_i(t, x^k, \dot{x}^k) - g_{ij}(t, x^k, \dot{x}^k)\ddot{x}^j. \tag{4.3}$$

(Note that the property of the E_i's to be affine in the second derivatives is well defined over overlapping adapted charts.) We denote the module of dynamical forms on $R \times T^2M$, with components affine in the second derivatives, over the ring of functions on $R \times TM$ by $\mathcal{D}^{\mathrm{af}}(R \times T^2M)$.

A semispray connection $\Gamma : R \times TM \to R \times T^2M$ is called *associated* with a dynamical form $E \in \mathcal{D}^{\mathrm{af}}(R \times T^2M)$ if

$$\Gamma^* E = 0. \tag{4.4}$$

Apparently, a dynamical form E can generally possess more associated semispray connections. E is called *regular* if equation (4.4) has a *unique* solution Γ.

Proposition 4.1. *$E \in \mathcal{D}^{\mathrm{af}}(R \times T^2M)$ is regular if and only if at each point of $R \times TM$*

$$\det(g_{ij}) \neq 0. \tag{4.5}$$

Proof. Writing equation (4.4) in coordinates, we get at each point $x \in R \times TM$ the following system of m linear non-homogeneous equations for the components Γ^j of Γ:

$$A_i - g_{ij}\Gamma^j = 0. \tag{4.6}$$

These equations have a unique solution $(\Gamma^1, \dots, \Gamma^m)$ if and only if the matrix $g = (g_{ij})$ is regular at x. In this case,

$$\Gamma^i = g^{ik}A_k, \tag{4.7}$$

where (g^{ik}) is the inverse matrix to g. □

Local uniqueness of the solution of (4.4) in the regular case immediately gives us a global solution:

Proposition 4.2. *If E is a global (i.e. defined on $R \times T^2M$) regular dynamical form then the connection Γ given by (4.4) is a global section of the fibered manifold $R \times T^2M \to R \times TM$.*

In view of the above, equations for paths of a *regular* dynamical form E can be regarded as a *'covariant form'* of equations for geodesics of a semispray connection Γ.

5. Lagrangian systems

Let us consider a fibered manifold $\pi : R \times M \to R$ and its r-jet prolongation $\pi_r : J^r(R\times M) \to R$, $r \geq 1$. Note that $J^r(R\times M) \approx R\times T^rM$, where T^rM is the manifold of r-jets of curves from a neighborhood of $0 \in R$ into M. A *local Lagrangian* of order r is defined to be a horizontal 1-form λ on an open set $W \subset R \times T^rM$. Since $\lambda = L\,dt$, where t is a global coordinate on R, a Lagrangian can equivalently be given by a function L on W. The (unique) 1-form θ_λ such that $h\theta_\lambda = \lambda$, and $p_1 d\theta_\lambda$ is a dynamical form, is called the *Cartan form* of λ. The forms $d\theta_\lambda$ and $E_\lambda = p_1 d\theta_\lambda$ are called the *Cartan 2-form* and the *Euler–Lagrange form* of the Lagrangian λ, respectively. A Lagrangian λ is called *global* if $W = R \times T^r M$. For $r = 1$ one has $L = L(t, x^i, \dot{x}^i)dt$,

$$\theta_\lambda = L\,dt + \frac{\partial L}{\partial \dot{x}^i}(dx^i - \dot{x}^i dt) = L\,dt + \frac{\partial L}{\partial \dot{x}^i}\omega^i, \tag{5.1}$$

and $E_\lambda = E_i(L)\omega^i \wedge dt = E_i(L)dx^i \wedge dt$, where

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} \tag{5.2}$$

are the *Euler–Lagrange expressions* of λ.

Two Lagrangians λ_1 and λ_2 are called *equivalent* if their Euler–Lagrange forms coincide (up to a possible projection). A Lagrangian is called *trivial* if $E_\lambda = 0$. Apparently, two Lagrangians are equivalent if and only if they differ by a trivial Lagrangian.

Let E be a dynamical form on $R \times T^2 M$. E is called *locally variational*, if for every point $x \in R \times T^2 M$ there exists a neighborhood U and a local Lagrangian $\lambda = L\,dt$ such that $E|_U = E_\lambda$. This means that on U, the components E_i of E coincide with the Euler–Lagrange expressions of L. Hence, for locally variational dynamical forms equations for paths (4.1) (resp. (4.2)) are *Euler–Lagrange equations*. E is called *globally variational* if $E = E_\lambda$ where λ is a global Lagrangian. It is known that local variationality does not imply global variationality, and that obstructions for global variationality are determined by the topology of M.

We have an important theorem expressing a close connection between variational and *closed* 2-forms:

Theorem 5.1 ([18]). *Let E be a dynamical form on $R \times T^2 M$. The following conditions are equivalent:*

(1) *E is locally variational.*

(2) *In every adapted chart, the components E_i of E satisfy the following conditions:*

$$\begin{aligned}
&\frac{\partial E_i}{\partial \ddot{x}^k} - \frac{\partial E_k}{\partial \ddot{x}^i} = 0, \\
&\frac{\partial E_i}{\partial \dot{x}^k} + \frac{\partial E_k}{\partial \dot{x}^i} - 2\frac{d}{dt}\frac{\partial E_k}{\partial \ddot{x}^i} = 0, \\
&\frac{\partial E_i}{\partial x^k} - \frac{\partial E_k}{\partial x^i} + \frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}^i} - \frac{d^2}{dt^2}\frac{\partial E_k}{\partial \ddot{x}^i} = 0.
\end{aligned} \tag{5.3}$$

(3) *There exists a unique closed* 2*-form* α *on* $R \times TM$ *such that* $E = p_1\alpha$.

For a proof we refer to [18] (see also [20]). Formulas (5.3) are called *Helmholtz conditions* (first obtained in [11], [23]). It is easy to see that they imply $\partial^2 E_i/\partial\ddot{x}^j\partial\ddot{x}^k = 0$, i.e. $E \in \mathcal{D}^{\mathrm{af}}(R \times T^2M)$. Using expression (4.3) for the E_i's we arrive at an equivalent form of these conditions, as follows:

$$g_{ik} = g_{ki}, \quad \frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}, \tag{5.4}$$

$$\frac{\partial A_i}{\partial \dot{x}^k} + \frac{\partial A_k}{\partial \dot{x}^i} + 2\frac{\bar{d}g_{ik}}{dt} = 0, \quad \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} = \frac{1}{2}\frac{\bar{d}}{dt}\left(\frac{\partial A_i}{\partial \dot{x}^k} - \frac{\partial A_k}{\partial \dot{x}^i}\right), \tag{5.5}$$

where $\bar{d}/dt = \partial/\partial t + \dot{x}^i\partial/\partial x^i$ is the 'reduced' total derivative operator. The formula for α reads

$$\begin{aligned}\alpha &= E_i\omega^i \wedge dt + \frac{1}{4}\left(\frac{\partial E_i}{\partial \dot{x}^k} - \frac{\partial E_k}{\partial \dot{x}^i}\right)\omega^i \wedge \omega^k + \frac{\partial E_i}{\partial \ddot{x}^k}\omega^i \wedge \dot{\omega}^k \\ &= A_i\omega^i \wedge dt + \frac{1}{4}\left(\frac{\partial A_i}{\partial \dot{x}^k} - \frac{\partial A_k}{\partial \dot{x}^i}\right)\omega^i \wedge \omega^k - g_{ik}\omega^i \wedge d\dot{x}^k.\end{aligned} \tag{5.6}$$

A *closed* two-form α such that $p_1\alpha$ is a *dynamical form* is called a *Lepagean two-form*.

The above theorem has several significant consequences. First, for any Lagrangian λ (of any order) for E, the Cartan 2-form $d\theta_\lambda$ is projectable onto an open set $W \subset R \times TM$, and $\alpha|_W = d\theta_\lambda$. Next, an appropriate version of the Poincaré lemma leads to an explicit construction of Lagrangians for a locally variational form E. Indeed, if (locally) $\alpha = d\rho$ then $\lambda = h\rho$ is a local Lagrangian for E. With help of the contact homotopy operator A (see [15]) one obtains $\rho = A\alpha$ and $\lambda = hA\alpha = Ap_1\alpha$. In coordinates this appears to be the celebrated Tonti–Vainberg–Volterra formula

$$L = x^i \int_0^1 (E_i \circ \chi)du, \tag{5.7}$$

(cf. [40]), where (for an appropriate open set $U \subset R \times T^2M$)

$$\chi : [0,1] \times U \ni (u, (t, x^i, \dot{x}^i, \ddot{x}^i)) \to (t, ux^i, u\dot{x}^i, u\ddot{x}^i) \in U. \tag{5.8}$$

Formula (5.7) gives a *second order* Lagrangian affine in the second derivatives (since the E_i's are affine in the second derivatives). It can be shown, however, that this Lagrangian is locally equivalent to a first order one. This means that second order locally variational forms correspond to *first order Lagrangian systems*.

Theorem 5.1 also gives that a Lagrangian is trivial if and only if $L = df/dt$ for a function f (indeed, $E_\lambda = 0$ means that $\alpha = d\rho = 0$, i.e. $\rho = df$; conversely $L = df/dt$ implies $\theta_\lambda = df$, i.e. $E_\lambda = p_1 d\theta_\lambda = 0$).

Note that the above theorem provides a *bijective correspondence between locally variational forms and Lepagean two-forms.* In view of this result we may state the following definition.

Definition 5.2 ([18]). By a *first-order Lagrangian system* on a fibered manifold $\pi : R\times M \to R$ we mean a locally variational form E on $R\times T^2M$, or equivalently, a Lepagean two-form on $R \times TM$.

By this definition, a first-order Lagrangian system is understood to be the *class of all equivalent Lagrangians* (not as a single global first-order Lagrangian, as usual). The equivalence class contains generally local Lagrangians of all finite orders ≥ 1 (a global Lagrangian need not exist).

Consider the vector field $\xi_t = \partial/\partial t$ on R, which is the (global) generator of the 1-parameter group of transformations $T_a : R \ni t \to t + a \in R$ (called *translations*). It can be naturally lifted to $R \times M$ and to its prolongations. A Lagrangian system is called *time-independent*, or *autonomous* if ξ_t is a symmetry of the Euler–Lagrange form E, i.e., $\partial_{\xi_t} E = 0$. In coordinates,

$$\frac{\partial E_i}{\partial t} = 0, \text{ or equivalently, } \quad \frac{\partial A_i}{\partial t} = 0, \ \frac{\partial g_{ij}}{\partial t} = 0 \tag{5.9}$$

for all i, j. This implies that there exists a family of equivalent local *time-independent first-order Lagrangians* for E (however, time-dependent Lagrangians do exist, as well).

6. Variational metric structures

Denote by $\tau_2^0 : T_2^0M \to M$ the bundle of all tensors of type $(0,2)$ over M. A morphism $g : R \times TM \to T_2^0M$ between the fibered manifolds $R \times TM \to M$ and $T_2^0M \to M$ over the identity of M is called *regular* (resp. *symmetric*) if for every $x \in M$, all the tensors $g_x \in (\tau_2^0)^{-1}(x)$ are regular (resp. symmetric). Every regular and symmetric morphism g will be called an *M-pertinent metric on $R \times TM$*. Thus, g can be thought of as a 'time, space and velocity dependent metric on M'. We denote by $\mathcal{M}(R \times TM)$ the set of M-pertinent metrics on $R \times TM$.

$g \in \mathcal{M}(R \times TM)$ is called *projectable* onto TM (resp. M), if there is a fibered morphism $\tilde{g} : TM \to T_2^0 M$ over id_M (resp. a metric $\hat{g}$ on M), such that the following diagram commutes:

$$\begin{array}{ccc} R \times TM & \xrightarrow{g} & T_2^0 M \\ \downarrow & & \downarrow \mathrm{id} \\ TM & \xrightarrow{\tilde{g}} & T_2^0 M \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} R \times TM & \xrightarrow{g} & T_2^0 M \\ \downarrow & & \downarrow \mathrm{id} \\ M & \xrightarrow{\hat{g}} & T_2^0 M \end{array}$$

Hence, $\tilde{g}$ is a 'space and velocity dependent metric on M', while $\hat{g}$ is a 'usual metric'. A projectable (onto TM or M) M-pertinent metric g on $R \times TM$ is also called *time-independent*.

Theorem 6.1 ([9]). *Let $\Gamma : R \times TM \to R \times T^2 M$ be a semispray connection. There arises a mapping $\mathcal{D}_\Gamma : \mathcal{M}(R \times TM) \to \mathcal{M}(R \times TM)$ defined by the formula*

$$(\mathcal{D}_\Gamma g)_{ij} = D_\Gamma g_{ij} + \frac{1}{2}\left(g_{ik} \frac{\partial \Gamma^k}{\partial \dot{x}^j} + g_{jk} \frac{\partial \Gamma^k}{\partial \dot{x}^i} \right). \tag{6.1}$$

PROOF. It is sufficient to check the transformation properties of $\mathcal{D}_\Gamma g$ under a change of adapted coordinates. This is, however, a routine calculation (for details see [19]). □

The operator $\mathcal{D}_\Gamma$ can be in a straightforward way extended to morphisms $t : R \times TM \to T_s^r M$ over id_M, where $T_s^r M \to M$ is the bundle of r-times contravariant and s-times covariant tensors over M. It will be called Γ-*derivative*. Indeed, $\mathcal{D}_\Gamma$ is defined to be the Lie derivative of a morphism t with respect to the vector field D_Γ, spanning the horizontal distribution H_Γ of the semispray connection Γ. Hence,

$$\mathcal{D}_\Gamma t(z) = \left(\frac{d}{du} (\phi_u^{\Gamma *} t)(z) \right)_{u=0}, \tag{6.2}$$

where $\{\phi_u^\Gamma\}$ denotes the local one-parameter group related with D_Γ.

Definition 6.2. Let g be an M-pertinent metric on $R \times TM$. A semispray connection $\Gamma : R \times TM \to R \times T^2M$ is called *g-compatible* if

$$\mathcal{D}_\Gamma g = 0. \tag{6.3}$$

Denote by $\mathcal{S}(R \times TM)$ and $\mathcal{D}(R \times TM)$ the module of soldering forms and dynamical forms on $R \times TM$, respectively. If M is a manifold endowed with a metric $g \in \mathcal{M}(R \times TM)$ then there is a canonical isomorphism

$$F_g : \mathcal{S}(R \times TM) \ni s \to F_g(s) = \Phi \in \mathcal{D}(R \times TM) \tag{6.4}$$

of modules. It is defined in each adapted chart on $R \times M$ by the formula

$$\Phi_i = g_{ij} s^j. \tag{6.5}$$

With the help of this isomorphism we can identify soldering forms with first-order dynamical forms.

To a given M-pertinent metric g on $R \times TM$ and a semispray connection $\Gamma : R \times TM \to R \times T^2M$ one has a *unique associated regular dynamical form E on $R \times T^2M$* such that

$$E_i = g_{ij}(\Gamma^j - \ddot{x}^j). \tag{6.6}$$

There is an important class of metrics generated by *Lagrangian systems* (cf. Definition 5.2).

Proposition 6.3. *Let E be a locally variational form on $R \times T^2M$. If E is regular then*

$$g_{ij} = -\frac{\partial E_i}{\partial \ddot{x}^j}, \quad 1 \le i, j \le m, \tag{6.7}$$

where E_i are components of E, is an M-pertinent metric on $R \times TM$.

Proof. The matrix (g_{ij}) is regular since E is regular, and symmetric, since E satisfies the Helmholtz conditions (cf. (5.4)). It remains to check the transformation properties of the functions g_{ij}. Consider two overlaping charts (x^i) and $(\bar{x}^i)$ on M. Then in the associated charts it holds $E = E_i dx^i \wedge dt = \bar{E}_i d\bar{x}^i \wedge dt$, hence

$$\bar{E}_i = \frac{dx^k}{d\bar{x}^i} E_k, \tag{6.8}$$

and

$$\bar{g}_{ij} = -\frac{\partial \bar{E}_i}{\partial \ddot{\bar{x}}^j} = -\frac{dx^k}{d\bar{x}^i}\frac{\partial E_k}{\partial \ddot{\bar{x}}^j} = -\frac{dx^k}{d\bar{x}^i}\frac{dx^l}{d\bar{x}^j}\frac{\partial E_k}{\partial \ddot{x}^l} = \frac{dx^k}{d\bar{x}^i}\frac{dx^l}{d\bar{x}^j}g_{kl}, \tag{6.9}$$

proving that formula (6.7) defines a (global) M-pertinent metric on $R \times TM$. □

As it is clear from formula (6.7), *the metric g does not depend on the choice of a particular Lagrangian for E*. However, g *can* be expressed by means of Lagrangians: For example, if L is *(any) first-order* Lagrangian for E, it holds

$$g_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}, \tag{6.10}$$

and if L is a *second-order* Lagrangian for E then we have

$$\begin{aligned} g_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} - \left(\frac{\partial^2 L}{\partial x^i \partial \ddot{x}^j} + \frac{\partial^2 L}{\partial \ddot{x}^i \partial x^j}\right) \\ - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \ddot{x}^j} + \frac{\partial^2 L}{\partial \ddot{x}^i \partial \dot{x}^j}\right). \end{aligned} \tag{6.11}$$

If the Lagrangian system represented by E is *autonomous* then the metric g is projectable onto TM (hence *time-independent*).

In view of the above proposition we can state the following

Definition 6.4 ([19]). We say that an M-pertinent metric g on $R \times TM$ is *variational* if there exists a regular locally variational form E on $R \times T^2M$ such that (in every adapted chart)

$$g_{ij} = -\frac{\partial E_i}{\partial \ddot{x}^j}, \quad 1 \leq i, j \leq m. \tag{6.12}$$

Any dynamical form E satisfying (6.12) is called a *generating form* for the metric g.

Obviously, a generating form is non-unique: the definition of a variational metric only assures that the class of all generating dynamical forms contains at least one locally variational form.

Necessary and sufficient conditions for a metric g to be variational are given by the following theorem.

Theorem 6.5 ([9]). *An M-pertinent metric g on $R \times TM$ is variational if and only if*

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j}, \quad 1 \leq i, j, k \leq m. \tag{6.13}$$

Proof. If g is a variational metric then relations (6.13) follow from Helmholtz conditions (5.4).

We shall prove the converse (cf. [19]). Consider an open ball $W \subset R^m$ with the center at the origin, and denote by (x^i) the canonical coordinates on W. Let g be a metric on $R \times TW$ satisfying (6.13). Define a mapping $\bar{\chi} : [0, 1] \times (R \times TW) \to R \times TW$ setting

$$\bar{\chi}(v, (t, x^i, \dot{x}^i)) = (t, x^i, v\dot{x}^i), \tag{6.14}$$

and put

$$T = \dot{x}^i \dot{x}^j \int_0^1 \Big(\int_0^1 (g_{ij} \circ \bar{\chi})\, dv \Big) \circ \bar{\chi}\, v\, dv. \tag{6.15}$$

We shall show that $T dt$ is a Lagrangian on the fibered manifold $\pi : R \times W \to R$ satisfying

$$g_{ij} = \frac{\partial^2 T}{\partial \dot{x}^i \partial \dot{x}^j} = -\frac{\partial E_i(T)}{\partial \ddot{x}^j}, \tag{6.16}$$

where $E_i(T)$, $1 \leq i \leq m$, are the Euler–Lagrange expressions of $T dt$. Denote

$$F_{ij} = \int_0^1 (g_{ij} \circ \bar{\chi})\, dv, \tag{6.17}$$

and notice that $F_{ij} = F_{ji}$ and $\partial F_{ij} / \partial \dot{x}^k = \partial F_{ik} / \partial \dot{x}^j$. Then, with help of Lemma A.1 (see Appendix),

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}^i} &= 2\dot{x}^j \int_0^1 (F_{ji} \circ \bar{\chi})\, v\, dv + \dot{x}^k \dot{x}^j \int_0^1 \left(\frac{\partial F_{kj}}{\partial \dot{x}^i} \circ \bar{\chi} \right) v^2 dv = \dot{x}^j F_{ji}, \\ \frac{\partial^2 T}{\partial \dot{x}^i \partial \dot{x}^j} &= F_{ij} + \dot{x}^k \frac{\partial F_{ik}}{\partial \dot{x}^j} = \int_0^1 (g_{ij} \circ \bar{\chi})\, dv \\ &\quad + \dot{x}^k \int_0^1 \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \circ \bar{\chi} \right) v\, dv = g_{ij}. \end{aligned} \tag{6.18}$$

Now, if g is an M-pertinent metric on $R \times TM$ satisfying (6.13), one has an open covering $\mathcal{U}$ of $R \times TM$ with the Lagrangian $T dt$ (6.15) defined on each of the open sets of $\mathcal{U}$. However, from the transformation properties of the components g_{ij} of g and of the coordinates $\dot{x}^i$, $1 \leq i \leq m$, it follows immediately that the local Lagrangians $T dt$ can be glued together into a *global* Lagrangian λ_g on $R \times TM$, i.e. such that for each $U \in \mathcal{U}$, $\lambda_g|_U = T dt$. Hence, the Euler–Lagrange form E_g of the Lagrangian λ_g is a generating form for the metric g, meaning that g is variational. □

Definition 6.6 ([19]). If g is a variational metric then the global Lagrangian λ_g defined in the proof of Theorem 6.5 is called the *kinetic energy* of the metric g. The Euler–Lagrange form E_g of λ_g is called the *canonical dynamical form* of the metric g. A manifold M endowed with a variational metric g is called *quasifinslerian manifold*.

Theorem 6.7 ([19]). *Let (M, g) be a quasifinslerian manifold. There exists a unique semispray connection $\Gamma_g : R \times TM \to R \times T^2M$ such that the geodesics of Γ_g coincide with the extremals of the kinetic energy λ_g, i.e., with the solutions of the Euler–Lagrange equations*

$$\frac{\partial T}{\partial x^i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} = 0, \quad 1 \leq i \leq m. \tag{6.19}$$

This connection is determined by the relation

$$\Gamma_g^* E_g = 0, \tag{6.20}$$

where E_g is the canonical dynamical form of g. The components Γ^j, $1 \leq j \leq m$, of Γ_g are given by the following formulas:

$$\Gamma^j = g^{ji}\Gamma_i, \tag{6.21}$$

where (g^{ij}) is the inverse matrix to (g_{ij}), and the functions Γ_i, $1 \leq i \leq m$, are given by

$$-\Gamma_i = \Gamma_{ijk}\,\dot{x}^j\dot{x}^k + \dot{x}^j \int_0^1 \left(\frac{\partial g_{ij}}{\partial t} \circ \bar{\chi}\right) dv, \tag{6.22}$$

where

$$\begin{aligned}\Gamma_{ijk} = {} & \frac{1}{2}\int_0^1 \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - 2\frac{\partial g_{jk}}{\partial x^i}\right) \circ \bar{\chi}\, dv \\ & + \int_0^1 \left(\frac{\partial g_{jk}}{\partial x^i} \circ \bar{\chi}\right) v\, dv\,.\end{aligned} \tag{6.23}$$

Proof. Let E_g be the canonical dynamical two-form of the variational metric g, and consider a section $\Gamma : R \times TM \to R \times T^2M$. Since E_g is *regular*, we get that the equation $\Gamma^* E_g = 0$ has a unique global solution, Γ_g, and the components of Γ_g are of the form $\Gamma^j = g^{jk} A_k$, where the A_k's are defined (using the components of E_g) by $E_k = A_k - g_{ki}\ddot{x}^i$. We shall show that the functions $A_i = \Gamma_i$ are given by the formulas (6.22), (6.23) above. Since E_g comes from the Lagrangian (6.15), we have

$$\Gamma_i = \frac{\partial T}{\partial x^i} - \frac{\partial^2 T}{\partial t\, \partial \dot{x}^i} - \frac{\partial^2 T}{\partial x^k \partial \dot{x}^i}\dot{x}^k. \tag{6.24}$$

In keeping notation (6.17), computing the derivatives of T on the right-hand side, we obtain

$$\begin{aligned}
\frac{\partial T}{\partial x^i} &= \dot{x}^j \dot{x}^k \int_0^1 \Big(\int_0^1 \Big(\frac{\partial g_{jk}}{\partial x^i} \circ \bar{\chi} \Big)\, dv \Big) \circ \bar{\chi}\, v\, dv \\
&= \dot{x}^j\, \dot{x}^k \Big(\int_0^1 \Big(\frac{\partial g_{jk}}{\partial x^i} \circ \bar{\chi} \Big)\, dv - \int_0^1 \Big(\frac{\partial g_{jk}}{\partial x^i} \circ \bar{\chi} \Big)\, v\, dv \Big), \\
\frac{\partial^2 T}{\partial t \partial \dot{x}^i} &= \dot{x}^j \frac{\partial F_{ji}}{\partial t} = \dot{x}^j \int_0^1 \Big(\frac{\partial g_{ji}}{\partial t} \circ \bar{\chi} \Big)\, dv, \\
\frac{\partial^2 T}{\partial x^k \partial \dot{x}^i} \dot{x}^k &= \dot{x}^j \dot{x}^k \frac{\partial F_{ji}}{\partial x^k} = \dot{x}^j \dot{x}^k \int_0^1 \Big(\frac{\partial g_{ji}}{\partial x^k} \circ \bar{\chi} \Big)\, dv \\
&= \frac{1}{2} \dot{x}^j \dot{x}^k \int_0^1 \Big(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} \Big) \circ \bar{\chi}\, dv.
\end{aligned} \tag{6.25}$$

Substituting into (6.24) we get the result.

It remains to show that Γ_g is the unique semispray connection with the desired property. However, this is obvious: semispray connections with the same geodesics have the same horizontal distributions, hence they coincide. □

Definition 6.8 ([19]). The semispray connection Γ_g (6.20) is called the *canonical connection* of g.

We have seen that on every quasifinslerian manifold (M, g) there exists a unique canonical semispray connection Γ_g, a unique canonical (global) Lagrangian λ_g (the kinetic energy), and, consequently, a unique canonical dynamical 2-form E_g (the Euler–Lagrange form of λ_g). Conversely, every manifold M endowed with a *regular* locally variational dynamical form on $R \times T^2M$ (*in particular*, with a global regular Lagrangian on $R \times TM$) is a quasifinslerian manifold.

It is clear, however, that the correspondence between the regular first order Lagrangian systems on a fibered manifold $\pi : R \times M \to R$ and the quasifinslerian structures on M is not one-to-one. We say that two Lagrangians are *metrically equivalent* if the quasifinslerian metrics defined by these Lagrangians coincide. From (6.10) we can immediately see that metrically equivalent Lagrangians are characterized by the following proposition:

Proposition 6.9. *Two first-order Lagrangians L_1, L_2 are metrically equivalent if and only if*

$$L_2 = L_1 + a_i(t, x^k)\dot{x}^i + b(t, x^k), \tag{6.26}$$

where $a_i(t, x^k), b(t, x^k)$ are arbitrary functions, i.e., if the difference $L_2 - L_1$ is an affine function in the "velocities" $\dot{x}^i$.

Corollary 6.10. *Let (M, g) be a quasifinslerian manifold. The class of all g-equivalent Lagrangians is locally of the form*

$$L = T - V \tag{6.27}$$

where T is the kinetic energy of the metric g, and V is any (local) function on $R \times TM$, affine in the $\dot{x}^i$'s.

Naturally, V is called *potential energy.* Contrary to the kinetic energy Lagrangian, potential energy generally is not global (it is defined on an open subset of $R \times TM$).

7. Examples of quasifinslerian structures

7.1. Riemannian metric. Every 'usual' metric g on a manifold M is trivially a variational metric (any signature is admissible). The formulas for the kinetic energy $\lambda_g = T dt$ and the canonical connection Γ_g of g take the standard form

$$T = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \quad \Gamma^i = -\Gamma^i_{jk}\,\dot{x}^j\dot{x}^k, \tag{7.1}$$

respectively, where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) \tag{7.2}$$

are the Christoffel symbols of the *Levi–Civita connection* ∇ of g. The geodesics of Γ coincide with the graphs of geodesics of ∇.

7.2. Time-dependent Riemannian metric. If g is a time-dependent metric on M we get from (6.15) and (6.22) the kinetic energy

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{7.3}$$

and the canonical connection

$$\Gamma^i = g^{ip} \frac{\partial g_{pj}}{\partial t} \dot{x}^j - \Gamma^i_{jk} \dot{x}^j \dot{x}^k, \quad \Gamma^i_{jk} = \frac{1}{2} g^{ip} \left(\frac{\partial g_{pj}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p} \right), \tag{7.4}$$

where the 'Christoffel symbols' Γ^i_{jk} do not depend on $\dot{x}$'s, but are dependent on t. Note that the components Γ^i of the canonical connection are polynomials of degree 2 in the velocities.

7.3. Finsler metric. An M-pertinent metric on $R \times \overset{\circ}{T}M$, where $\overset{\circ}{T}M = TM \setminus \{0\}$, is called *Finsler metric* if it is projectable into TM, and satisfies the 'integrability condition' (6.13) and the 'homogeneity condition'

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \tag{7.5}$$

(See e.g. [4], [22], or [34] for foundations of Finsler geometry.) Thus, every Finsler metric is variational by definition. The *kinetic energy* $\lambda_g = T dt$ of a Finsler metric g reads

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \tag{7.6}$$

since using (A.2), (A.3) (see Appendix) and the homogeneity condition we get from (6.15)

$$\begin{aligned} T &= \dot{x}^i\dot{x}^j \int_0^1 \Big(\int_0^1 (g_{ij} \circ \bar{\chi})\, dv \Big) \circ \bar{\chi}\, v\, dv \\ &= \dot{x}^i\dot{x}^j \int_0^1 \Big(g_{ij} - \int_0^1 \Big(\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k \Big) \circ \bar{\chi}\, dv \Big) \circ \bar{\chi}\, v\, dv \\ &= \dot{x}^i\dot{x}^j \int_0^1 (g_{ij} \circ \bar{\chi}) v\, dv \\ &= \frac{1}{2}\dot{x}^i\dot{x}^j \Big(g_{ij} - \int_0^1 \Big(\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k \Big) \circ \bar{\chi}\, v\, dv \Big) = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j. \end{aligned} \tag{7.7}$$

Taking into account Proposition 6.9 we get that for a Finsler metric g all g-equivalent time-independent first-order Lagrangians are of the form

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j + a_i(x^k)\dot{x}^i + b(x^k). \tag{7.8}$$

Apparently, the kinetic energy of a Finsler metric is a function positively homogeneous of degree 2 in the $\dot{x}^k$'s. Conversely, any smooth function T on $\overset{\circ}{T}M$ such that

$$\frac{\partial T}{\partial \dot{x}^k}\dot{x}^k = 2T \tag{7.9}$$

is a kinetic energy of a Finsler metric. Indeed, $g_{ij} = \partial^2 T/\partial \dot{x}^i \partial \dot{x}^j$ satisfies the homogeneity condition (7.5). Note that the correspondence between Finsler metrics and positively homogeneous functions of degree 2 in the $\dot{x}^k$'s is one-to-one, since if a class of g-equivalent Lagrangians contains a function satisfying (7.9), then this function is unique.

Formulas (6.22), (6.23) for the *canonical connection* Γ_g of a Finsler metric g simplify to

$$\Gamma^i = -\Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = -g^{il}\Gamma_{ljk}\dot{x}^j\dot{x}^k, \tag{7.10}$$

where

$$\begin{aligned} \Gamma_{ljk} &= \frac{1}{2}\int_0^1 \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - 2\frac{\partial g_{jk}}{\partial x^l} \right) \circ \bar{\chi}\, dv + \int_0^1 \left(\frac{\partial g_{jk}}{\partial x^l} \circ \bar{\chi} \right) v\, dv \\ &= \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - 2\frac{\partial g_{jk}}{\partial x^l} \right) + \frac{1}{2}\frac{\partial g_{jk}}{\partial x^l} + \cdots \end{aligned}$$

where $\cdots$ refer to terms vanishing due to homogeneity, i.e.,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{7.11}$$

Thus $\Gamma^i = 2G^i$, where G^i are the *geodesic coefficients* on the Finsler manifold (M, g) (cf. [34]). Finsler geodesics now appear as the solutions of the equations

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0. \tag{7.12}$$

The above functions Γ^i_{jk} generally depend on the $\dot{x}^k$'s, hence the components Γ^i of the canonical connection need not be quadratic in the $\dot{x}^k$'s (if the Γ^i's are quadratic functions in the $\dot{x}^k$'s then g is called a *Berwald metric*). However, homogeneity of g gives us

$$\frac{\partial \Gamma^i}{\partial \dot{x}^k} \dot{x}^k = 2\Gamma^i. \tag{7.13}$$

7.4. Time-dependent Finsler metric. If g is a variational metric on $R \times \overset{\circ}{T}M$ satisfying the homogeneity condition (7.5) then g defines a time-dependent Finsler structure on M. Similarly as above we obtain that the corresponding kinetic energy and canonical connection are of the form

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{7.14}$$

and

$$\Gamma^i = g^{ip} \frac{\partial g_{pj}}{\partial t} \dot{x}^j - \Gamma^i_{jk} \dot{x}^j \dot{x}^k, \quad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \tag{7.15}$$

respectively. Of course, the Γ^i_{jk}'s may depend on t and the $\dot{x}^l$'s.

7.5. Quasifinslerian metrics generated by functions. Any smooth function L on $R \times TM$ (defined possibly on an open subset $W \subset R \times TM$) satisfying the condition

$$\det \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right) \neq 0 \tag{7.16}$$

defines on W a quasifinslerian metric g by

$$(g_{ij}) = \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right). \tag{7.17}$$

The relation between the kinetic energy T of g and L is

$$L = T - V, \tag{7.18}$$

where V is an affine function in the $\dot{x}^i$'s. The relation between the canonical dynamical form E_g and the Euler–Lagrange form of L then reads

$$E = E_g + \Phi, \tag{7.19}$$

where Φ is a locally variational form on W. Explicitly, Φ takes a form of 'covariant Lorentz-type force'

$$\Phi = \Phi_i dx^i \wedge dt, \quad \text{where } \Phi_i = \alpha_{ij}\dot{x}^j + \beta_i, \tag{7.20}$$

with the α_{ij}'s and β_i's not depending upon the x^k's

$$\alpha_{ij} + \alpha_{ji} = 0, \quad \frac{\partial \alpha_{ij}}{\partial x^k} + \frac{\partial \alpha_{ki}}{\partial x^j} + \frac{\partial \alpha_{jk}}{\partial x^i} = 0, \quad \frac{\partial \beta_i}{\partial x^j} + \frac{\partial \beta_j}{\partial x^i} + \frac{\partial \alpha_{ij}}{\partial t} = 0. \tag{7.21}$$

Accordingly, the semispray connection Γ related with L differs from the canonical connection Γ_g of g by the soldering form

$$s = h_\Gamma - h_{\Gamma_g} = g^{ij}\Phi_j \frac{\partial}{\partial \dot{x}^i} \otimes dt. \tag{7.22}$$

If $\partial L/\partial t = 0$ then the arising quasifinslerian structure is time independent. A pair (M, L) of this kind is sometimes called a *Lagrange space.* A Lagrange space (M, L), such that

$$L = T + a_i\dot{x}^i + b \tag{7.23}$$

where T is a (regular) function on $\overset{\circ}{T}M$ *positively homogeneous of degree* 2 *in the* $\dot{x}^k$'s and a_i, b do not depend upon $\dot{x}^k$'s, is a *Finsler space.*

Any smooth function F on $\overset{\circ}{T}M$, *positively homogeneous in the* $\dot{x}^k$'s such that F^2 is regular, is a generating function for a Finsler space (F is called *fundamental function*). Indeed, putting

$$T = \frac{1}{2}F^2 \tag{7.24}$$

we get a regular, nonnegative, positively homogeneous function of degree 2 on $\overset{\circ}{T}M$. With help of the corresponding Finsler metric g, F takes the form

$$F = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}. \tag{7.25}$$

Similar arguments hold for the time-dependent case.

7.6. Variational metrics generated by dynamical forms. Let $E \in \mathcal{D}^{\mathrm{af}}(R \times T^2M)$ be a *regular* dynamical form, $E = E_i dx^i \wedge dt$,

$$E_i = A_i - g_{ij}\ddot{x}^j. \tag{7.26}$$

By Theorem 6.5, if the matrix (g_{ij}) is symmetric and $g_{ij} = -\partial E_i/\partial \ddot{x}^j$ satisfy the integrability conditions (6.13) then E gives rise to a variational metric, g, on $R \times TM$. Dynamical forms with this property will be called *metrical*. Thus, with a metrical dynamical form E on $R \times T^2M$, (M, g) becomes a quasifinslerian manifold, with the kinetic energy $\lambda_g = T dt$, the canonical dynamical form E_g, and the canonical connection Γ_g. Putting $\Phi = E - E_g$ we get a first-order dynamical form, called a *force* related with the quasifinslerian metric g and the dynamical form E. Recall that the (unique) semispray connection Γ associated with E, i.e., $\Gamma^* E = 0$, then differs from the canonical connection Γ_g by a soldering form s, defined by $s = h_\Gamma - h_{\Gamma_g}$. However, by the isomorphism (6.4), forces on a quasifinslerian manifold can be identified with soldering forms. This means that *a metrical dynamical form E on $R \times T^2M$ generates a quasifinslerian manifold (M, g) endowed with a force* (= a first order dynamical form Φ or equivalently a soldering form s, related by $s = {F_g}^{-1}(\Phi)$).

Equations for paths of E (respectively, equations for geodesics of Γ) then read

$$\frac{\partial T}{\partial x^i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} = \Phi_i, \qquad \text{resp.} \qquad \Gamma^i_g = s^i, \tag{7.27}$$

where $s^i = g^{ij}\Phi_j$. Accordingly, one gets a physical interpretation for the dynamical form E as a *mechanical system with the kinetic energy $\lambda_g = T dt$, moving in a force field* Φ. Naturally, $E = E_g$ has the meaning of a *free particle* for the quasifinslerian manifold (M, g).

Keeping the above notations, we get immediately from Theorem 5.1 the following result, characterizing *locally variational forms* on $R \times T^2M$ by means of their related quasifinslerian structure:

Proposition 7.1. *The following conditions are equivalent:*

(1) *$E \in \mathcal{D}^{\mathrm{af}}(R \times T^2M)$ is locally variational.*

(2) *The dynamical form $\Phi = E - E_g$ is locally variational.*

(3) *The Φ_i's are affine functions in the $\dot{x}^k$'s, $\Phi_i = \beta_i + \alpha_{ij}\dot{x}^j$, and the*

two-form

$$\alpha = \beta_i dx^i \wedge dt + \alpha_{ij} dx^i \wedge dx^j \tag{7.28}$$

is closed.

(4) *There is a unique closed* 2*-form* α *on* $R \times M$ *such that* $p_1\alpha = E - E_g$.

Hence, a locally variational form E differs from the corresponding canonical dynamical form E_g by a first order dynamical form (force) Φ, which is *potential* (i.e., locally variational), satisfying Helmholtz conditions. This means, however, that Φ is a (covariant) *Lorentz-type force.* The soldering form $s = h_\Gamma - h_{\Gamma_g}$ (the components of which are $s^i = g^{ij}\Phi_j$) then is a 'contravariant Lorentz-type force'. Denoting a Lagrangian for Φ by V, we have (first order) Lagrangians for E expressed by $L = T - V$, where V is affine in the $\dot{x}^k$'s.

As an example, consider $M = R^3$ with the canonical global chart (x^i), and a dynamical form $E = E_i dx^i \wedge dt$, where

$$-E_i = f\delta_{ij}\ddot{x}^j + \frac{df}{dt}\delta_{ij}\dot{x}^j \tag{7.29}$$

(f is an arbitrary nowhere zero function on R, and (δ_{ij}) is the unit matrix). Note that the equations of the paths of E are the Newton equations for a free particle with nonconstant mass $m(t) = f(t)$, i.e. $dp_i/dt = 0$ with $p_i = f(t)\dot{x}^i$. E defines a time-dependent quasifinslerian metric on R^3 by

$$g_{ij} = -\frac{\partial E_i}{\partial \ddot{x}^j} = f(t)\delta_{ij}. \tag{7.30}$$

For the kinetic energy, the canonical dynamical form and the canonical connection we get $T = \frac{1}{2}f\delta_{ij}\dot{x}^i\dot{x}^j$, $E_g = E$, and

$$\ddot{x}^i \circ \Gamma_g = -\frac{1}{f}\frac{df}{dt}\dot{x}^i, \tag{7.31}$$

respectively. Hence, $\Phi = 0$ and E is a free particle for the quasifinslerian manifold $(R^3, f\delta_{ij})$. Considering a dynamical form Φ on $(R^3, f\delta_{ij})$ we get the mechanical system $\bar{E} = E_g + \Phi$, representing a particle with nonconstant mass $m = f$ moving in the force field Φ.

8. Semispray connections on quasifinslerian manifolds

Let (M, g) be a quasifinslerian manifold, E_g the canonical dynamical form and Γ_g the canonical connection. Consider a semispray connection $\Gamma : R \times TM \to R \times T^2 M$. We have as associated objects a *soldering form* $s = h_\Gamma - h_{\Gamma_g}$, a *dynamical form* $E \in \mathcal{D}^{\mathrm{af}}(R \times T^2 M)$ defined by

$$\frac{\partial E_i}{\partial \ddot{x}^j} = -g_{ij}, \quad \text{and} \quad \Gamma^* E = 0, \tag{8.1}$$

and a *dynamical form* Φ *on* $R \times TM$ defined by $\Phi = E - E_g$.

Similarly as above, it holds $s = F_g{}^{-1}(\Phi)$, and we call s (resp. Φ) the *force* associated with the connection Γ and the metric g. The equations for the geodesics of Γ (resp. the equations for the paths of E) then take the form of (contravariant resp. covariant) Euler–Lagrange equations for non-conservative mechanical systems on (M, g) (7.27).

Theorem 8.1 ([19]). *A semispray connection Γ on a quasifinslerian manifold (M, g) is g-compatible if and only if $h_\Gamma - h_{\Gamma_g} = s$, where $F_g(s) \in \mathcal{D}^{\mathrm{af}}(R \times TM)$, and the components of Φ are of the form*

$$\Phi_i = \beta_i(t, x^k) + \alpha_{ij}(t, x^k)\dot{x}^j, \quad \alpha_{ij} = -\alpha_{ji}. \tag{8.2}$$

Proof. Putting $\Gamma_i = g_{ij}\Gamma^j$, and using (6.13), the g-compatibility condition $\mathcal{D}_\Gamma g = 0$ takes the form

$$\frac{\partial g_{ij}}{\partial t} + \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k + \frac{1}{2}\left(\frac{\partial \Gamma_i}{\partial \dot{x}^j} + \frac{\partial \Gamma_j}{\partial \dot{x}^i}\right) = 0, \tag{8.3}$$

which is one of the Helmholtz conditions (5.5). (Note that this means that the dynamical form E associated to Γ by (8.1) satisfies all but the last set of the Helmholtz conditions.) Substituting $\Phi = E - E_g$ into (8.3) and using that E_g is locally variational we get that components Φ_i of the $\Phi = F_g(s)$ satisfy

$$\frac{\partial \Phi_i}{\partial \dot{x}^j} + \frac{\partial \Phi_j}{\partial \dot{x}^i} = 0. \tag{8.4}$$

Denote

$$a_{ijk} = \frac{\partial^2 \Phi_i}{\partial \dot{x}^j \partial \dot{x}^k}. \tag{8.5}$$

Then differentiating (8.4) with respect to $\dot{x}^k$ we get

$$a_{ijk} + a_{jik} = 0. \tag{8.6}$$

Cycling the indices in (8.6), summing up the arising three relations with appropriate signs and using $a_{ijk} = a_{ikj}$ we obtain

$$a_{ijk} = 0. \tag{8.7}$$

Hence, the Φ_i's are affine in the velocities, and in view of (8.4) they are of the form (8.2).

Conversely, if Φ_i are affine in velocities with skewsymmetric coefficients α_{ij} then $E = E_g + \Phi$ satisfies (8.3), meaning that $\mathcal{D}_\Gamma g = 0$. □

Theorem 8.1 has an interesting application in physics: it provides a characterization of all admissible (possibly time-dependent) g-compatible forces on quasifinslearian manifolds.

Corollary 8.2. *On a quasifinslerian manifold (M, g) the only admissible covariant g-compatible forces are of the form (8.2), i.e. affine in velocities, with skewsymmetric coefficients at $\dot{x}$. In particular, on a Riemannian manifold both covariant and contravariant admissible g-compatible forces are of the form (8.2).*

Let us discuss in more detail the g-compatibility in the time-independent case (i.e. for metrics and connections on tangent bundles). Assume that Γ is a *time-independent semispray connection*, and g is a *time-independent metric* on TM. In this case, formula (6.1) for the Γ-derivative of g reads

$$(\mathcal{D}_\Gamma g)_{ij} = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k + \frac{\partial g_{ij}}{\partial \dot{x}^k}\Gamma^k + \frac{1}{2}\Big(g_{ik}\frac{\partial \Gamma^k}{\partial \dot{x}^j} + g_{jk}\frac{\partial \Gamma^k}{\partial \dot{x}^i}\Big). \tag{8.8}$$

Now, recall that by a *connection on TM* one usually means a fibered morphism $TM \to \mathcal{C}M$ over id_M, where $\mathcal{C}M \to M$ is the *bundle of linear connections over M* [16]. Locally a connection γ on TM is represented by its components γ^i_{jk}, which obey the same transformation rules as the components of a linear connection on M, the functions γ^i_{jk} however may depend on all the variables x^l and $\dot{x}^l$. The covariant derivative of g is locally expressed by the formula

$$(\nabla g)_{ijk} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial \dot{x}^s}\gamma^s_{rk}\dot{x}^r - g_{is}\gamma^s_{jk} - g_{js}\gamma^s_{ik}. \tag{8.9}$$

Geodesic curves of γ are then given by the equations

$$\ddot{x}^i + \gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{8.10}$$

We can see that to any connection γ on TM there corresponds a *unique* time-independent semispray connection Γ. It is determined by the requirement that geodesics of Γ and γ coincide. In coordinates,

$$\Gamma^i = -\gamma^i_{jk}\dot{x}^j\dot{x}^k. \tag{8.11}$$

Of course, the same semispray connection can arise from different connections on TM. We say that connections γ and $\bar{\gamma}$ on TM are *equivalent* if their associated semispray connections coincide. In view of (8.11)

$$\bar{\gamma} \sim \gamma \quad \Longleftrightarrow \quad \bar{\gamma}^i_{(jk)} = \gamma^i_{(jk)} + \varphi^i_{(jk)} \text{ where } \varphi^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{8.12}$$

Above and in what follows (rs) denotes symmetrization in the indicated indices.

Writing (8.8) in terms of γ we obtain

$$\begin{aligned}(\mathcal{D}_\Gamma g)_{ij} &= \left(\frac{\partial g_{ij}}{\partial x^s} - \frac{\partial g_{ij}}{\partial \dot{x}^k}\gamma^k_{rs}\dot{x}^r - \frac{1}{2}\left(g_{ik}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^j} + g_{jk}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^i}\right)\dot{x}^r \right.\\ &\qquad \left. - g_{ik}\gamma^k_{(js)} - g_{jk}\gamma^k_{(is)}\right)\dot{x}^s \\ &= (\nabla g)_{ijs}\dot{x}^s - \frac{1}{2}\left(g_{ik}\frac{\partial \bar{\gamma}^k_{rs}}{\partial \dot{x}^j} + g_{jk}\frac{\partial \bar{\gamma}^k_{rs}}{\partial \dot{x}^i}\right)\dot{x}^r\dot{x}^s,\end{aligned} \tag{8.13}$$

where ∇ denotes the covariant derivative related to a torsion free connection $\bar{\gamma}$ equivalent with γ.

Note that, as expected, the right-hand side of (8.13) does not depend upon a choice of a connection in the equivalence class of γ.

Formula (8.13) gives us a relation between the g-compatible semispray connections Γ and the *torsion free* connections on TM associated with Γ:

$$(\mathcal{D}_\Gamma g)_{ij} = 0 \quad \Longleftrightarrow \quad (\nabla g)_{ijs}\dot{x}^s = \frac{1}{2}\left(g_{ik}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^j} + g_{jk}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^i}\right)\dot{x}^r\dot{x}^s. \tag{8.14}$$

In particular, we have the following result:

Proposition 8.3. *Let γ be a torsion free connection on TM, Γ the associated semispray connection. If γ is metrizable with a quasifinslerian metric g such that Γ is the canonical connection of g, then γ satisfies the homogeneity condition*

$$\frac{1}{2}\left(g_{ik}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^j} + g_{jk}\frac{\partial \gamma^k_{rs}}{\partial \dot{x}^i}\right)\dot{x}^r\dot{x}^s = 0. \tag{8.15}$$

We can see immediately, that, for example, the *Cartan connection* of a Finsler metric possesses the homogeneity property (8.15).

Next, for Riemannian metrics we can conclude the following:

Proposition 8.4. *Let (M, g) be a Riemannian manifold. Then the mapping* (8.11), *assigning to a linear connection ∇ on M a time-independent semispray connection Γ such that Γ^i are quadratic in the $\dot{x}^i$'s, is one-to-one on the quotient set of linear connections modulo torsion. For a semispray connection Γ and the related torsion free linear connection ∇ we have*

$$\mathcal{D}_\Gamma g = 0 \quad \Longleftrightarrow \quad \nabla g = 0. \tag{8.16}$$

Proof. The first assertion is obvious, since

$$\Gamma^i = \gamma^i_{jk}(x)\dot{x}^i\dot{x}^k \quad \Longleftrightarrow \quad \gamma^i_{(jk)} = \frac{1}{2}\frac{\partial^2\Gamma^i}{\partial\dot{x}^j\partial\dot{x}^k}. \tag{8.17}$$

Next, writing (8.13) for a torsion free linear connection ∇ on M we get

$$(\mathcal{D}_\Gamma g)_{ij} = (\nabla g)_{ijk}\dot{x}^k, \quad \text{and} \quad (\nabla g)_{ijk} = \frac{\partial(\mathcal{D}_\Gamma g)_{ij}}{\partial\dot{x}^k}. \tag{8.18}$$

□

Corollary 8.5. *Let (M, g) be a Riemannian manifold. A time-independent semispray connection Γ such that the Γ^i's are quadratic in the velocities is g-compatible if and only if the associated torsion free linear connection is the Levi–Civita connection of g.*

Of course, the above assertion also follows immediately from Theorem 8.1, which more generally answers the question about *all* g-compatible semispray connections on a Riemannian manifold (M, g).

Let (M, g) be a quasifinslerian manifold, Γ a (possibly time-dependent) semispray connection on $R\times TM$. We say that Γ is *variational with respect to the metric g*, if the dynamical form E defined by Γ and g (cf. (8.1)) is locally variational.

Summarizing our results obtained so far we have the following classification of variational semispray connections on quasifinslerian manifolds.

Theorem 8.6. *Let (M, g) be a quasifinslerian manifold, let Γ be a semispray connection on (M, g). The connection Γ is variational with* *respect to the metric g if and only if $h_\Gamma = h_{\Gamma_g} + s$, where Γ_g is the canonical connection of g, and the dynamical form $F_g(s)$ is locally variational.*

Recall that '$F_g(s)$ is locally variational' means that it is affine in the velocities, and the 2-form (7.28) is closed (i.e. the coefficients satisfy (7.21)), or, equivalently that there is a unique closed 2-form α on $R \times M$ such that $F_g(s) = p_1\alpha$ (cf. Section 7.5 and 7.6).

A special case of Theorem 8.6, classifying variational forces in classical and relativistic particle mechanics has been first obtained by E. Engels and W. Sarlet [7], and J. Novotný [29].

We say that a semispray connection Γ is *metrizable* if there exists a variational metric g on $R \times TM$ such that $\Gamma = \Gamma_g$, the canonical connection of g.

Taking into account our above results, we easily conclude

Proposition 8.7. *For a semispray connection Γ on $R \times TM$ we have:*

(1) *Γ is metrizable $\Rightarrow$ Γ is variational $\Rightarrow$ Γ is compatible with a quasifinslerian metric.*

(2) *If Γ is associated with a linear connection ∇ on M then Γ is metrizable with a metric g on M $\Leftrightarrow$ Γ is variational with variational multipliers defined on M $\Leftrightarrow$ Γ is compatible with a metric g on M $\Leftrightarrow$ $\nabla g = 0$.*

Various results of this kind were obtained by I. Anderson and G. Thompson [3], J. Grifone and Z. Muzsnay [9], D. Krupka and A. Sattarov [16], O. Krupková [18], [19], J. Klein [12], [13] and W. Sarlet [31]. Other related significant results on metrizability of connections are due to R. Miron [24], [25], A. Rapcsák [30], J. Szilasi and Z. Muzsnay [35], L. Tamássy [36]–[39], and others.

Finally, let us mention a *geometric interpretation of Helmholtz conditions* (cf. (5.4), (5.5)), arising from the study of quasifinslerian metric structures.

Let E be a regular dynamical form on $R \times T^2M$, and Γ the associated semispray connection (cf. (8.1)).

• The first set of (5.4) means that the dynamical form E is *metrical.*

• (5.4) means that E is metrical and the associated metric is *quasifinslerian.*

• the first set of (5.5) means that there exists an M-pertinent metric g on $R \times TM$ such that Γ is *g-compatible*.

• (5.4) + the first set of (5.5) mean that Γ is *compatible* with a *quasifinslerian* metric g.

• All the Helmholtz conditions mean that Γ is compatible with a quasifinslerian metric g, and the arising force Φ is a *Lorentz-type force* (7.20)–(7.21).

Appendix

Lemma A.1. *Let $I \subset R$ be a neighborhood of the zero, $\mathcal{B}$ an open ball in $R^m \times R^m$ with the center at the origin, and F a smooth function on $I \times \mathcal{B}$. Define the mapping $\bar{\chi} : [0,1] \times I \times \mathcal{B} \to I \times \mathcal{B}$ by*

$$\bar{\chi}(v, (t, x^i, \dot{x}^i)) = (t, x^i, v\dot{x}^i). \tag{A1}$$

Then the following identities hold:

$$F = \int_0^1 (F \circ \bar{\chi}) dv + \dot{x}^i \int_0^1 \left(\frac{\partial F}{\partial \dot{x}^i} \circ \bar{\chi} \right) v \, dv, \tag{A.2}$$

$$F = 2 \int_0^1 (F \circ \bar{\chi}) v \, dv + \dot{x}^i \int_0^1 \left(\frac{\partial F}{\partial \dot{x}^i} \circ \bar{\chi} \right) v^2 \, dv, \tag{A.3}$$

$$\int_0^1 \Big(\int_0^1 (F \circ \bar{\chi}) dv \Big) \circ \bar{\chi} \, v \, dv = \int_0^1 (F \circ \bar{\chi}) dv - \int_0^1 (F \circ \bar{\chi}) v \, dv. \tag{A.4}$$

Proof. The first two formulas follow from the following identities:

$$\begin{aligned} F &= \int_0^1 d(v(F \circ \bar{\chi})) = \int_0^1 (F \circ \bar{\chi}) dv + \int_0^1 v d(F \circ \bar{\chi}), \\ F &= \int_0^1 d(v^2(F \circ \bar{\chi})) = 2 \int_0^1 (F \circ \bar{\chi}) v \, dv + \int_0^1 v^2 d(F \circ \bar{\chi}). \end{aligned} \tag{A.5}$$

The third one is obtained as follows: Substituting (A.2) into the second integral on the right-hand side of (A.4) we get for this integral the expression

$$\begin{aligned}\int_0^1 (F\circ\bar\chi)v\,dv &= \int_0^1 \Big(\int_0^1 (F\circ\bar\chi)dv\Big)\circ\bar\chi\,v\,dv\\ &\quad + \dot x^i \int_0^1 \Big(\int_0^1 \Big(\frac{\partial F}{\partial \dot x^i}\circ\bar\chi\Big)v\,dv\Big)\circ\bar\chi\,v^2\,dv.\end{aligned} \tag{A.6}$$

Denote

$$G = \int_0^1 (F\circ\bar\chi)dv. \tag{A.7}$$

Then applying (A.3) to G yields

$$\begin{aligned}\int_0^1 (F\circ\bar\chi)dv &= 2\int_0^1 \Big(\int_0^1 (F\circ\bar\chi)dv\Big)\circ\bar\chi\,v\,dv\\ &\quad + \dot x^i \int_0^1 \Big(\frac{\partial}{\partial \dot x^i}\int_0^1 (F\circ\bar\chi)dv\Big)\circ\bar\chi\,v^2\,dv\\ &= 2\int_0^1 \Big(\int_0^1 (F\circ\bar\chi)dv\Big)\circ\bar\chi\,v\,dv\\ &\quad + \dot x^i \int_0^1 \Big(\int_0^1 \Big(\frac{\partial F}{\partial \dot x^i}\circ\bar\chi\Big)v\,dv\Big)\circ\bar\chi\,v^2\,dv.\end{aligned} \tag{A.8}$$

Now, (A.8) and (A.6) give (A.4). □

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