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Almost Kählerian structures determined by Riemannian structures

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Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday

Abstract. We prove that for a given Riemannian metric g on an $2n$-dimensional differentiable manifold $\widetilde{M}$ which admits an n-dimensional foliation $\mathcal{F}$, there exists an almost Hermitian structure $(\mathbb{G}, \mathbb{F})$ on $\widetilde{M}$ determined by the pairing $(g, \mathcal{F})$. In particular, we investigate the case when it is almost Kählerian or Kählerian.

0. Introduction

In [3], the first and third authors proved that for every Riemannian structure $\mathbb{G}$ on the total space of the tangent bundle TM of an n-dimensional real differentiable manifold there exists an almost Hermitian structure $(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}})$ determined only by $\mathbb{G}$.

One of the reasons which justifies the existence of the structure $(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}})$ determined by the given Riemannian metric $\mathbb{G}$ is that there exists a vertical distribution V on TM (which is $2n$-dimensional), which is an integrable distribution (foliation).

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The following problem arises: Given a Riemannian metric g on a $2n$-dimensional differentiable manifold $\widetilde{M}$ which admits an n-dimensional foliation $\mathcal{F}$, does there exist an almost Hermitian structure $(\mathbb{G}, \mathbb{F})$ on $\widetilde{M}$ determined by the pairing $(g, \mathcal{F})$?

If $(\mathbb{G}, \mathbb{F})$ exists, determine one such pairing. When it is almost Kählerian or Kählerian?

In the following, we will prove that the answer is affirmative and we will point-out an almost Hermitian structure $(\mathbb{G}, \mathbb{F})$ determined by the Riemannian metric g and the foliation $\mathcal{F}$.

The method which we use is suggested by that of the paper written by K. Matsumoto and R. Miron [3]. In fact, one remarks that the tensor g_1 induced by g on the foliation $\mathcal{F}$ is a symmetric, positive definite d-tensor field on $\widetilde{M}$ (d means distinguished). The distribution N of type $(0, 2)$ orthogonal to the foliation $\mathcal{F}$ defines a non-linear connection on $\widetilde{M}$, determined only by g. Then the lift $\mathbb{G}$ (of Sasakian type) of g_1 is a Riemannian metric on $\widetilde{M}$ determined only by g. But N determines an almost complex structure $\mathbb{F}$, which depends only on g. Then the pairing $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure, constructed only with the help of the pairing $(g, \mathcal{F})$. The cases when $(\mathbb{G}, \mathbb{F})$ is almost Kählerian or Kählerian are easy to establish. Obviously, they impose new conditions to the geometric objects $(g, \mathcal{F})$.

The problem under discussion was stated by Koji Matsumoto at the Conference organized by Prof. L. Verstraelen at the Catholic University of Leuven (in 1999), in honor of Prof. Radu Rosca.

1. Foliations on Riemannian manifolds

Let $\widetilde{M}$ be a $2n$-dimensional real differentiable manifold and $\mathcal{F}$ an n-dimensional foliation on $\widetilde{M}$. Then we choose local coordinates $(x^1, \dots, x^n; y^1, \dots, y^n)$ such that $x^i = x^i_0$ (= constant) give the leaves of $\mathcal{F}$, y^i are variables on the leaves and the coordinate transformations of this type are

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, \dots, x^n), \quad \det\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right) \neq 0, \\ \widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j}(x) y^j, \end{cases} \tag{1.1}$$

where $i, j, k \in \{1, 2, \dots, n\}$.

The distribution $\mathcal{D}_{\mathcal{F}}$ tangent to the leaves of $\mathcal{F}$ is locally spanned by the vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$.

Obviously $\mathcal{D}_{\mathcal{F}}$ has geometric character. Indeed, from (1.1) we obtain the transformation of the natural frames $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ of the form

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \widetilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \widetilde{x}^j} + \dfrac{\partial \widetilde{y}^j}{\partial x^i}\dfrac{\partial}{\partial \widetilde{y}^j}, \\ \dfrac{\partial}{\partial y^i} = \dfrac{\partial \widetilde{y}^j}{\partial y^i}\dfrac{\partial}{\partial \widetilde{y}^j}, \end{cases} \tag{1.2}$$

where $\frac{\partial \widetilde{y}^j}{\partial y^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}$, $\frac{\partial \widetilde{y}^j}{\partial x^i} = \frac{\partial^2 \widetilde{x}^j}{\partial x^i \partial x^k} y^k$.

Let g be a Riemannian metric on the manifold $\widetilde{M}$. In each point $u = u(x, y)$, g has the local components

$$\begin{cases} g\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right) = \overset{(1)}{g}_{ij}(x, y), \quad g\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^j}\right) = \overset{(2)}{g}_{ij}(x, y), \\ g\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial x^j}\right) = \overset{(3)}{g}_{ij}(x, y), \quad g\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) = g_{ij}(x, y). \end{cases} \tag{1.3}$$

It is clear that in any point $u \in \widetilde{M}$, we have

$$\overset{(1)}{g}_{ij} = \overset{(1)}{g}_{ji}, \quad \overset{(2)}{g}_{ij} = \overset{(3)}{g}_{ji}, \quad g_{ij} = g_{ji}. \tag{1.4}$$

With respect to (1.1), $\overset{(1)}{g}_{ij}$, $\overset{(2)}{g}_{ij}$, $\overset{(3)}{g}_{ij}$ and g_{ij} are transformed as follows

$$\begin{cases} \overset{(1)}{g}_{ij} = \dfrac{\partial \widetilde{x}^r}{\partial x^i}\dfrac{\partial \widetilde{x}^s}{\partial x^j}\overset{(1)}{\widetilde{g}}_{rs} + \dfrac{\partial \widetilde{x}^r}{\partial x^i}\dfrac{\partial \widetilde{y}^s}{\partial x^j}\overset{(2)}{\widetilde{g}}_{rs} + \dfrac{\partial \widetilde{y}^r}{\partial x^i}\dfrac{\partial \widetilde{x}^s}{\partial x^j}\overset{(3)}{\widetilde{g}}_{rs} + \dfrac{\partial \widetilde{y}^r}{\partial x^i}\dfrac{\partial \widetilde{y}^s}{\partial x^j}\widetilde{g}_{rs}, \\ \overset{(2)}{g}_{ij} = \dfrac{\partial \widetilde{x}^r}{\partial x^i}\dfrac{\partial \widetilde{y}^s}{\partial y^j}\overset{(2)}{\widetilde{g}}_{rs} + \dfrac{\partial \widetilde{y}^r}{\partial x^i}\dfrac{\partial \widetilde{y}^s}{\partial y^j}\widetilde{g}_{rs}, \\ \overset{(3)}{g}_{ij} = \dfrac{\partial \widetilde{y}^r}{\partial y^i}\dfrac{\partial \widetilde{x}^s}{\partial x^j}\overset{(3)}{\widetilde{g}}_{rs} + \dfrac{\partial \widetilde{y}^r}{\partial y^i}\dfrac{\partial \widetilde{y}^s}{\partial x^j}\widetilde{g}_{rs}, \\ g_{ij} = \dfrac{\partial \widetilde{x}^r}{\partial x^i}\dfrac{\partial \widetilde{x}^s}{\partial x^j}\widetilde{g}_{rs}. \end{cases} \tag{1.5}$$

The last equation shows the tensorial character of g_{ij} (the components of g on $\mathcal{F}$). It is called a d-tensor field. From the last equation (1.3), it follows that $g_{ij}(x,y)$ are the components of a d-tensor field g_1 induced by g on the foliation $\mathcal{F}$. g_1 is a symmetric, positive definite d-tensor field of type $(0,2)$. Then

$$\operatorname{rank}\|g_{ij}\| = n \tag{1.6}$$

on the manifold $\widetilde{M}$. We may consider its contravariant tensor g^{ij} by the equation

$$g_{ir}g^{rj} = \delta_i^j. \tag{1.7}$$

Let

$$M = \{(x^1,\dots,x^n,y^1,\dots,y^n) \in \widetilde{M} \mid y^1 = \dots = y^n = 0\}.$$

It is easily seen that M is an n-dimensional submanifold of $\widetilde{M}$. Then $(\widetilde{M},\pi,M)$ is a differentiable fibre bundle, where the map $\pi:\widetilde{M}\to M$ is defined by

$$\pi(x^1,\dots,x^n,y^1,\dots,y^n) = (x^1,\dots,x^n), \quad \forall (x,y)\in\widetilde{M}.$$

The foliation $\mathcal{F}$ is given by the integrable distribution $\mathcal{D}_{\mathcal{F}}$, which is the kernel of the differential of π. We will denote $\mathcal{D}_{\mathcal{F}}$ by V and we will call it the vertical distribution on the manifold $\widetilde{M}$.

In the Lagrangian geometry [5], [8], the pairing $(M, g_{ij}(x,y)) = GL^n$ is said to be a generalized Lagrange space.

A necessary condition for the existence of a function $L(x,y)$ (called Lagrangian) such that

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{1.8}$$

is that the d-tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is totally symmetric.

If the above property holds, the space GL^n is called reducible to a Lagrange space. If the space GL^n is reducible to a Lagrange space, denoted $L^n = (M, L(x,y))$, and the d-tensor $g_{ij}(x,y)$ has homogeneous components of degree 0 with respect to y^i, then the function

$$L(x,y) = F^2(x,y) = g_{ij}(x,y)y^iy^j$$

is a solution of the equation (1.8) and $F(x,y) = \sqrt{g_{ij}(x,y)y^iy^j}$ is the fundamental function of a Finsler space $F^n = (M, F(x,y))$.

In this case, we say that GL^n is reducible to a Finsler space.

In the following, we will use fundamental geometric objects, for instance, non-linear connections, distinguished connections on GL^n, in order to solve the proposed problem [3].

2. Non-linear connections on the manifold $\widetilde{M}$

Definition. A non-linear connection on the differentiable manifold $\widetilde{M}$ is a regular distribution N on $\widetilde{M}$ complementary to the vertical distribution V, i.e.,

$$T_u\widetilde{M} = N(u) \oplus V(u), \quad \forall u \in \widetilde{M}.$$

It follows that the local dimension of the distribution N is n. Since $V(u)$ is spanned by $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})_u$, then N is locally spanned by vector fields of the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \quad (i = \overline{1,n}\,), \tag{2.1}$$

such that, under a transformation (1.1) on $\widetilde{M}$, they satisfy

$$\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{x}^j}. \tag{2.2}$$

The system of functions $N_i^j(x,y)$ is called the system of the coefficients of the non-linear connection N. By (2.1) and (2.2) one derives:

A transformation of local coordinates (1.1) determines the following transformation of the coefficients N_i^j

$$\widetilde{N}_j^i = \frac{\partial \widetilde{x}^i}{\partial x^s} \frac{\partial x^r}{\partial \widetilde{x}^j} N_r^s + \frac{\partial \widetilde{x}^i}{\partial x^r} \frac{\partial y^r}{\partial \widetilde{x}^j}. \tag{2.3}$$

The converse statement holds too.

As is known, the d-curvature tensors R_{jk}^i and d-torsion tensors t_{jk}^i of the non-linear connection N are given by

$$\begin{cases} R_{jk}^i = \dfrac{\delta N_j^i}{\delta x^k} - \dfrac{\delta N_k^i}{\delta x^j}, \\[2ex] t_{jk}^i = \dfrac{\partial N_j^i}{\partial y^k} - \dfrac{\partial N_k^i}{\partial y^j}. \end{cases} \tag{2.4}$$

The Berwald connection [7] determined by the non-linear connection N has the coefficients

$$B^i_{jk} = \frac{\partial N^i_j}{\partial y^k}. \tag{2.5}$$

One has

$$t^i_{jk} = B^i_{jk} - B^i_{kj}. \tag{2.6}$$

Using (2.3), it follows that under a transformation of coordinates (1.1) on the manifold $\widetilde{M}$, the coefficients $B^i_{jk}(x,y)$ are transformed in the same way as the coefficients of a linear connection on the manifold M, i.e.

$$\widetilde{B}^i_{jk} = \frac{\partial \widetilde{x}^i}{\partial x^s}\frac{\partial x^r}{\partial \widetilde{x}^j}\frac{\partial x^p}{\partial \widetilde{x}^k}B^s_{rp} + \frac{\partial \widetilde{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \widetilde{x}^j \partial \widetilde{x}^k}.$$

Thus, we have the following

Theorem 2.1. *There exist non-linear connections on the manifold $\widetilde{M}$ determined only by the Riemannian metric g and the foliation $\mathcal{F}$. One of them (denoted by N and called canonical) has the coefficients*

$$N^i_j(x,y) = \overset{(2)}{G}_{jm}(x,y) g^{mi}(x,y). \tag{2.7}$$

Proof. Under a transformation (1.1), from (1.5) one gets

$$\overset{(2)}{\widetilde{g}}_{ji} = \frac{\partial x^r}{\partial \widetilde{x}^j}\frac{\partial x^s}{\partial \widetilde{x}^i}\overset{(2)}{g}_{rs} + \frac{\partial x^s}{\partial \widetilde{x}^j}\frac{\partial y^r}{\partial \widetilde{x}^j} g_{sr}.$$

Multiplying by

$$\widetilde{g}^{im} = \frac{\partial \widetilde{x}^i}{\partial x^p}\frac{\partial \widetilde{x}^m}{\partial x^q} g^{pq},$$

we obtain for the coefficients N^i_j the transformation law (2.7) □

The adapted basis of the distribution N is given by (2.1), where N^j_i are given by (2.7).

Then, one has the following geometric interpretation for N.

Proposition 2.1. *The non-linear connection N is characterized by the condition that the distributions N and V are orthogonal with respect to the Riemannian metric g.*

Proof. Indeed, if $\frac{\delta}{\delta x^i}$ $(i=\overline{1,n})$ is the adapted basis of the distribution N, then $N \perp V$ if and only if

$$g\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0, \quad \forall i,j = 1,\dots,n.$$

Then one derives (2.7). □

Applying a known result [7], one obtains:

Theorem 2.2. *The canonical non-linear connection N is integrable if and only if the equations*

$$\frac{\delta(\overset{(2)}{g}_{jm}\, g^{mi})}{\delta x^k} - \frac{\delta(\overset{(2)}{g}_{km}\, g^{mi})}{\delta x^j} = 0 \tag{2.8}$$

are satisfied.

Proof. Indeed, $R^i_{jk} = 0$, by (2.4), with N^i_j given by (2.3), get the necessary and sufficient condition for the integrability. □

Let $(dx^i, \delta y^i), (i=\overline{1,n})$ be the dual basis of the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $(i=\overline{1,n})$. Then, in any point $u \in \widetilde{M}$, one has

$$\delta y^i = dy^i + N^i_j(x,y)dx^j. \tag{2.9}$$

The following result is easy to prove.

Proposition 2.2. 1° *With respect to the adapted basis, the Riemannian metric g is given by the symmetric and positive definite covariant d-tensor fields $h_{ij}(x,y)$ and $g_{ij}(x,y)$ from*

$$g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = h_{ij}(x,y), \quad g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}(x,y). \tag{2.10}$$

2° *The tensor g has the expression*

$$g = h_{ij}(x,y)dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j. \tag{2.11}$$

3° *h_{ij} is given by*

$$h_{ij} = \overset{(1)}{g}_{ij} - N^s_j \overset{(2)}{g}_{is} - N^s_i \overset{(3)}{g}_{sj} + g_{rs}N^r_i N^s_j. \tag{2.12}$$

It is obvious that the Levi–Civita connection of the metric g with respect to the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ of the distributions N and V may be determined as in [3].

We will use the above results for proving the existence of an almost Hermitian structure determined by the Riemannian metric g and the foliation $\mathcal{F}$.

3. Almost Hermitian structures determined by the metric g and the foliation $\mathcal{F}$

Consider the canonical non-linear connection N, with the coefficients (2.3) and the local adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ to the distributions N and V. The dual cobasis is $(dx^i, \delta y^i)$.

It is easy to see, by (1.1), (1.2) and (2.2), that $(dx^i, \delta y^i)$ are transformed by the rule

$$d\widetilde{x}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} dx^j, \quad \delta \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} \delta y^j. \tag{3.1}$$

Taking account that $g_{ij}(x, y)$ is a d-tensor field, we construct the tensor field on $\widetilde{M}$

$$\mathbb{G}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j. \tag{3.2}$$

Since N is the canonical non-linear connection, $g_{ij}(x)$ give the restriction of the Riemannian metric g to the leaves of the foliation $\mathcal{F}$ and (3.1) holds, we have the following:

Theorem 3.1. $\mathbb{G}$ *defined by* (3.2) *is a Riemannian structure on the manifold* $\widetilde{M}$ *determined only by the Riemannian metric* g *and the foliation* $\mathcal{F}$.

Next, we consider the $\mathcal{F}(\widetilde{M})$-linear map

$$\mathbb{F} : \chi(\widetilde{M}) \to \chi(\widetilde{M}),$$

defined by

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = -\frac{\delta}{\delta x^i}, \quad (i = 1, \ldots, n). \tag{3.3}$$

Remark. In [3], $\mathbb{F}$ is defined by

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n).$$

It is easily seen that $\mathbb{F}$ is invariant under coordinates transformations on $\widetilde{M}$ and one has

Theorem 3.2. *The following properties holds:*
1° $\mathbb{F}$ *is an almost complex structure globally defined on* $\widetilde{M}$

$$\mathbb{F} \circ \mathbb{F} = -I. \tag{3.4}$$

2° $\mathbb{F}$ *is a tensor filed on* $\widetilde{M}$ *having the local expression*

$$\mathbb{F} = \frac{\partial}{\partial y^i} \otimes dx^i - \frac{\delta}{\delta x^i} \otimes \delta y^i. \qquad (3.3)\text{'}$$

3° *The structure* $\mathbb{F}$ *depends only on the Riemannian metric* g *and the foliation* $\mathcal{F}$.

Proof. 1° Using (1.2) and (2.2), it follows that $\mathbb{F}$ defined by (3.3) does not depend on the choice of the local coordinates (x^i, y^i). (3.4) holds because for any $X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$, we have $\mathbb{F} \circ \mathbb{F}(X) = -X$.

2° Obviously (3.3) and (3.3)' are equivalent.

3° $\frac{\delta}{\delta x^i}$ and δy^i depend only on g and $\mathcal{F}$. □

Theorem 3.3. $\mathbb{F}$ *is a complex structure if and only if the curvature d-tensors* R^i_{jk} *and the torsion d-tensors* t^i_{jk} *of the canonical non-linear connection* N *vanish identically.*

Proof. With respect to the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, the Nijenhuis tensor $\mathcal{N}_{\mathbb{F}}$ of the almost complex structure $\mathbb{F}$

$$\mathcal{N}_{\mathbb{F}}(X, Y) = -[X, Y] + [\mathbb{F}X, \mathbb{F}Y] - \mathbb{F}[X, \mathbb{F}Y] - \mathbb{F}[\mathbb{F}X, Y]$$

vanishes identically if and only if

$$\mathcal{N}_{\mathbb{F}}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \quad \mathcal{N}_{\mathbb{F}}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0, \quad \mathcal{N}_{\mathbb{F}}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

The above system holds if and only if $R^i_{jk} = 0$ and $t^i_{jk} = 0$ [7]. □

Finally, the proposed problem is solved by the help of the following theorem.

Theorem 3.4. 1° *The pairing* $(\mathbb{G}, \mathbb{F})$ *is an almost Hermitian structure on* $\widetilde{M}$*, determined only by the Riemannian metric* g *and the foliation* $\mathcal{F}$.

2° *The associated* 2*-form* θ *to the pairing* $(\mathbb{G}, \mathbb{F})$ *has the local expression*

$$\theta = g_{ij} dx^i \wedge \delta y^j. \tag{3.5}$$

3° θ *is an almost symplectic structure on* $\widetilde{M}$ *determined only by* g *and* $\mathcal{F}$.

Proof. 1° The equation $\mathbb{G}(\mathbb{F}X, \mathbb{F}Y) = \mathbb{G}(X, Y)\ \forall X, Y \in \chi(\widetilde{M})$ is satisfied on the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$.

2° $\theta(X, Y) = \mathbb{G}(\mathbb{F}X, Y)$ leads to

$$\theta\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \qquad \theta\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = g_{ij},$$

$$\theta\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = -g_{ij}, \quad \theta\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

3° θ given by (3.5) is a 2-form on $\widetilde{M}$, with $\det\|\theta\| = 2n$. □

Theorem 3.5. *The almost symplectic structure* θ *defined by* (3.5) *is integrable (or, equivalently, symplectic) if and only if the following equations hold:*

$$\begin{cases} g_{ri} R^r_{jk} + g_{rj} R^r_{ki} + g_{rk} R^r_{ij} = 0, \\ \nabla_k g_{ij} - \nabla_j g_{ik} - g_{ir} t^r_{jk} = 0, \\ \overset{\circ}{\nabla}_k\, g_{ij} - \overset{\circ}{\nabla}_j\, g_{ik} = 0, \end{cases} \tag{3.6}$$

where

$$\nabla_k g_{ij} = \frac{\delta g_{ij}}{\delta x^k} - g_{sj} B^s_{ki} - g_{is} B^s_{kj}, \tag{3.7}$$

∇_k is the covariant h-derivative of g_{ij} with respect to the transpose of the Berwald connection B^i_{jk} and

$$\overset{\circ}{\nabla}_k\, g_{ij} = \frac{\partial g_{ij}}{\partial y^k}.$$

The proof runs similarly as in Section 4 of [3]. The conditions (3.5) represent the necessary and sufficient condition $d\theta = 0$ for the integrability of the structure θ.

Corollary 3.1. *The structure* $(\mathbb{G}, \mathbb{F})$ *on* $\widetilde{M}$ *is almost Kählerian if and only if the equations* (3.6) *hold.*

Corollary 3.2. *The structure* $(\mathbb{G}, \mathbb{F})$ *on* $\widetilde{M}$ *is Kählerian if and only if the Riemannian metric* g *and the foliation* $\mathcal{F}$ *have the following properties:*

1° $\mathbb{F}$ *is a complex structure on* $\widetilde{M}$ *(i.e.* $R^i_{jk} = t^i_{jk} = 0$.*)*

2° *The following equations hold*

$$\begin{cases} \nabla_k g_{ij} - \nabla_j g_{ik} = 0, \\ \overset{\circ}{\nabla}_k\, g_{ij} - \overset{\circ}{\nabla}_j\, g_{ik} = 0. \end{cases} \tag{3.8}$$

Remark. ∇_k is the covariant h-derivative of the d-field g_{ij} with respect to Berwald connection.

In particular, if the generalized Lagrange space $GL^n = (\widetilde{M}, g_{ij}(x, y))$ is reducible to a Riemannian space [5], that is, $g_{ij}(x, y)$ depends only on the variables (x^i), then from $g_{ij}(x, y) = g_{ij}(x)$ it follows that we may determine a non-linear connection $\overset{\circ}{N}$, which is different from the canonical non-linear connection N. The coefficients of $\overset{\circ}{N}$ are

$$\overset{\circ}{N}{}^i_j = \gamma^i_{jk}(x) y^k, \tag{3.9}$$

where γ^i_{jk} are the Christoffel symbols of the d-tensor field $g_{ij}(x)$.

The curvature of this non-linear connection is given by

$$\overset{\circ}{R}{}^i_{jk}\,(x, y) = y^h r^i_{hjk}(x), \tag{3.10}$$

where $r^i_{jk}(x)$ is the curvature tensor of $g_{ij}(x)$.

The Berwald connection of $\overset{\circ}{N}$ is

$$B^i_{jk} = \gamma^i_{jk}(x),$$

and the torsion $t^i_{jk} = 0$.

The Riemannian structure $\mathbb{G}$ is defined by

$$\mathbb{G} = g_{ij}(x)dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j,$$

where $\delta y^i = dy^i + \gamma^i_{jk}(x)y^k dx^j$.

The almost complex structure $\mathbb{F}$ is given by (3.3)'.

The pairing $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure determined only by the foliation $\mathcal{F}$ and the d-tensor g_1 induced by g on the foliation $\mathcal{F}$, g_1 having the components $g_{ij}(x)$. Then the pairing $(M, g_{ij}(x))$ is a Riemannian space.

The 2-form θ associated to the pairing $(\mathbb{G}, \mathbb{F})$ is integrable, because, by using the Bianchi identity and the fact that

$$\nabla_k g_{ij} = 0, \quad \overset{\circ}{\nabla}_k\, g_{ij} = 0,$$

it follows that the equations (3.6) are satisfied.

Consequently, we may state the following.

Theorem 3.6. 1° *The structure* $(\mathbb{G}, \mathbb{F})$ *determined on the manifold* $\widetilde{M}$ *by a Riemannian d-structure* $g_{ij}(x)$ *and a foliation* $\mathcal{F}$ *is an almost Kählerian structure.*

2° *The structure* $(\mathbb{G}, \mathbb{F})$ *determined by* $g_{ij}(x)$ *and the foliation* $\mathcal{F}$ *is Kählerian if and only if the Riemannian space* $(M, g_{ij}(x))$ *is locally flat.*

Proof. 1° We already saw that the equations (3.6) hold. Then we apply the Corollary 3.1. 2° The tensor $\overset{\circ}{R}{}^i_{jk} = y^h r^i_{hjk}(x)$ vanishes if and only if the curvature tensor $r^i_{hjk}(x)$ of the metric $g_{ij}(x)$ on M vanishes.

Since $t^i_{jk} = \gamma^i_{jk}(x) - \gamma^i_{kj}(x) = 0$ and the equations (3.8) hold, then the Corollary 3.2 achieves the proof. □

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